Quantum Chromodynamics

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QCD-2

Plan

- Form Factors in Gauge Theories
- Infrared Structure
  - Soft
  - Collinear
- Multi-leg, Multi-loop amplitudes
  - K+G equation
  - Catani's proposal
- Factorisation and Resummation
- Casimir Duality
- UV from IR

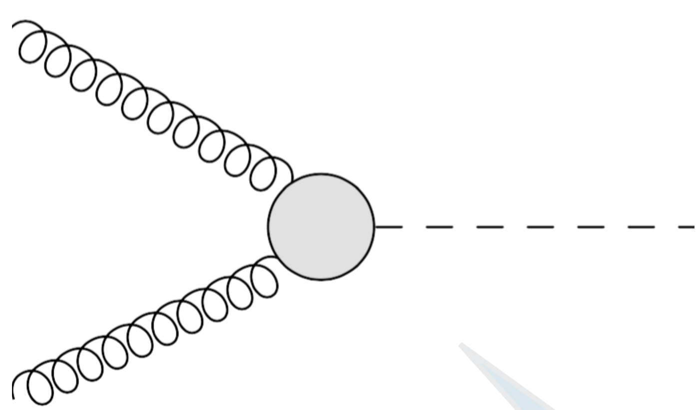
Form Factor

Form Factor : On-shell matrix elements of composite operators

$$< p'|\mathcal{O}|p >$$

Gauge boson form factor

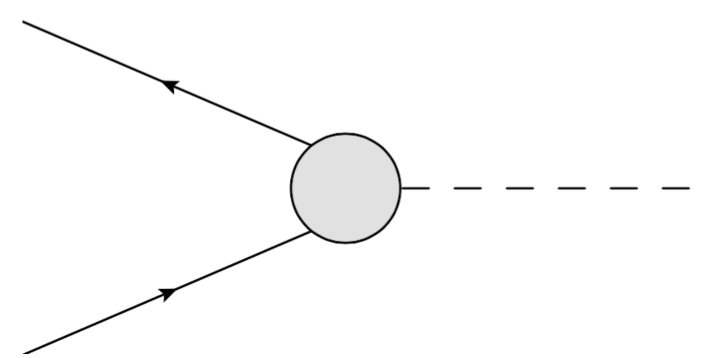
$$< g(p')|G^a_{\mu\nu}G^{\mu\nu a}|g(p) >$$



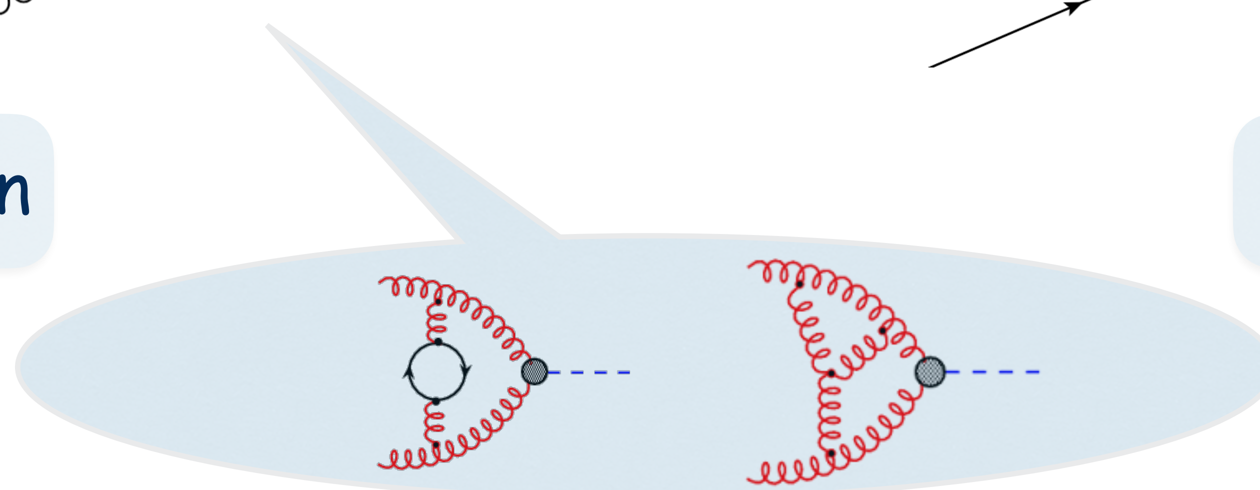
Higgs production

Fermion form factor

$$< e(p')|\overline{\psi}\gamma_\mu\psi|e(p) >$$



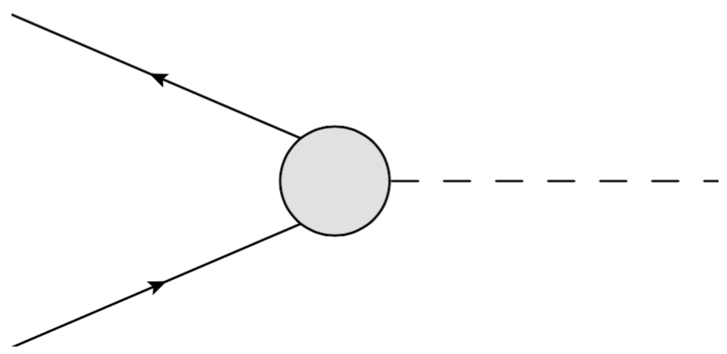
Drell-Yan



Sudakov Form Factor

[Sudakov, Sen, Sterman, Collins, Magnea]

Large $q^2 = (p+p')^2$ behaviour of form factors



Sudakov: $< e(p')|\overline{\psi}\gamma_\mu\psi|e(p) >$

$$\exp\left(-\frac{g^2}{8\pi^2}\ln^2(\frac{q^2}{m^2})\right)$$

Sen: QCD Leading and subleading logs exponentiate:

$$\left(\frac{g_s^2}{8\pi^2}\right)^n \ln^\nu\left(\frac{q^2}{m^2}\right), \qquad 2\nu \le n$$

$< q(p')|\overline{\psi}\gamma_\mu\psi|q(p) >$

Infrared divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

Quantum Field Theories with massless (even almost) particles encounter two kinds of infrared divergences:

Soft :

Soft divergences arise due to the presence of massless gauge bosons.

Collinear :

Collinear divergences arise when atleast two massless particles become collinear to each other

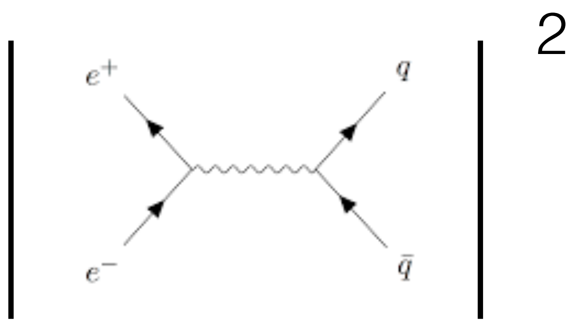
- Collinear gauge fields
- When the hard scale of the process is much larger than mass of the matter fields,

Electron-Positron annihilation

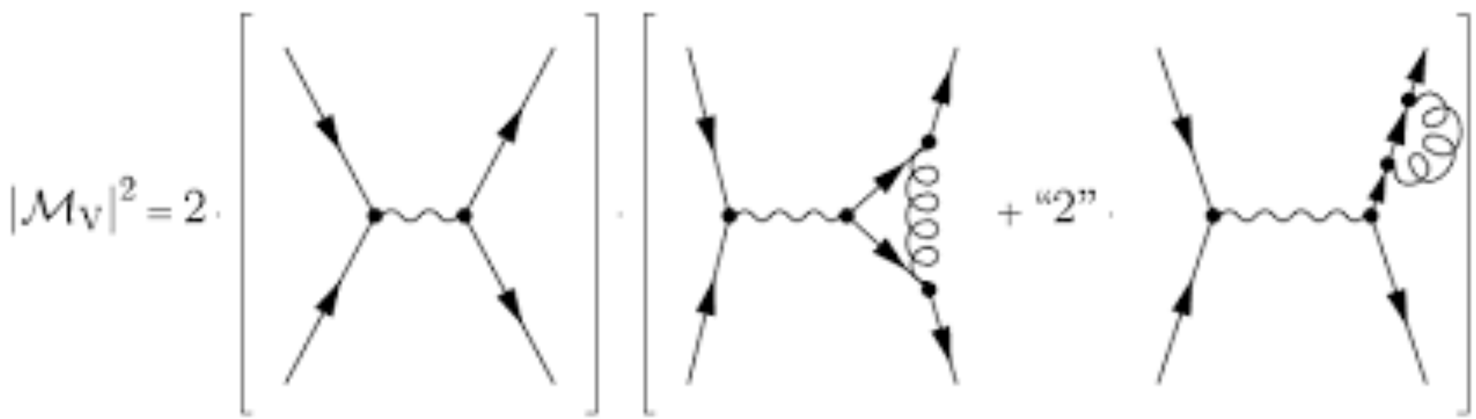
Inclusive hadroproduction in Electron-Positron scattering

$e^- + e^+ \to$ hadrons

$|\mathcal{M}_B|^2 =$



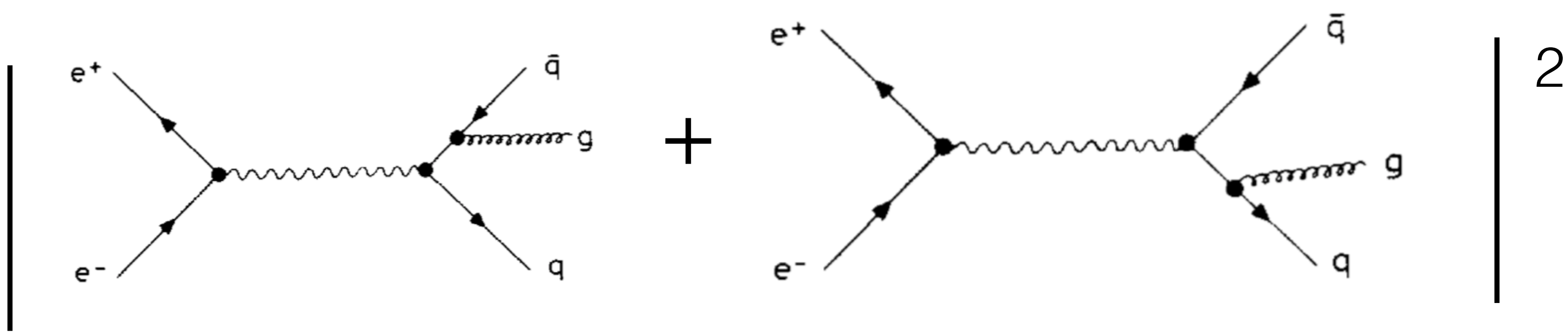
Born



$a_s(\mu_R^2)$

Virtual

$|\mathcal{M}_R|^2 =$



$a_s(\mu_R^2)$

Real emission

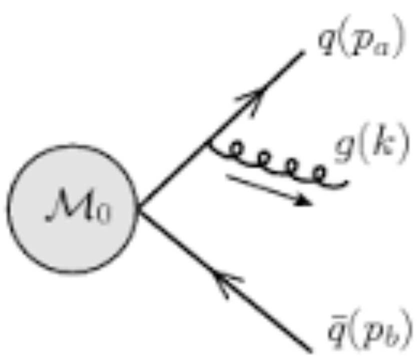
$$\sigma^h(s) = \sigma_B(s)\Big[1 + a_s(\mu_R^2)\sigma^{(1)}(s) + \mathcal{O}(a_s^3)\Big]$$

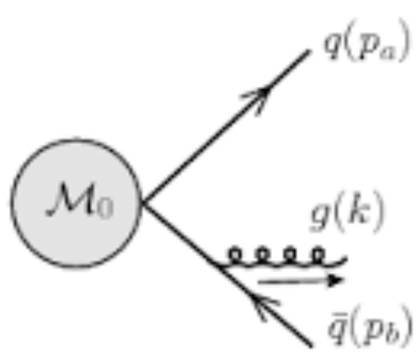
Infrared divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

In the Limit $k \to p \quad (p_a \text{ or } \mathrm{p_b})$ $m_a, m_b << Q$

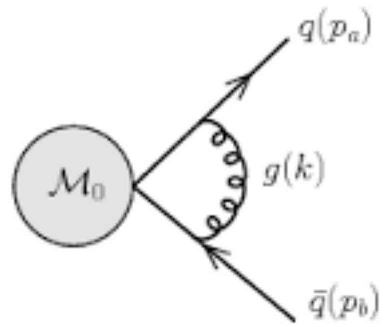
Real emission





$$\frac{1}{(p+k)^2} = \frac{1}{2p^0 k^0 (1 - \cos\theta)}$$

Virtual



$k^0 \to 0$ Soft divergence

$\cos\theta \to 0$ Collinear divergence

Infrared divergences

[S Weinberg]

"In (Yang-Mills theory) a soft photon (gluon) emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons (gluons), and so on, building up a cascade of soft massless particles each of which contributes an infra-red divergence.

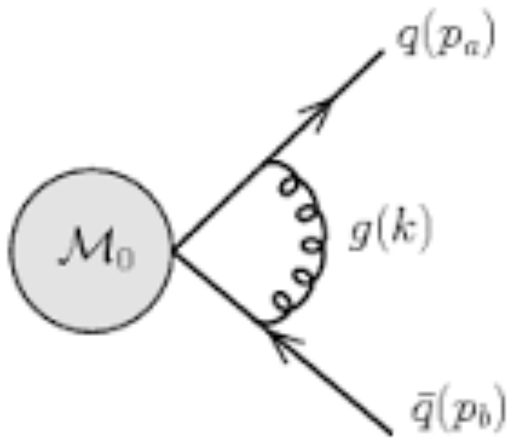
The elimination of such complicated interlocking infra-red divergences would certainly be a Herculean task, and might not even be possible."

S. Weinberg, Phys. Rev. 140B (1965)

Sudakov Form Factor

One loop on-shell form factor

$$(p-k)^2 = 0,$$
$$p_a^2 = p_b^2 = m^2 << q^2$$



Soft

$k \to 0$

Collinear

$p_a || k \;\; or \;\; p_b || k$

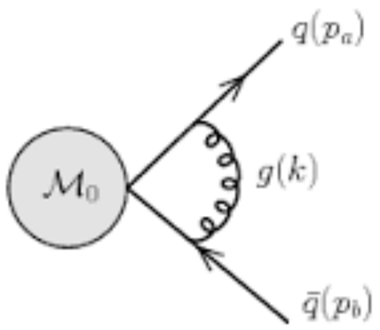
$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2((p_a+k)^2 - m^2)((p_b-k)^2 - m^2)} \to \infty$$

Ill defined

Virtual effect

One loop on-shell form factor

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2((p_a+k)^2-m^2)((p_b-k)^2-m^2)} \to \infty$$



$$p_a^2 = p_b^2 = m^2 << q^2$$

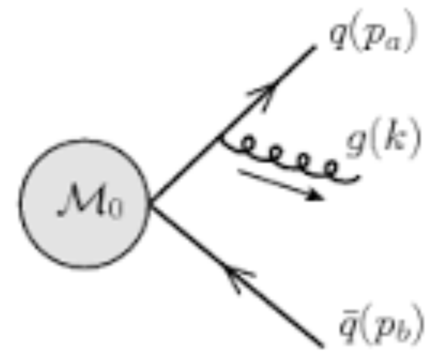
Summing to all orders in g^2

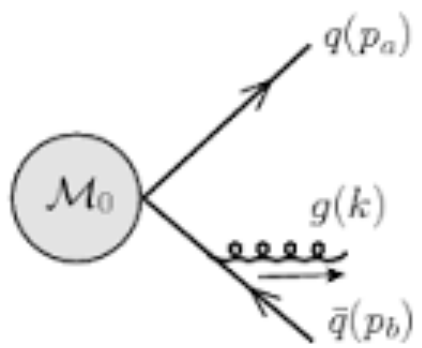
$$1 - g^2\Big(\infty\Big) + \frac{1}{2!}g^4\Big(\infty\Big) - \frac{1}{3!}g^6\Big(\infty\Big) + \cdots = exp(-g^2\infty)$$

Probability to happen this is ZERO

Real emission

Real photon emission:





$$\int \frac{d^4k}{(2\pi)^4} \frac{\delta^+(k^2)}{((p_a+k)^2-m^2)((p_b-k)^2-m^2)} \to \infty$$

Summing multiple emissions

$$p_a^2 = p_b^2 = m^2 << q^2$$

$$1 + g^2\infty + g^4\infty + \cdots$$

Probability grows uncontrollably

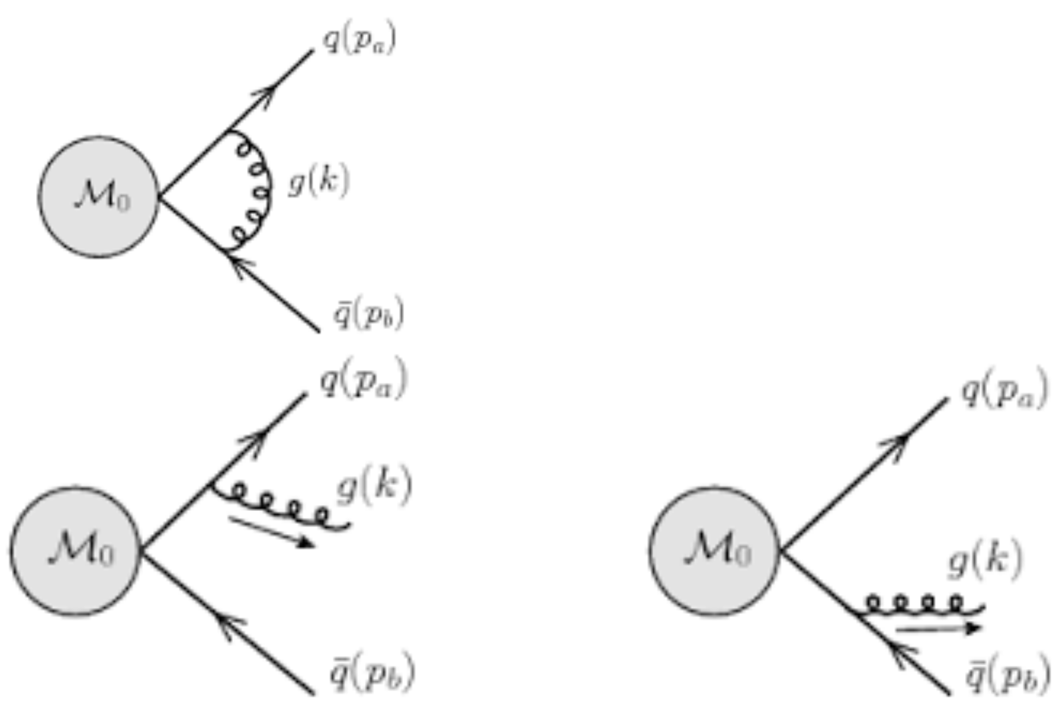
"Weinberg Fear"

Indistinguishable states

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

We know that the detectors are not sensitive to photons below certain energy Es (soft ones)

Below this energy the Detector can not distinguish these two processes when the gluons are soft/collinear

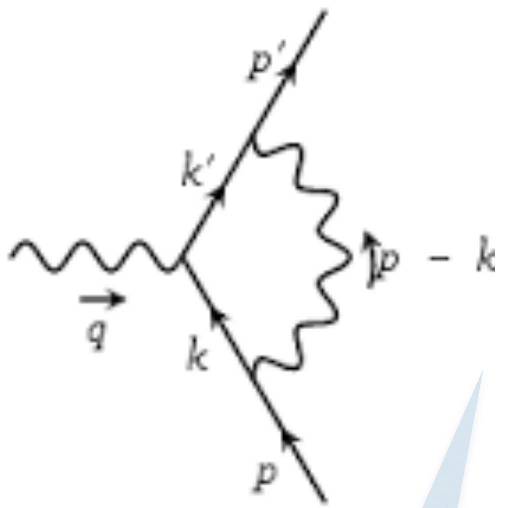


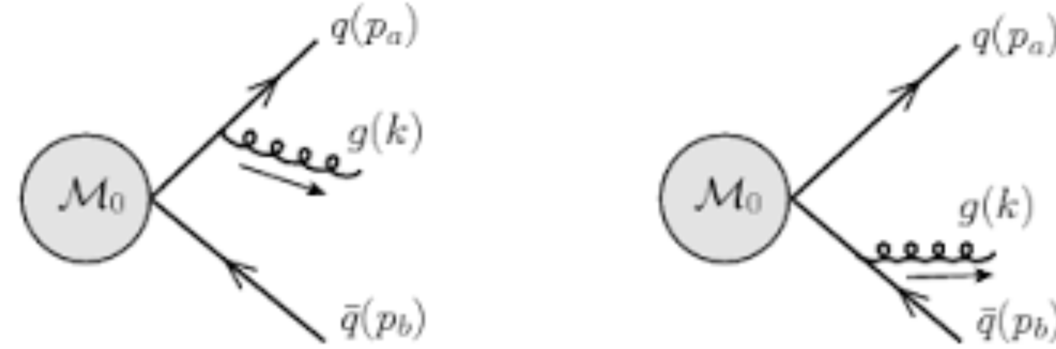
Indistinguishable when soft or collinear

Sum their contributions and it is finite but dependent on Es !

[Bloch, Nordsieck,Yannie, Suura]

If the detector is not sensitive to photons below certain energy Es (soft ones)





$$|\mathcal{B}|^2 \exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\}$$

$$\exp\left\{+\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{\Delta E}{\lambda}\right)\right\}$$

$$\exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\}\exp\left\{+\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{\Delta E}{\lambda}\right)\right\} \longrightarrow \left(\frac{\Delta E}{E}\right)^{\alpha\mathcal{K}/\pi}$$

Probability with no energy loss is Zero

Infrared Safety

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

Physical processes that happen at Long distances are responsible for these divergences.

Long distance physics is associated to configurations that are experimentally indistinguishable

Measurable quantities are not sensitive to soft and Collinear divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

Bloch and Nordsieck Theorem

Soft Singularities cancel between real and virtual processes when one adds up all states which are indistinguishable by virtue of the energy resolution of the apparatus.

$$\exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\}\exp\left\{+\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{\Delta E}{\lambda}\right)\right\} \longrightarrow \left(\frac{\Delta E}{E}\right)^{\alpha\mathcal{K}/\pi}$$

Kinoshita, Lee and Nauenberg Theorem

Both soft and collinear singularities cancel when the summation is carried out among all the mass degenerate states.

Infrared Safety

[Kulish, Fadeev]

Alternate formalism in QED was proposed by Kulish and Fadeev:

Evolution operator can be factorised into Asymptotic and Regular ones

Fock states are dressed with soft photons giving Coherent states.

S-matrix elements between these Coherent states give IR finite results.

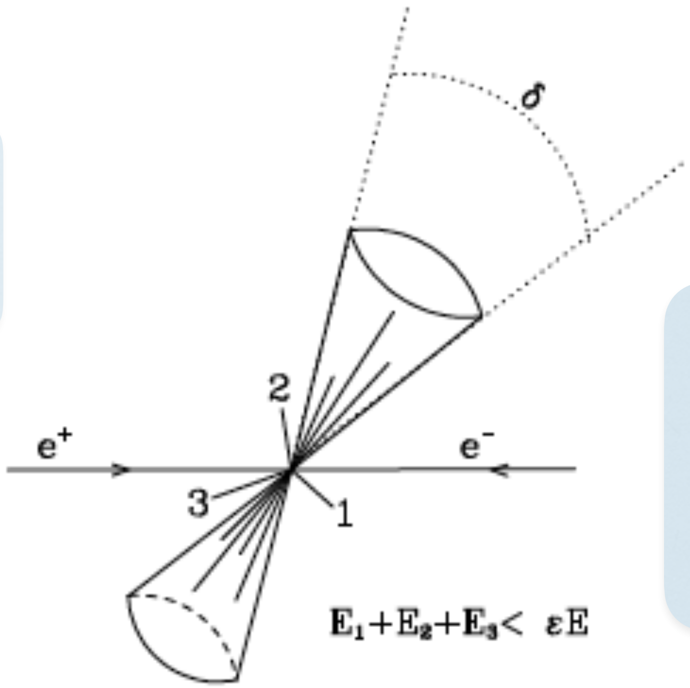
Sterman-Weinberg Jet in QCD

[Sterman, Weinberg]

Any event in electron-positron collision containing

Two cones of opening angle δ that contain all the energy of the event, excluding atmost ϵ fraction of the total.

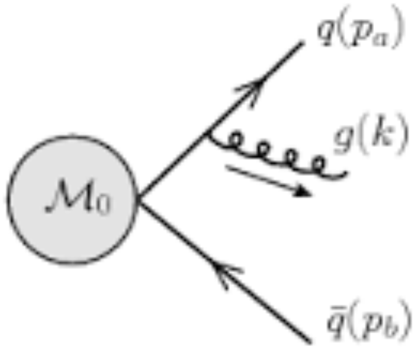
Infra-red Safe

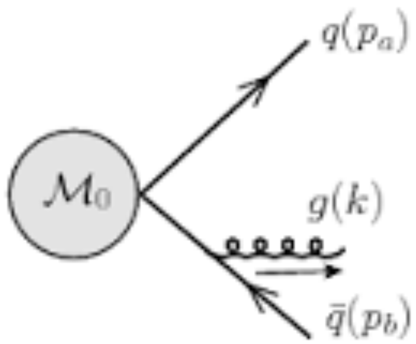


$$= \sigma_0 \left(1 - \frac{4\alpha_s C_F}{2\pi} \log \epsilon \log \delta^2 \right)$$

[Yennie, Frautschi, Suura, Weinberg]

Eikonal approximation:





$$g_s T^a \frac{p_\mu}{p \cdot k + i\epsilon} \mathcal{M}_0^{\mu a}$$

Universal current

Born amplitude

Infrared divergences FACTORISE

Infrared divergences

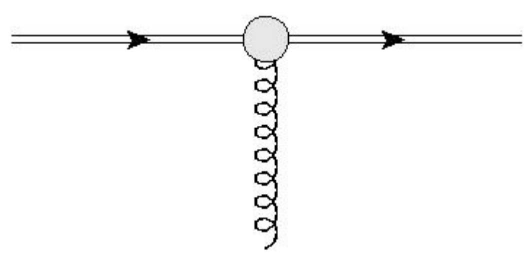
[Yennie, Frautschi, Subram; Weinberg]

Phase space integrals Diverge

$$\int \frac{dk_0}{k_0} |M|^2_{s,c} \to \infty$$

Loop integrals Diverge

$$|M|^2_{s,c} \approx (g_s T^a \frac{p_\mu}{p\cdot k + i\epsilon}) \; (g_s T^a \frac{p^\mu}{p\cdot k + i\epsilon})^* |\mathcal{M}_0|^2$$



$$i g_s \, \mathbf{T}^a \beta^\mu \times \frac{i}{\beta \cdot k + i\varepsilon}$$

Wilson Line Captures IR:

$$\mathcal{W}_\beta(\infty, 0) = \mathcal{P} \exp\left[i g_s \int_0^\infty \beta_\mu \, A^\mu(\lambda\beta) \, d\lambda \right] \qquad \beta^\mu = \frac{p^\mu}{\sqrt{p^2}}$$

Sudakov Equation (K+G Eqn.)

$$\mathcal{F}^{\lambda}_{\beta} =< \beta|\mathcal{O}^{\lambda}|\beta >$$

[Sen, Sterman, Moch, Vogt, Vermaseren; Ravindran; Magnea]

$$d = 4 + \varepsilon$$

$$Q^2 \frac{d}{dQ^2} \ln \mathcal{F}^{\lambda}_{\beta}(\hat{a}_s, Q^2, \mu^2, \epsilon) = \frac{1}{2} \left[K^{\lambda}_{\beta}(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon) + G^{\lambda}_{\beta}(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon) \right]$$

poles

No poles

RG invariance

$$\mu_R^2 \frac{d}{d\mu_R^2} K^{\lambda}_{\beta}(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon) = -\mu_R^2 \frac{d}{d\mu_R^2} G^{\lambda}_{\beta}(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon) = -A^{\lambda}_{\beta}(a_s(\mu_R^2))$$

Cusp (soft) Anomalous dim.

Casimir Duality

$$A_q = \frac{C_F}{C_A} A_g$$

Upto 3 loops

Single Pole mystery

[Ravindran, Smith, van Neerven; Moch et. al.]

UV Anomalous dim.

$$C^{\lambda}_{\beta,i} = \sum_j s_j C^{\lambda}_{\beta,j}, j < i$$

$$G^{\lambda}_{\beta,i}(\epsilon) = 2\left(B^{\lambda}_{\beta,i} - \gamma^{\lambda}_{\beta,i}\right) + f^{\lambda}_{\beta,i} + C^{\lambda}_{\beta,i} + \sum_{k=1}^{\infty} \epsilon^k g^{\lambda,k}_{\beta,i}$$

Collinear Anomalous dim.

Soft Anomalous dim.

Casimir Duality

$$f_q = \frac{C_F}{C_A} f_g$$

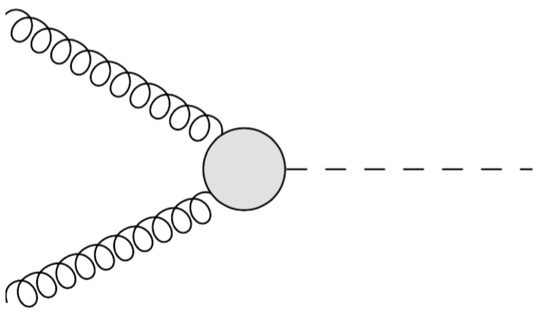
Upto 3 loops

Form Factor

[Moch, Vogt, Vermaseren,VR, Smith, v Neerven]

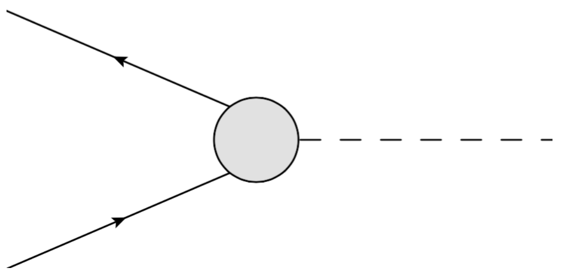
Gluon form factor

$$< g(p')|G^a_{\mu\nu}G^{\mu\nu a}|g(p) >$$



Quark form factor

$$< q(p')|\overline{\psi}\gamma_\mu\psi|q(p) >$$



Anomalous dimension

$$\gamma_q, \qquad \gamma_g$$

UV

$$A_q = \frac{C_F}{C_A} A_g$$

Cusp

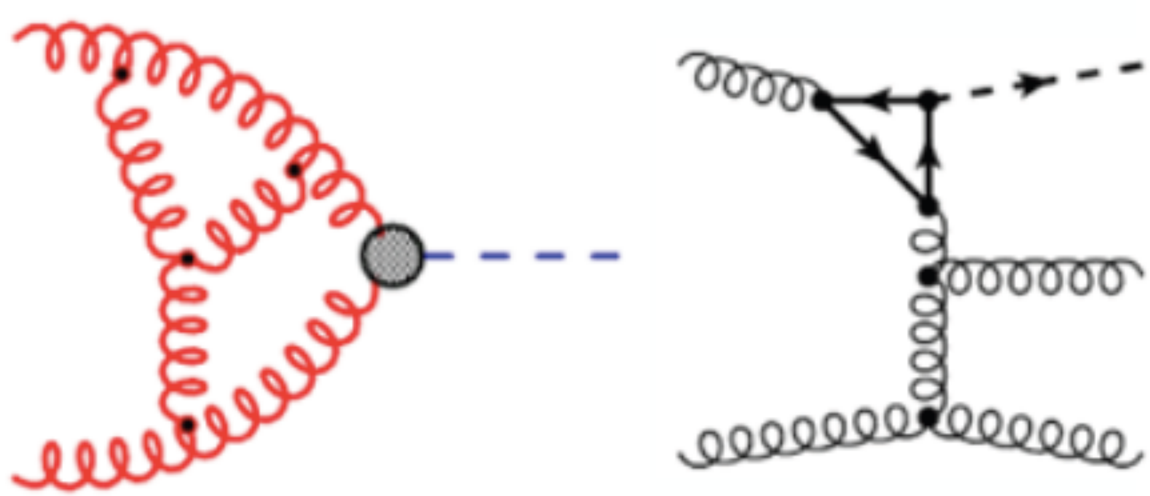
$$f_q = \frac{C_F}{C_A} f_g$$

Soft

$$B_q, \qquad B_g$$

Collinear

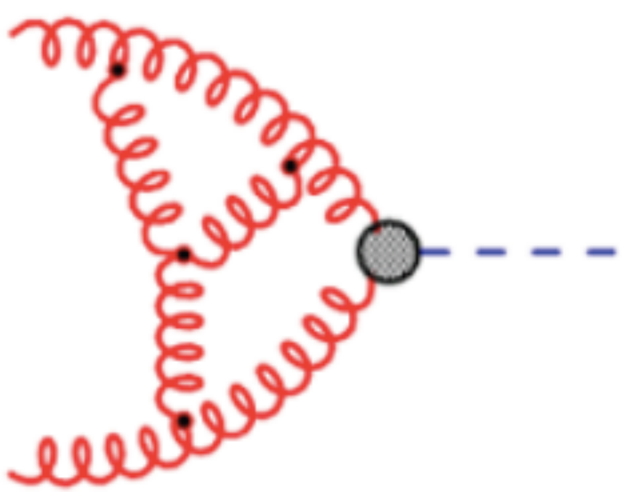
Multi-loops and Multi-legs

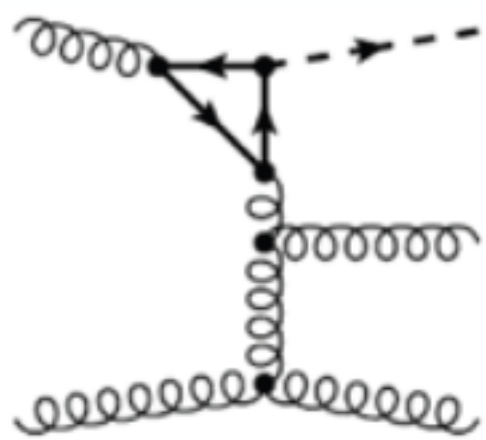


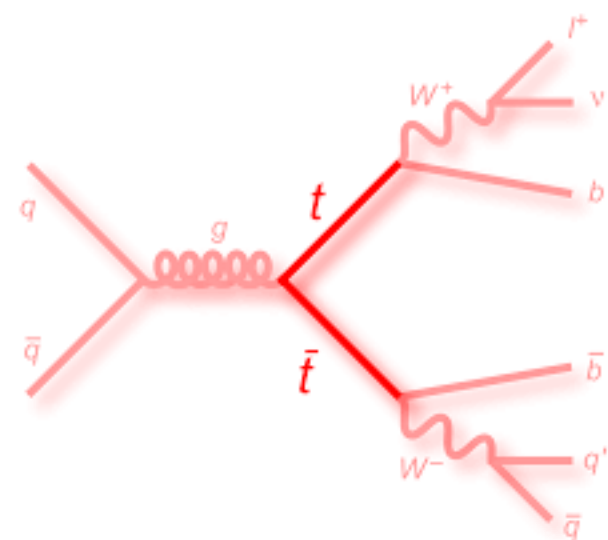
[Yennie, Frautschi, Subram; Weinberg]

UV Renormalised on-shell QCD amplitudes

$$|\mathcal{M}_n(\epsilon, \{p\})\rangle$$







Universal Infrared Structure

Catani's proposal

[S. Catani]

Universal IR Subtraction Operator

Up to Two loop !

$$\left[1 - \frac{\alpha_s}{2\pi}\, \boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon)\right] |\mathcal{M}_n(\epsilon, \{p\})\rangle$$

IR Finite

$$\boldsymbol{I}^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{g_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right) \sum_{j\neq i} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}$$

$$\boldsymbol{I}^{(2)}(\epsilon) = \frac{e^{-\epsilon\gamma_E}\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(K + \frac{\beta_0}{2\epsilon}\right)\boldsymbol{I}^{(1)}(2\epsilon) - \frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\left(\boldsymbol{I}^{(1)}(\epsilon) + \frac{\beta_0}{\epsilon}\right) + \boldsymbol{H}^{(2)}_{\mathrm{R.S.}}(\epsilon)$$

Colour matrices satisfy $\sum_i \boldsymbol{T}_i\, |\mathcal{M}_n(\epsilon, \{p\})\rangle = 0.$

Catani's proposal

[Catani]

Upto Two loop !

$$\left[1 - \frac{\alpha_s}{2\pi} \boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon)\right] |\mathcal{M}_n(\epsilon, \{p\})\rangle$$

Universal IR Subtraction Operators
depend only on
Process independent

Soft and Collinear

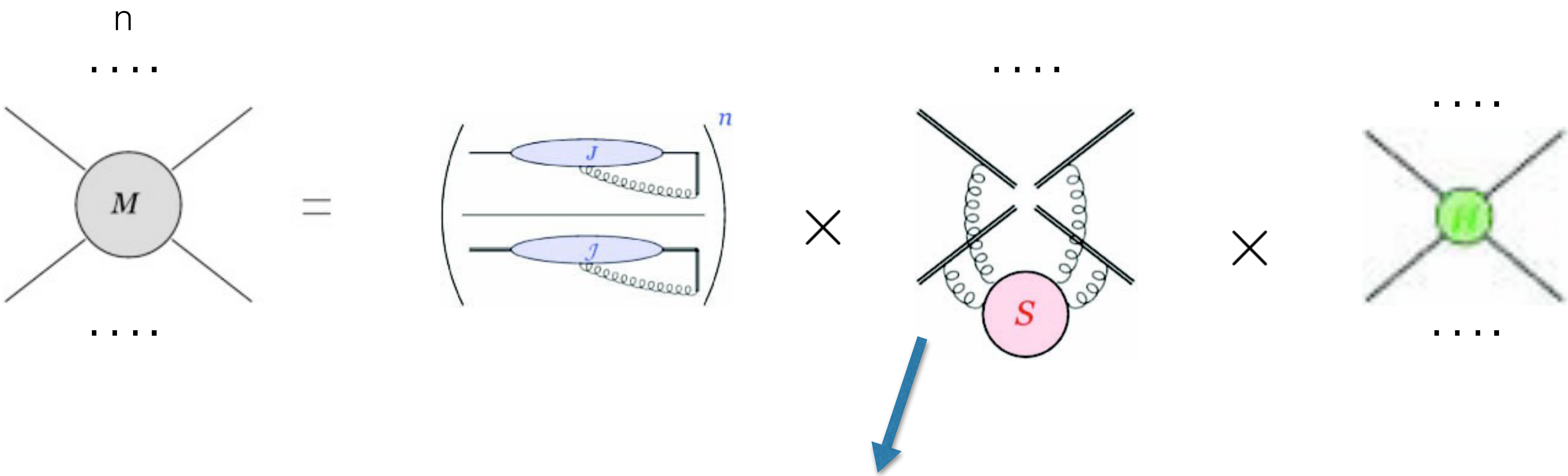
Anomalous Dimensions

Sterman's proof using factorisation

On-shell QCD amplitude in color basis:

[G. Sterman, M Tejeda-Yeomans]

$$\mathcal{M}^{[\mathrm{f}]}_{\{r_i\}}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \sum_{L=1}^{N^{[\mathrm{f}]}} \mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) (c_L)_{\{r_i\}}$$



$$\mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \prod_{i=1}^{n+2} J^{[i]}\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) S^{[\mathrm{f}]}_{LI}\left(\beta_j, \frac{Q'^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2), \epsilon\right) H^{[\mathrm{f}]}_{I}\left(\beta_j, \frac{Q^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2)\right)$$

Collinear Soft Hard

Renormalisation Group for IR

[Sterman]

Factorisation of IR singularities:

IR singular

$$\mathcal{M}\left(\frac{p_i \cdot p_j}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{Z}\left(\frac{p_i \cdot p_j}{\mu_f^2}, \alpha_s(\mu_f^2), \epsilon\right) \times \mathcal{H}\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{\mu^2}{\mu_f^2}, \alpha_s(\mu^2)\right)$$

Introduces Arbitrary Factorisation Scale μ_F

Amplitudes are independent of this scale

Renormalisation Group Invariance (RGE)

$$\mu_f \frac{d}{d\mu_f} \mathcal{Z}\left(\frac{p_i . p_j}{\mu_f^2}, \alpha_s(\mu_f^2), \epsilon\right) = -\mathcal{Z}\left(\frac{p_i . p_j}{\mu_f^2}, \alpha_s(\mu_f^2), \epsilon\right) \Gamma\left(\frac{p_i . p_j}{\mu_f^2}, \alpha_s(\mu_f^2)\right)$$

Three loop conjecture in QCD

[Becher, Neubert, Gardi, Magnea]

Matrix valued solution

$$\mathcal{Z}\left(\frac{p_i . p_j}{\mu_f^2}, \alpha_s(\mu_f^2), \epsilon\right) = \mathcal{P}exp\left[-\int_0^{\mu_f^2} \frac{d\lambda}{\lambda} \Gamma\left(\frac{p_i . p_j}{\lambda}, \alpha_s(\lambda)\right)\right]$$

Conjecture for IR anomalous dimension in QCD

$$\boldsymbol{\Gamma} = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

Di-pole

Soft

Soft +Collinear

Only Di-pole part Depends on Kinematics

Cusp Anomalous Dimension

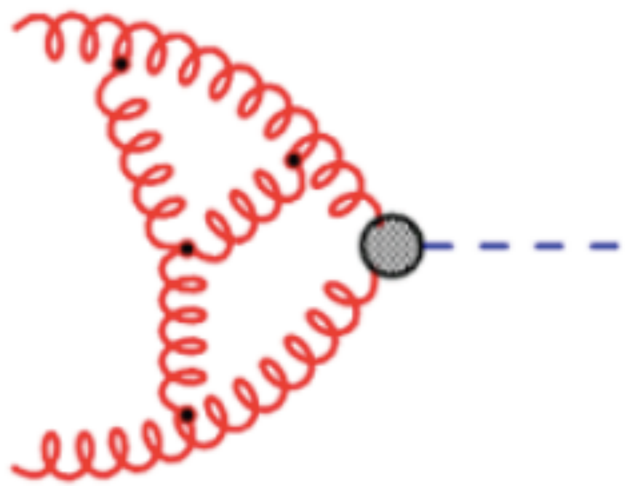
$$A_q = \frac{C_F}{C_A} A_g$$

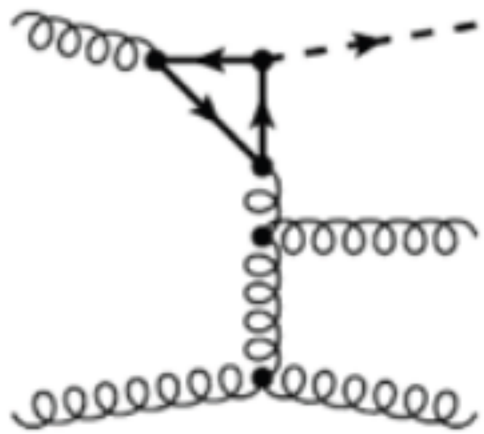
Soft Anomalous Matrix

$$\Gamma_q = \frac{C_F}{C_A} \Gamma_g$$

Upto 3- loops in QCD!

Multi-parton amplitude





Anomalous dimension

$$\gamma_q, \qquad \gamma_g$$

$$A_q = \frac{C_F}{C_A} A_g$$

$$\Gamma_q = \frac{C_F}{C_A} \Gamma_g$$

$$B_q, \qquad B_g$$

UV

Cusp

Soft Matrix

Collinear

Three loop conjecture

All order Conjecture of QCD:

$\gamma^i(\alpha_s)$ are independent of s_{ij}

$$\mathcal{Z}\left(\frac{p_i.p_j}{\mu_f^2}, \alpha_s(\mu_f^2), \epsilon\right) = \mathcal{P}exp\left[-\int_0^{\mu_f^2} \frac{d\lambda}{\lambda}\Gamma\left(\frac{p_i.p_j}{\lambda}, \alpha_s(\lambda)\right)\right]$$

known

$$\frac{1}{4}\sum_{L=1}^{\infty}\alpha^L\left[\frac{\gamma_c^{(L)}}{L^2\epsilon^2}\mathbf{D_0} - \frac{\gamma_c^{(L)}}{L\epsilon}\mathbf{D} + \frac{4}{L\epsilon}\gamma_J^{(L)}\mathbb{I} + \frac{1}{L\epsilon}\mathbf{\Delta}^{(L)}\right]$$

Three loop non-planar in $\mathcal{N} = 4$ SYM

$$\mathbf{\Delta}^{(1)} = \mathbf{\Delta}^{(2)} = \mathbf{0}$$

$$\mathbf{\Delta}_4^{(3)} = \frac{1}{4} f_{abe} f_{cde}\Big[\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\,\mathcal{S}(x) + \mathbf{T}_4^a\mathbf{T}_1^b\mathbf{T}_2^c\mathbf{T}_3^d\,\mathcal{S}(1/x)\Big],$$

$$\mathbf{\Delta}_3^{(3)} = -C\, f_{abe} f_{cde} \sum_{\substack{i=1\ldots 4\\ 1\le j<k\le 4\\ j,k\neq i}} \{\mathbf{T}_i^a, \mathbf{T}_i^d\}\, \mathbf{T}_j^b \mathbf{T}_k^c\,.$$

$\mathcal{S}(x)$ are dependent on s_{ij} at three loop level

Breakdown of Conjucture

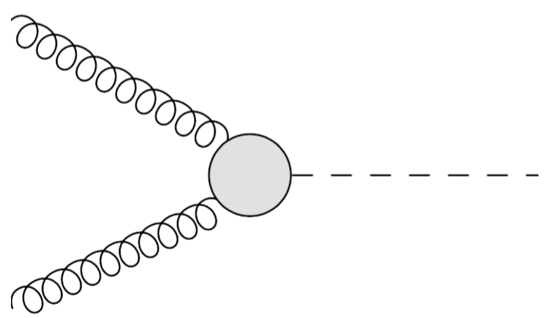
Universality of IR structure

Form Factor

[Moch, Vogt, Vermaseren,VR, Smith, v Neerven]

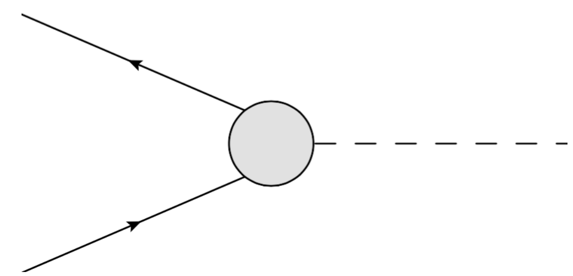
Gluon form factor

$$< g(p')|G^a_{\mu\nu}G^{\mu\nu a}|g(p) >$$



Quark form factor

$$< q(p')|\overline{\psi}\gamma_\mu\psi|q(p) >$$



$A_q, \quad A_g$

$f_q, \quad f_g$

$B_q, \quad B_g$

Anomalous dimension

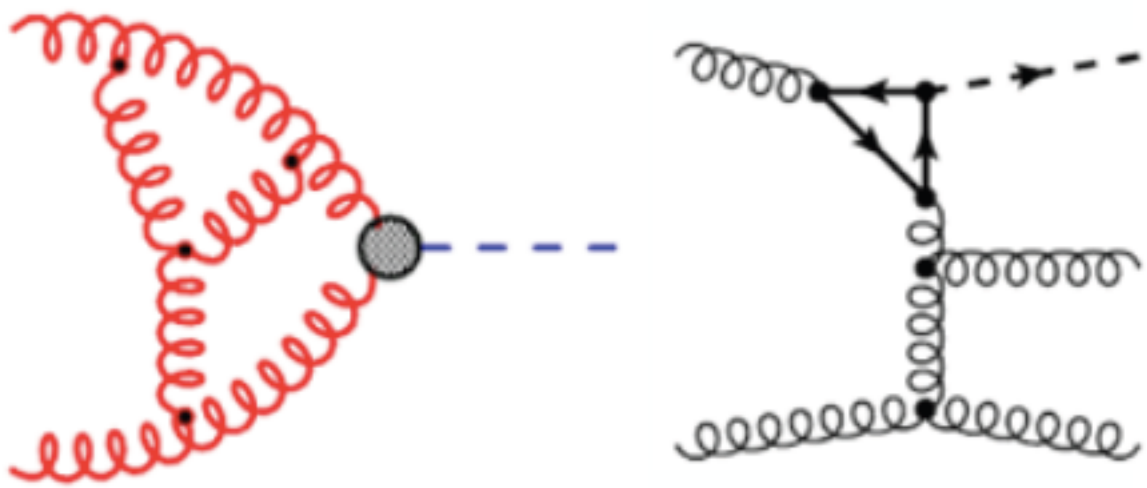
Cusp

Soft

Collinear

Universal independent of Operators, depends only external legs

Multi-parton amplitude



Anomalous dimension

$A_q, \quad A_g$

Cusp

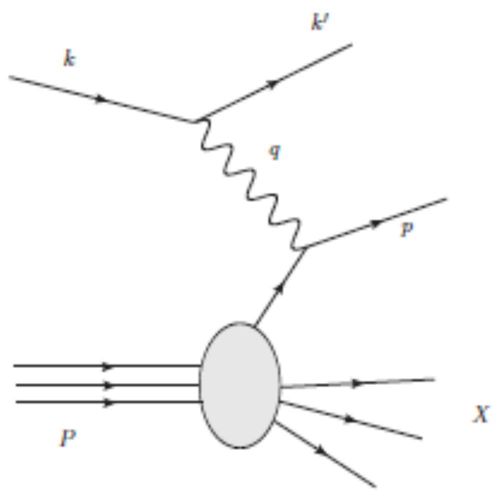
$\Gamma_q, \quad \Gamma_g$

Soft Matrix

$B_q, \quad B_g$

Collinear

DIS: Naive Parton Model



$$d\sigma^{DIS}(P,q) = \sum_i \int_x^1 dz\, f_i(z)\, d\hat{\sigma}_i(zP,q)$$

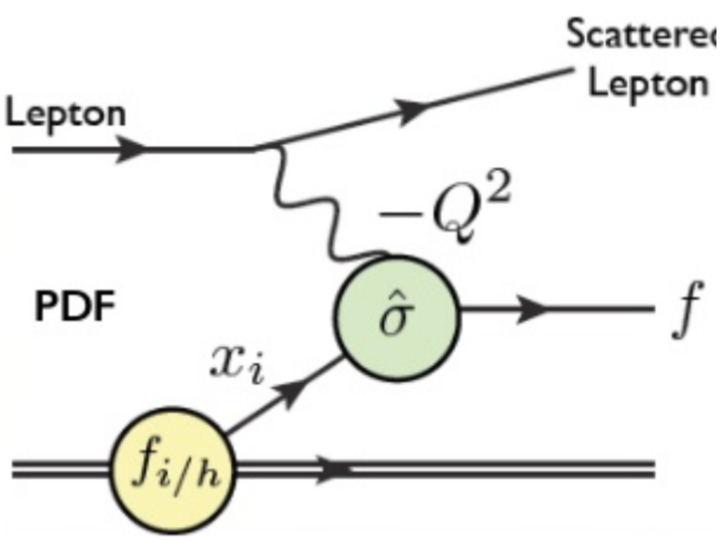
- Elastic scattering cross section with i-th parton
- Does not depend on the details of the target proton - Target Independent

$f_i(z)$ Parton Distribution Function (PDF)

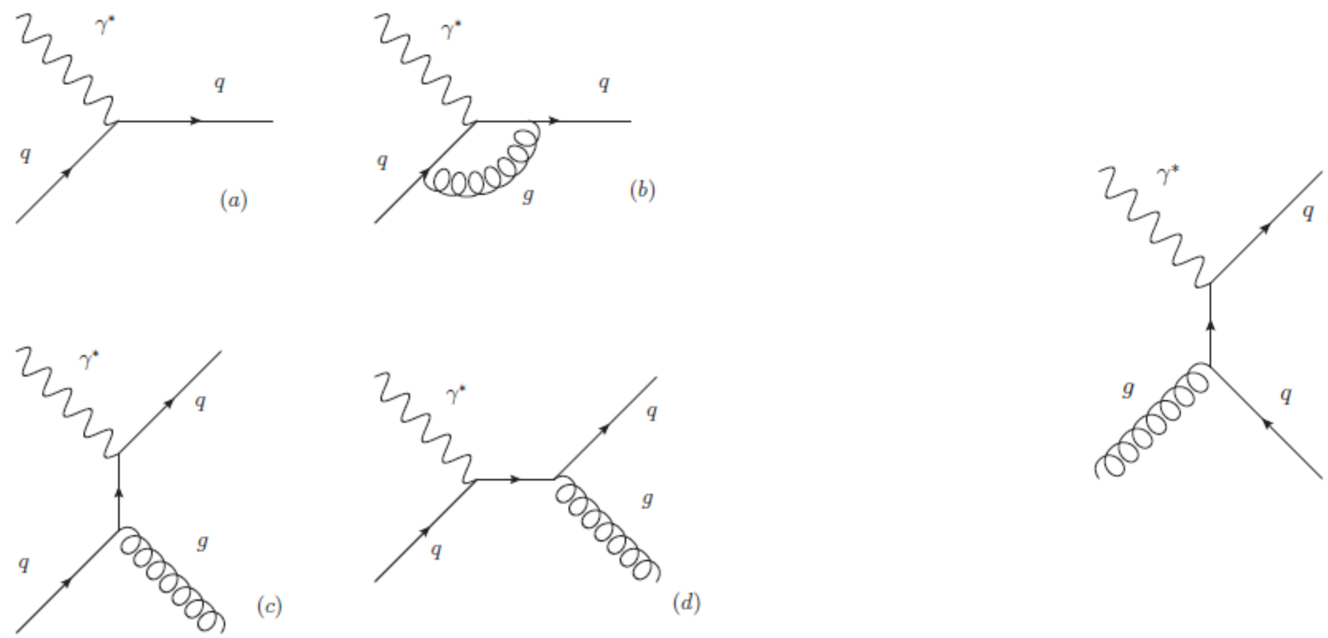
- Probability of finding i-th parton with momentum fraction z of proton

- Does not depend on the future course of action of the i-th parton - Process Independent

DIS: QCD Factorisation



Higher Order QCD corrections:



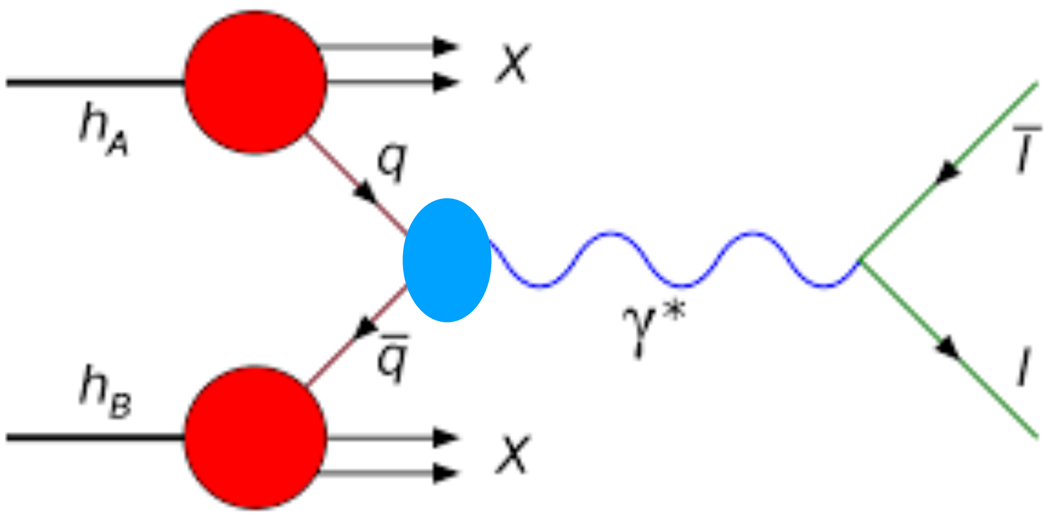
Mass factorization:

IR finite Coefficient function

$$\sigma^P(x, Q^2) = \sum_{i=q,\bar{q},g} \int_x^1 \frac{dz}{z} C_i\left(z, Q^2, \mu_R^2, \mu_F^2\right) f_{i/P}\left(\frac{x}{z}, \mu_F^2\right)$$

μ_F- Factorisation Scale
μ_R- Renormalisation Scale

Drell-Yan (DY) / Higgs boson production in Hadron collisions

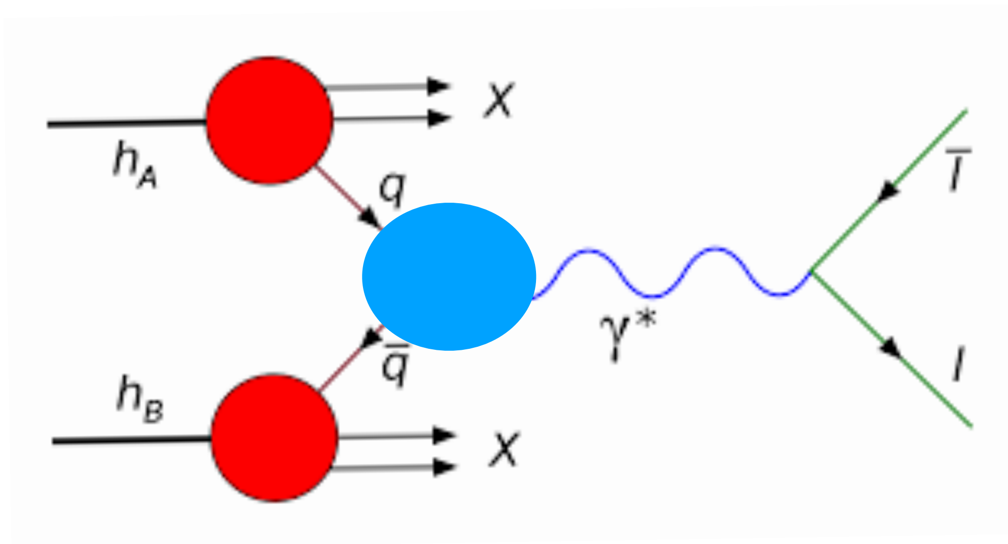


$$\sigma(q^2, \tau) = \sigma_0(\mu_R^2) \int \frac{dz}{z} \Phi_{ab}\left(\frac{\tau}{z}, \mu_F^2\right) \Delta_{ab}(q^2, \mu_F^2, z)$$

Partonic flux

$$\Phi_{ab}(\mu_F^2, z) = \int \frac{dy}{y} f_a(y, \mu_F^2) f_b\left(\frac{z}{y}, \mu_F^2\right)$$

For Drell-Yan process:



Dimension Regularisation:

$$n = 4 + \epsilon$$

$$\frac{1}{\tau}\sigma^H(\tau, q^2) = \sigma_0(q^2, \mu_R^2) \int \frac{dx_1}{x_1} \hat{f}_a(x_1) \int \frac{dz}{z} \hat{\sigma}_{ab}(z, q^2, \mu_R^2) \int \frac{dx_2}{x_2} \hat{f}_b(x_2) \delta\left(z - \frac{\tau}{x_1 x_2}\right)$$

- Higher Radiative corrections in QCD are IR singular
- Soft singularities and final state collinear singularities go away after summing up degenerate final states
- Initial state Collinear singularities remain as poles in ϵ

Mass Factorisation

Hadronic Cross section:

μ_R - Renormalisation Scale

$$\frac{1}{\tau} \sigma^H(\tau, q^2) = \sigma_0(q^2, \mu_R^2) \left[\hat{f}_a(\tau) \otimes \hat{\sigma}_{ab}(\tau, q^2, \mu_R^2) \otimes \hat{f}_b(\tau) \right]$$

$\hat{\sigma}_{ab}(\tau, q^2, \mu_R^2)$ Bare parton cross section Collinear Singular

$\hat{f}_c(\tau)$ Bare parton distribution function

Mass factorization

$$\frac{1}{\tau} \hat{\sigma}_{ab}(\tau, q^2, \mu_R^2) = \Gamma_{a'a}(\tau, \mu_F^2, \epsilon) \otimes \left[\Delta_{a'b'}(\tau, q^2, \mu_R^2, \mu_F^2) \right] \otimes \Gamma_{b'b}(\tau, \mu_F^2, \epsilon)$$

$\Gamma_{ab}(\tau, \mu_F^2, \epsilon)$

- Altarelli - Parisi Splitting Kernel
- Collinear Singular
- Universal - Process Independent

$\Delta_{ab}(\tau, q^2, \mu_R^2, \mu_F^2)$ Finite - Coefficient function

Mass Factorisation

$$\frac{1}{\tau}\sigma^H = \sigma_0 \quad \hat{f}_a \otimes \left[\Gamma_{a'a}(\mu_F^2,\varepsilon) \otimes \Delta_{a'b'}(\mu_F^2,\varepsilon) \otimes \Gamma_{b'b}(\mu_F^2,\varepsilon)\right] \otimes \hat{f}_b$$

Renormalise PDF: $f_a(\tau,\mu_F^2) = \Gamma_{aa'}(\tau,\mu_F^2) \otimes \hat{f}_{a'}(\tau)$

Finite Parton distribution functions

$$\frac{1}{\tau} \quad \sigma^H(\tau,q^2) = \sigma_0(q^2,\mu_F^2)\Big[f_a(\tau,\mu_F^2) \otimes \Delta_{a'b'}(\tau,q^2,\mu_R^2,\mu_F^2) \otimes f_b(\tau,\mu_F^2)\Big]$$

Finite Coefficient functions

$$\frac{1}{\tau}\sigma^H(\tau,q^2) = \sigma_0(q^2,\mu_R^2)\int\frac{dx_1}{x_1}f_a(x_1,\mu_F^2)\int\frac{dz}{z}\Delta_{ab}(z,q^2,\mu_R^2,\mu_F^2)$$

$$\otimes\int\frac{dx_2}{x_2}\hat{f}_b(x_2,\mu_F^2)\delta\left(z-\frac{\tau}{x_1x_2}\right)$$

DGLAP evolution Equation

Altarelli-Parisi Evolution Equation

Renormalised PDF: $$f_a(\tau,\mu_F^2) = \Gamma_{aa'}(\tau,\mu_F^2) \otimes \hat{f}_{a'}(\tau)$$

RG invariance $$\mu_F^2 \frac{d\hat{f}_a(\tau)}{d\mu_F^2} = 0$$

RG equation $$\mu_F^2 \frac{d}{d\mu_F^2} f_a(\tau,\mu_F^2) = \frac{1}{2} P_{aa'}(\tau,\mu_F^2) \otimes f_{a'}(\tau,\mu_F^2)$$

AP Splitting functions: $$\frac{1}{2} P_{aa'}(\tau,\mu_F^2) = \left(\Gamma^{-1}(\tau,\mu_F^2) \otimes \mu_F^2 \frac{d}{d\mu_F^2} \Gamma(\tau,\mu_F^2)\right)_{aa'}$$

finite

Altarelli-Parisi Evolution Equation

Collinear
Renormalisation Group Equation

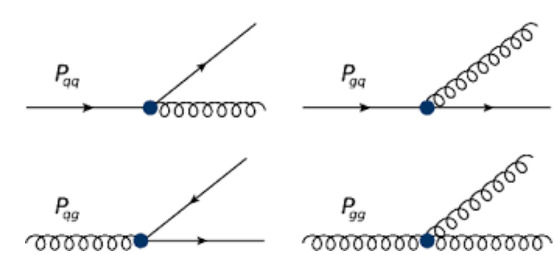
Alatarelli-Parisi (DGLAP) Evolution Equation

$$\frac{\partial}{\partial \log \mu^2} \begin{pmatrix} q_i \\ g \end{pmatrix} (x, \mu^2) = \frac{\alpha_s}{2\pi} \int_x^1 \sum_{j=q,\bar{q}} \frac{d\xi}{\xi} \begin{pmatrix} P_{ij}\left(\frac{x}{\xi}, \alpha_s\right) & P_{ig}\left(\frac{x}{\xi}, \alpha_s\right) \\ P_{gj}\left(\frac{x}{\xi}, \alpha_s\right) & P_{gg}\left(\frac{x}{\xi}, \alpha_s\right) \end{pmatrix} \begin{pmatrix} q_j \\ g \end{pmatrix} (\xi, \mu^2),$$

In QCD perturbation

$$P_{ij}^{N^m LO}(x, \mu^2) = \sum_{k=0}^{m} a_s^{k+1}(\mu^2) P_{ij}^{(k)}(x).$$

Leading Order



Altarelli-Parisi Evolution Equation

DGLAP kernel satisfies RGE

[Moch, Vogt, Vermaseren]

$$\mu_F^2 \frac{d}{d\mu_F^2}\Gamma(z,\mu_F^2,\varepsilon) = \frac{1}{2}P\left(z,\mu_F^2\right)\otimes\Gamma\left(z,\mu_F^2,\varepsilon\right)$$

Γ is available up to $\mathcal{O}(a_s^4)$

Altarelli-Parisi Splitting functions.

$$P_{cc}(z) = 2\left[A^c\mathcal{D}_0(z) + B^c\delta(1-z) + C^c\log(1-z) + D^c\right] + \mathcal{O}((1-z))\,.$$

$$\mathcal{D}_0(z) = \left(\frac{1}{1-z}\right)_+$$

Cusp Anomalous Dim Ac.

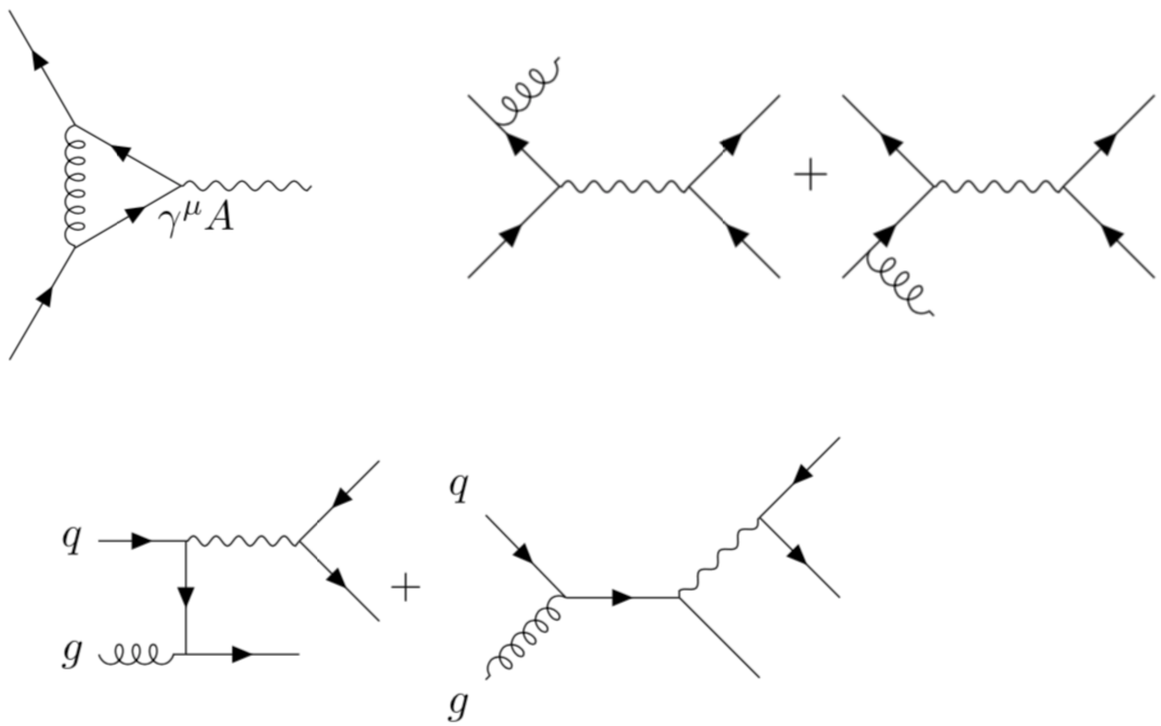
Collinear Anomalous Dim.

All order Solution: $$\Gamma_{cc}(z,\mu_F^2,\varepsilon) = \mathcal{C}\exp\left(\frac{1}{2}\int_0^{\mu_F^2}\frac{d\lambda^2}{\lambda^2}P_{cc}(\lambda^2,z,\varepsilon)\right)$$

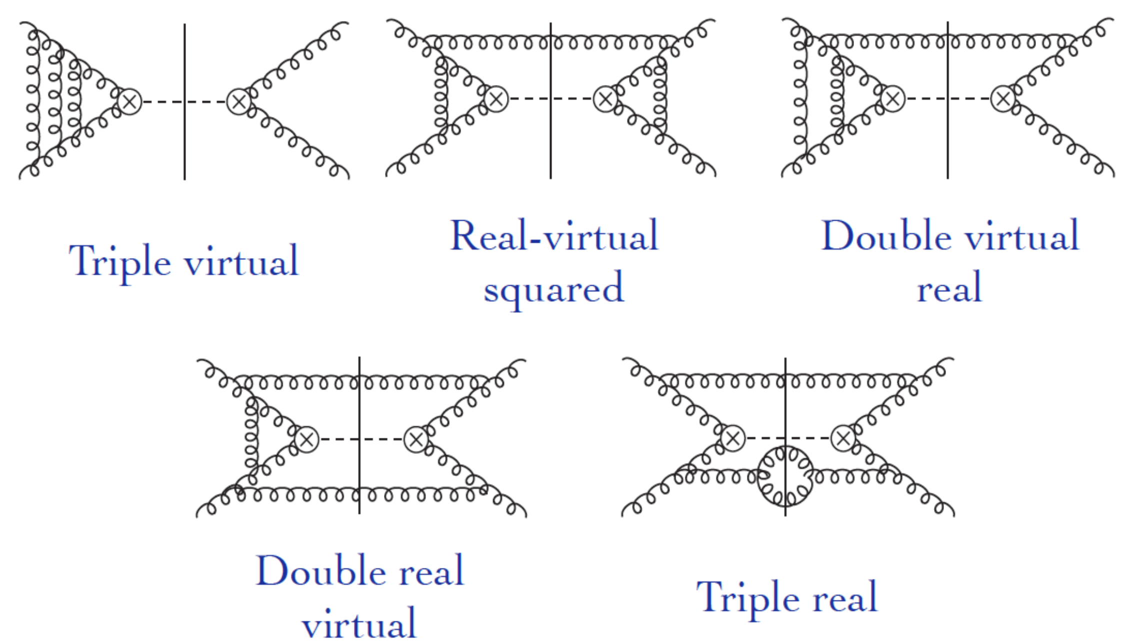
DY and Higgs production:

Coefficient function $\Delta(z, q^2) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2) \Delta^{(i)}(z, q^2, \mu_R^2)$

Drell-Yan Production



Higgs Production

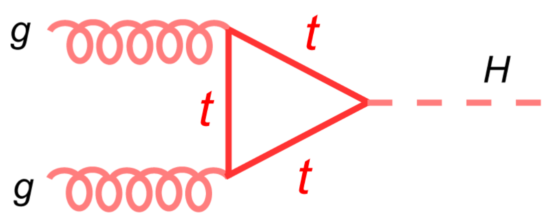


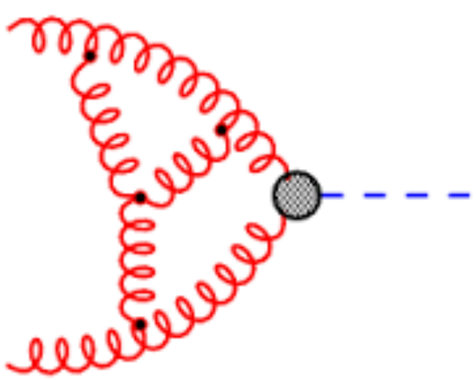
Inclusive Higgs production

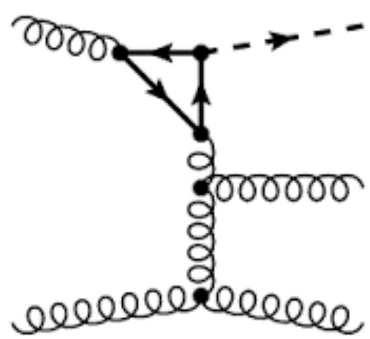
Anastasiou, Duhr, Dulat and Mistleberger (`19)

$$\hat{\sigma}(z) = \hat{\sigma}^{LO}(z) + \alpha_S \hat{\sigma}^{NLO}(z) + \alpha_S^2 \hat{\sigma}^{NNLO}(z) + \alpha_S^3 \hat{\sigma}^{N3LO}(z) + \mathcal{O}(\alpha_S^4)$$

✓ ✓ ✓ ✓

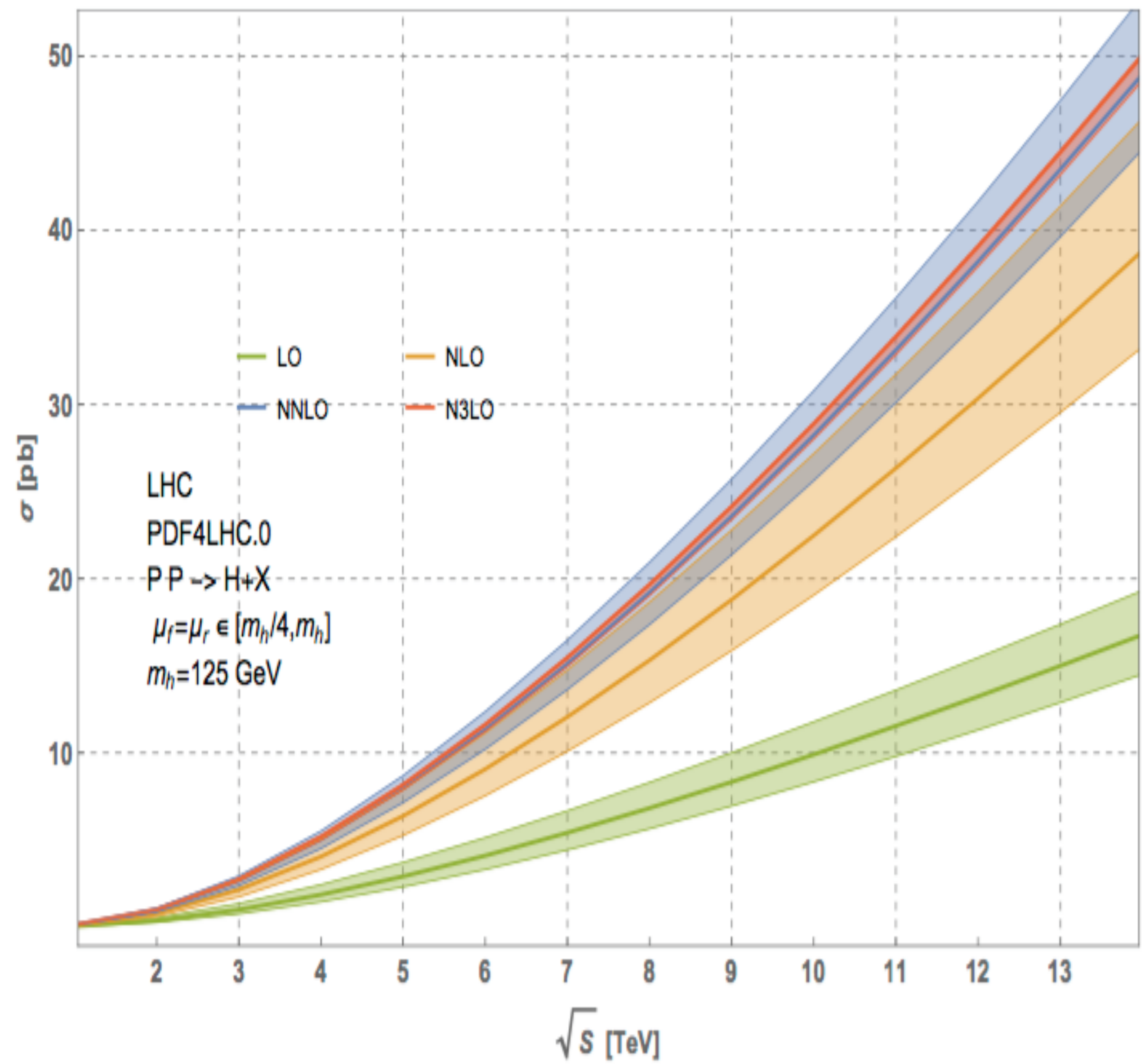






LO	$15.05 \pm 14.8\%$
NLO	$38.2 \pm 16.6\%$
NNLO	$45.1 \pm 8.8\%$
N3LO	$45.2 \pm 1.9\%$

pb



Soft+Virtual+Hard

DY/Higgs production cross section:

$$\sigma(q^2,\tau) = \sigma_0(\mu_R^2) \int \frac{dz}{z} \Phi_{ab}\left(\frac{\tau}{z}, \mu_F^2\right) \Delta_{ab}(q^2, \mu_F^2, z)$$

$$\Delta_{ab}(q^2,\mu_i^2,z) = \Delta_{ab}^{SV}(q^2,\mu_i^2,z) + \Delta_{ab}^{H}(q^2,\mu_i^2,z)$$

Soft + Virtual

Hard

Nielson PolyLogs

$$S_{np}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 dy \, \frac{\log^{n-1}(y) \log^p(1-zy)}{y}$$

$$\delta(1-z_i)$$

$$\left(\frac{\ln(1-z_i)}{(1-z_i)}\right)_+$$

- Next to SV

$$\Delta_{ab}^{NSV}(z) = \sum_{k=0}^{\infty} C_i^{NSV} \log^k(1-z)$$

- Next to next to....to soft

$$\Delta_{ab}^{N^n SV}(z) = \sum_{k=1}^{\infty} d_k (1-z)^k$$

Mellin Moments and large N

Mellin Moment:

$$f_N = \int_0^1 dz\ z^{N-1} f(z)$$

Threshold limit $z \to 1$ in z-Space translates to $N \to \infty$ in N-Space

$N \to \infty$ Taking into account SV and NSV terms

$$\left(\frac{\log(1-z)}{1-z}\right)_+ = \frac{\log^2 N}{N} - \frac{\log N}{2N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\log^k(1-z) = \frac{\log^k N}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

N-Space structure

Mellin moment of CFs

Sterman, Catani, Trentedue

$$\Delta_N^c = \int_0^1 dz\ z^{N-1} \Delta_c(z)$$

In N-Space $N \to \infty$ We can predict tower of log N s

$$\Delta_N^c = 1 + a_s \Big[c_1^2 \log^2 N + c_1^1 \log N + c_1^0 + d_1^1 \frac{\log N}{N} + \mathcal{O}(1/N) \Big]$$

$$+ a_s^2 \Big[c_2^4 \log^4 N + \cdots + c_2^0 + d_2^3 \frac{\log^3 N}{N} + \cdots + \mathcal{O}(1/N) \Big]$$

$$+ \cdots$$

$$+ a_s^n \Big[c_n^{2n} \log^{2n} N + \cdots + d_n^{2n-1} \frac{\log^{2n-1} N}{N} + \cdots + \mathcal{O}(1/N) \Big]$$

$a_s \log N$ is of order `one` when a_s is very small

Spoils the trucation of the series

Soft plus Virtual

$$\Delta_{ab}^{SV}(z) = \delta_{ab}\Delta_{a,\delta}\delta(1-z) + \delta_{ab}\sum_{j=0}^{\infty}\Delta_{a,\mathcal{D}_j}\left(\frac{\log^j(1-z)}{1-z}\right)_+$$

Perturbatively Calculable

$$\Delta_{a,\delta} = \sum_k a_s^k \Delta_{a,\delta}^{(k)}$$

$$\Delta_{a,\mathcal{D}_j} = \sum_k a_s^k \Delta_{a,\mathcal{D}_j}^{(k)}$$

Sensitive to Soft and Collinear parsons

$$\frac{1}{(p+k)^2} = \frac{1}{2p^0k^0(1-\cos\theta)}$$

$k^0 \to 0$ Soft

$\cos\theta \to 1$ Collinear

- Divergences are controlled by Soft and Collinear Anomalous dimensions - ex: Cusp A, collinear B, soft-f etc
- Soft and Collinear regions are Universal
- RGE in IR sector allows for All Order Prediction

Sterman-Catani-Trentedue

Sterman (`87),Catani,Trentedue `89

SUMMATION OF LARGE CORRECTIONS TO SHORT-DISTANCE HADRONIC CROSS SECTIONS

George STERMAN*

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RESUMMATION OF THE QCD PERTURBATIVE SERIES FOR HARD PROCESSES

S. CATANI

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Dipartimento di Fisica, Università di Parma, INFN, Gruppo Collegato di Parma, I-43100 Parma, Italy

$$\Delta_N(Q^2) \underset{N\to\infty}{=} \exp\left(2\int_0^1 \mathrm{d}x\, \frac{x^{N-1}-1}{1-x} \int_{Q^2}^{Q^2(1-x)} \frac{\mathrm{d}k^2}{k^2} A\left(\alpha_s((1-x)Q^2)\right) + \frac{3}{2}\frac{C_F}{\pi}\int_0^1 \mathrm{d}x\, \frac{x^{N-1}-1}{1-x}\,\alpha_s((1-x)Q^2)\right) + \mathrm{O}\left(\alpha_s(\alpha_s \ln N)^n\right). \tag{3.25}$$

Resums threshold logarithms in large N

Factorisation: Diagonal

Begin with Mass factorised cross section

$$\frac{1}{z}\hat{\sigma}_{ab}(\varepsilon) = \sigma_0 \sum_{a'b'} \Gamma^T_{aa'}(\mu_F^2, \varepsilon) \otimes \left(\frac{1}{z}\Delta_{a'b'}(\mu_F^2, \varepsilon)\right) \otimes \Gamma_{b'b}(\mu_F^2, \varepsilon)$$

For Drell-Yan process:

Diagonal Channel:

$$\frac{\hat{\sigma}_{q\bar{q}}}{z\sigma_0} = \Gamma^T_{qq} \otimes \frac{\Delta_{qq}}{z} \otimes \Gamma_{q\bar{q}} + \Gamma^T_{qq} \otimes \frac{\Delta_{qg}}{z} \otimes \Gamma_{g\bar{q}} + \cdots$$

In the threshold limit z -> 1 , keeping only $\left(\frac{\ln(1-z_i)}{(1-z_i)}\right)_+$ $\delta(1-z_i)$ SV

$\log^k(1-z_i), \quad k = 0, \cdots \infty$ next to SV

$$\frac{\hat{\sigma}^{\text{sv+nsv}}_{q\bar{q}}}{z\sigma_0} = \Gamma^T_{qq} \otimes \Delta^{\text{sv+nsv}}_{q\bar{q}} \otimes \Gamma_{\bar{q}\bar{q}} \,.$$

dropping

$(1-z_i)^k, \quad k = 1, \cdots \infty$

Remarkably Simple form !

Master Formula

$$\Psi^c(q^2, \mu_R^2, \mu_F^2, z, \varepsilon) = \mathcal{C} \log\left(\Delta_c(q^2, \mu_R^2, \mu_F^2, z)\right)$$

$$\Psi^c(q^2, \mu_R^2, \mu_F^2, z, \varepsilon) = \left(\ln\left(Z_{UV,c}(\hat{a}_s, \mu^2, \mu_R^2, \varepsilon)\right)^2 + \ln\left|\hat{F}_c(\hat{a}_s, \mu^2, Q^2, \varepsilon)\right|^2\right)\delta(1-z)$$
$$+2\Phi^c(\hat{a}_s, \mu^2, q^2, z, \varepsilon) - 2\mathcal{C}\ln\Gamma_{cc}(\hat{a}_s, \mu^2, \mu_F^2, z, \varepsilon)$$

- $\hat{F}_c(Q^2)$ Form Factor
- $Z_{UV,c}(\mu_R^2)$ UV Renormalisation const
- $\Phi^c(q^2, z)$ Soft+next to soft distribution fn. $\mathcal{S}_c = \mathcal{C}\exp\left(2\Phi^c\right)$
- $\Gamma_{cc}(\mu_F^2, z)$ Altarelli-Parisi kernel

Solution to K+G/Sudakov Equation

Solution to K+G equation for $\Phi^c(\hat{a}_s, q^2, \mu^2, \varepsilon)$

$$\Phi^c = \Phi^c_A + \Phi^c_B$$

SV

$$\Phi^c_A = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S^i_\varepsilon \left(\frac{i\varepsilon}{1-z}\right) \hat{\phi}^{c(i)}_{SV}(\varepsilon)\,,$$

Matrix element

Phase Space

NSV

$$\Phi^c_B = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S^i_\varepsilon \varphi^{(i)}_c(z, \varepsilon)$$

Φ^c = IR Divergent part + Finite part

IR Div cancels with rest ->Finite part = $\Phi^c_f(z)$

Structure of SV and NSV

$$\Phi^c(z) = \Phi^c_{SV}(z) + \Phi^c_{NSV}(z)$$

$\Phi^c_{SV}(z)$ $\left\{ \left(\frac{\log^k(1-z)}{1-z}\right)_+ , \ \delta(1-z) \right\}$ SV

$\Phi^c_{NSV}(z)$ $\left\{ \log^k(1-z) \right\}$ $k = 0, 1, \cdots$ NSV

Drop $(1-z)^\alpha$ $\alpha = 1, \cdots$

Properties:

$$\Phi^q_{SV} = \left(\frac{C_F}{C_A}\right) \Phi^g_{SV}$$

To all orders in the coupling
``Maximally non-abelian"

$$\Phi^q_{NSV} \neq \left(\frac{C_F}{C_A}\right) \Phi^g_{NSV}$$

$$\Phi^b_{NSV} \neq \Phi^q_{NSV}$$

Beaks down beyond second order!

Integral Representation

Integral representation:

$$\Delta_c(q^2, z) = C_0^c(q^2) \ \ \mathcal{C} \exp\left(2\Psi_{\mathcal{D}}^c(q^2, z)\right),$$

Exponent:

$$\Psi_{\mathcal{D}}^c(q^2, z) = \frac{1}{2}\int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} P'_{cc}(a_s(\lambda^2), z) + \mathcal{Q}^c(a_s(q^2(1-z)^2), z)$$

C_0^c	Process dependent constant
$P'_{cc}(z)$	Process independent (Universal): SV+NSV part of Alatarell-Parisi splitting function
$\mathcal{Q}^c(z)$	Process dependent function

Perturbative predictions

$$\Delta_c(z) = \mathcal{C}\exp\left(\Psi^c\left(q^2,\mu_R^2,\mu_F^2,z,\varepsilon\right)\right)\Big|_{\varepsilon=0}$$

$$= \sum_{k=0}^{\infty}\Delta^c_{\mathcal{D},k}\left(\frac{\log^k(1-z)}{1-z}\right)_+ + \Delta^c_\delta\,\delta(1-z) \qquad \text{SV}$$

$$+\sum_{k=0}^{\infty}\Delta^c_{L,k}\,\log^k(1-z) \qquad \text{NSV}$$

Perturbative expansion: $\Delta^c_{J,k} = \sum_{i=0}^{\infty} a_s^i \Delta^{c,(i)}_{J,k}$

Predictive Power:

Lower order results decide ``Certain`` higher order SV and NSV terms to all orders

All order perturbative predictions

All order exponentiation can predict to all orders from lower orders:

$$\Delta_c(z) = \mathcal{C} \exp\left(\Psi^c\left(q^2, \mu_R^2, \mu_F^2, z, \varepsilon\right)\right)\Big|_{\varepsilon=0}$$

$$= \sum_{i=0}^{\infty} a_s^i \; \Delta_c^{(i)}(z)$$

$$\mathcal{D}_k = \left(\frac{\log^k(1-z)}{1-z}\right)_+$$

$$L_z = \log(1-z)$$

GIVEN				PREDICTIONS		
$\Psi_c^{(1)}$	$\Psi_c^{(2)}$	$\Psi_c^{(3)}$	$\Psi_c^{(n)}$	$\Delta_c^{(2)}$	$\Delta_c^{(3)}$	$\Delta_c^{(i)}$
$\mathcal{D}_0, \mathcal{D}_1, \delta$ L_z^1, L_z^0				$\mathcal{D}_3, \mathcal{D}_2$ L_z^3	$\mathcal{D}_5, \mathcal{D}_4$ L_z^5	$\mathcal{D}_{(2i-1)}, \mathcal{D}_{(2i-2)}$ $L_z^{(2i-1)}$
	$\mathcal{D}_0, \mathcal{D}_1, \delta$ L_z^2, L_z^1, L_z^0				$\mathcal{D}_3, \mathcal{D}_2$ L_z^4	$\mathcal{D}_{(2i-3)}, \mathcal{D}_{(2i-4)}$ $L_z^{(2i-2)}$
		$\mathcal{D}_0, \mathcal{D}_1, \delta$ $L_z^3, \cdots, L_z^0$				$\mathcal{D}_{(2i-5)}, \mathcal{D}_{(2i-6)}$ $L_z^{(2i-3)}$
			$\mathcal{D}_0, \mathcal{D}_1, \delta$ $L_z^n, \cdots, L_z^0$			$\mathcal{D}_{(2i-(2n-1))}, \mathcal{D}_{(2i-2n)}$ $L_z^{(2i-n)}$

Fourth order prediction

Vogt,Moch et al,
DeFlorian et al, Das et al

4th order QCD prediction for Drell-Yan production

$$\Delta_q^{(4)} = \left(-\frac{4096}{3} C_F^4 \right) \log^7(1-x) + \left(\frac{39424}{9} C_F^3 C_A + \frac{19712}{3} C_F^4 - \frac{7168}{9} n_f C_F^3 \right)$$
$$\times \log^6(1-x) + \left(-\frac{123904}{27} C_F^2 C_A^2 - \frac{805376}{27} C_F^3 C_A + 9088 C_F^4 + \frac{45056}{27} n_f C_F^2 C_A \right.$$
$$\left. + \frac{139520}{27} n_f C_F^3 - \frac{4096}{27} n_f^2 C_F^2 + 3072 \zeta_2 C_F^3 C_A + 20480 \zeta_2 C_F^4 \right) \log^5(1-x)$$

4th order QCD prediction for Higgs production in gluon fusion

$$\Delta_g^{(4)} = \left(-\frac{4096}{3} C_A^4 \right) \log^7(1-x) + \left(\frac{98560}{9} C_A^4 - \frac{7168}{9} n_f C_A^3 \right) \log^6(1-x)$$
$$+ \left(-\frac{298240}{9} C_A^4 + \frac{174208}{27} n_f C_A^3 - \frac{4096}{27} n_f^2 C_A^2 + 23552 \zeta_2 C_A^4 \right) \log^5(1-x).$$

Structure of NSV logs

Structure of Next to SV terms

$$\Delta_N^c = 1 + a_s\Big[c_1^2 \log^2 N + c_1^1 \log N + c_1^0 + d_1^1 \frac{\log N}{N} + \mathcal{O}(1/N)\Big]$$

$$+ a_s^2\Big[c_2^4 \log^4 N + \cdots + c_2^0 + d_2^3 \frac{\log^3 N}{N} + \cdots + \mathcal{O}(1/N)\Big]$$

$$+ \cdots$$

$$+ a_s^n\Big[c_n^{2n} \log^{2n} N + \cdots + d_n^{2n-1} \frac{\log^{2n-1} N}{N} + \cdots + \mathcal{O}(1/N)\Big]$$

$a_s \log N$ is of order `one` when a_s is very small at every order 1/N

Can one resume terms of the form $\frac{\log^k N}{N}$ to all orders

Resummation of NSV logs

Using z-space Integral representation

$$\Delta_c(q^2, z) = C_0^c(q^2)\ \ \mathcal{C}\exp\left(2\Psi_{\mathcal{D}}^c(q^2, z)\right),$$

where

$$\Psi_{\mathcal{D}}^c(q^2, z) = \frac{1}{2}\int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} P'_{cc}(a_s(\lambda^2), z) + \mathcal{Q}^c(a_s(q^2(1-z)^2), z)$$

Mellin moment

$$\Delta_N^c(q^2) = \int_0^1 dz\ z^{N-1}\Delta_c(q^2, z)$$

$$= C_0^c(q^2)\ \ e^{\Psi_{sv,N}^c(q^2,\omega)+\Psi_{nsv,N}^c(q^2,\omega)}$$

where,

$$\Psi_{sv,N}^c = g_1(w)\log N + \sum_{n=0}^{\infty} a_s^n\ g_{n+1}(w)$$

Already known!

$$\Psi_{nsv,N}^c = \frac{1}{N}\sum_{n=0}^{\infty}\sum_{k=0}^{n} h_{nk}(w)\ a_s^n \log^k N$$

New Result

$$\omega = 2\ a_s(\mu_R^2)\ \beta_0 \log N$$

Integration By Parts (IBP)

[Tkachov, Chetyrkin]

- Generalization of **Gauss's theorem** in d dimension.

- Within dimensional regularization, all integrals in d dimension are well-defined and convergent.

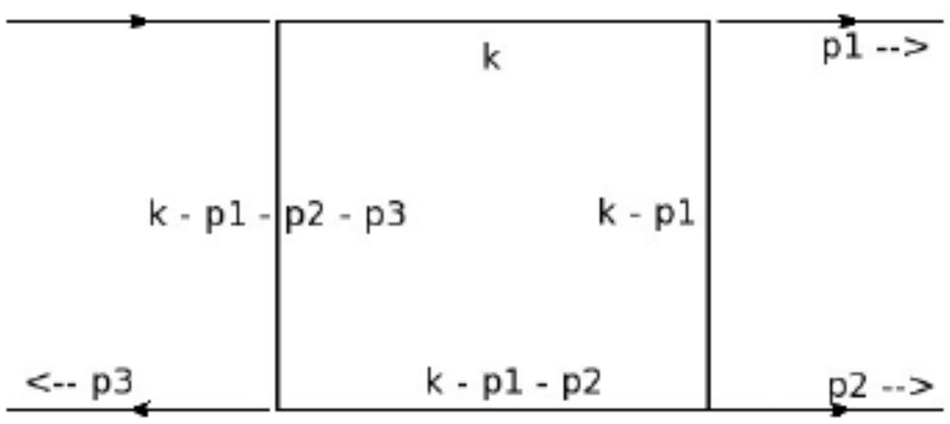
$$\Downarrow$$

the integrand must be zero at boundary
(*necessary condition for convergence*)

- to make it free from Lorentz index

$$\int \prod_{i=1}^{l} \mathcal{D}^d k_l \frac{\partial}{\partial k_j^\mu} \left(\frac{v^\mu}{D_1^{n_1} \dots D_m^{n_m}} \right) = 0 \qquad \Big|_{v \equiv k_i, p_i}$$

IBP at one-loop



Consider

$$\mathcal{I}(n) = \int \mathcal{D}^d k \frac{1}{\mathcal{D}_0\ \mathcal{D}_1\ \mathcal{D}_{12}^n} \qquad \{n = 1, 2, \ldots\}$$

$$\mathcal{D}_0 = k^2, \mathcal{D}_1 = (k - p_1)^2, \mathcal{D}_{12} = (k - p_1 - p_2)^2,$$
$$\mathcal{D}_{123} = (k - p_1 - p_2 - p_3)^2 \ .$$

IBP

$$0 = \int \mathcal{D}^d k \frac{\partial}{\partial k^\mu} \left(\frac{k^\mu}{\mathcal{D}_0\ \mathcal{D}_1\ \mathcal{D}_{12}^n} \right)$$
$$= \Big(d - (n+3)\Big)\ \mathcal{I}(n) + n\ s\ \mathcal{I}(n+1)$$

Master Integral

$$\mathcal{I}(n+1) = -\frac{d - (n+3)}{n\ s}\ \mathcal{I}(n)$$
$$= (-1)^n\ \frac{(d - (n+3)) \ldots (d-5)\ (d-4)}{n!\ s^n}\ \mathcal{I}(1)$$

Lorentz Invariance

[Gehrmann, Remiddi]

- Under Lorentz transformation of external momenta

$$p_i^\mu \to p_i^\mu + \delta p_i^\mu = p_i^\mu + \omega^\mu_\nu p_i^\nu \quad \text{with } \omega^\mu_\nu = -\omega^\nu_\mu$$

the integrals are invariant i.e.

$$\mathcal{I}(p_i) = \mathcal{I}(p_i + \delta p_i) = \mathcal{I}(p_i) + \omega^\nu_\mu \sum_j p_j^\mu \frac{\partial}{\partial p_j^\nu} \mathcal{I}(p_i)$$

- from which the identity can be derived

$$\sum_j \left(p_{j,\mu} \frac{\partial}{\partial p_j^\nu} - p_{j,\nu} \frac{\partial}{\partial p_j^\mu} \right) \mathcal{I}(p_i) = 0$$

Differential eqns. for Integrals

Simplest Integral :

$$\mathcal{I} = \int \mathcal{D}^d k \frac{1}{k^2 \, (k - p_1 - p_2)^2}$$

Satisfies differential equation $\Downarrow$

$$\frac{d}{ds_{12}} \mathcal{I} = \frac{(d-4)}{2 s_{12}} \mathcal{I}$$

A linear homogeneous differential equation to solve, so

$$\mathcal{I} = \mathbf{C} \, (s_{12})^{\frac{(d-4)}{2}}$$