Applications of Reciprocity Theorems to the calculation of Signals in Particle Detectors

W. Riegler, P. Windischhofer, 29.1. 2020
Signals in Particle Detectors

Academic Training Lectures
Werner Riegler, CERN,  werner.riegler@cern.ch
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https://indico.cern.ch/event/843083/
Electrostatics
Reciprocity theorem

Two arbitrary charge distributions $\rho(x)$ and $\bar{\rho}(x)$

\[
\varphi(x) = \int \frac{\rho(x')}{|x - x'|} d^3x'
\]

\[
\bar{\varphi}(x) = \int \frac{\bar{\rho}(x')}{|x - x'|} d^3x'
\]

\[
W = \int \bar{\rho}(x) \varphi(x) d^3x = \int \int \frac{\bar{\rho}(x)\rho(x')}{|x - x'|} d^3xd^3x' = \int \rho(x')\bar{\varphi}(x') d^3x'
\]

\[
\int \bar{\rho}(x) \varphi(x) d^3x = \int \rho(x)\bar{\varphi}(x) d^3x
\]

This is simply a result of the fact that the Coulomb force depends only on the relative distance of the charges and not on absolute position in space.

The expressions can be interpreted as the work needed to move one charge distribution in the electric field of the other charge distribution, actio = reactio

Sounds like a trivial statement, but has very practical consequences.
Theorem, induced charge

The charge induced on a grounded conducting electrode by a point charge $q$ at position $x$ can be calculated the following way:

Remove the point charge, put the electrode in question to potential $V_w$ while keeping all other electrodes at ground potential.

This defines the weighting potential $\psi_n(x)$ of this electrode and the induced charge is

$$Q_{n}^{\text{ind}} = -\frac{q}{V_w} \psi_n(x)$$

We therefore do not have to solve the Poisson equation for a point charge but we just have to solve the Laplace equation for the given boundary conditions on the electrodes.

For detectors with long electrode, like wire chambers, RPCs, silicon strip detectors, we only have to solve the 2D Laplace equation instead of the 3D Poisson equation.

Specifically for numeric field calculations this is much easier and numerically stable.
The current induced on a grounded conducting electrode by a point charge \( q \) moving along a trajectory \( x(t) \) can be calculated the following way:

\[
I_{n}^{\text{ind}}(t) = -\frac{Q_{x}^{\text{ind}}(x(t))}{dt} = \frac{q}{V_{w}} \nabla \psi_{n}(x(t)) \dot{x}(t) = -\frac{q}{V_{w}} E_{n}(x(t)) \dot{x}(t)
\]

This weighting field \( E_{n}(x) \) is given by

\[
E_{n}(x) = -\nabla \psi_{n}(x)
\]

→ Ramo-Shockley theorem
Currents to Conductors Induced by a Moving Point Charge

W. Shockley
Bell Telephone Laboratories, Inc., New York, N. Y.
(Received May 14, 1938)

General expressions are derived for the currents which flow in the external circuit connecting a system of conductors when a point charge is moving among the conductors. The results are applied to obtain explicit expressions for several cases of practical interest.

Summary—A method is given for computing the instantaneous current induced in neighboring conductors by a given specified motion of electrons. The method is based on the repeated use of a simple equation giving the current due to a single electron's movement and is believed to be simpler than methods previously described.

Method of Computation

The method is based on the following equation, whose derivation is given later:
Weak conductivity of the material
Conductivity, volume resistivity

Volume resistivity $\rho$ [\(\Omega\text{m}\)] – typically expressed as \(\Omega\text{cm}\)

Conductivity $\sigma = 1/\rho$ [Siemens]

Surface resistivity $R$ [\(\Omega/\text{square}\)]

\[
R_1 = \rho \frac{L}{A} \quad I = \frac{U}{R_1}
\]

Nonuniform conductivity (volume resistivity) relates the local current density to the local electric field:

\[
j(x) = \sigma(x)E(x) = -\sigma(x)\nabla \varphi(x)
\]
Quasi-static Approximation

In a medium with conductivity \( \sigma \) there will be a current flowing according to

\[
j(x, t) = \sigma(x)E(x, t) = -\sigma(x)\nabla \varphi(x)
\]

In addition to this current we can have and externally impressed current \( j_e(x, t) \), so the total current is

\[
j(x, t) = \sigma(x)E(x, t) + j_e(x, t) = -\sigma(x)\nabla \varphi(x, t) + j_e(x, t)
\]

Assuming the variation of the electric field to be slow, we can use the Poisson equation for a medium given by

\[
\nabla [\varepsilon(x)\nabla \varphi(x, t)] = -\rho(x, t)
\]

Performing the time derivative we have

\[
\nabla [\varepsilon(x)\nabla \frac{\partial \varphi(x, t)}{\partial t}] = -\frac{\partial \rho(x, t)}{\partial t}
\]

And using

\[
\nabla j(x, t) = -\frac{\partial \rho(x, t)}{\partial t}
\]

we have

\[
\nabla \left[ \varepsilon(x)\nabla \frac{\partial \varphi(x, t)}{\partial t} + \sigma(x)\nabla \varphi(x, t) \right] = -\frac{\partial \rho_e(x, t)}{\partial t}
\]

Where \( \rho_e \) is the ‘externally impressed’ charge density.
Quasi-static Approximation of Maxwell’s equations

Assuming a conductivity \( \sigma \) of the material we have a current according to

\[
j(x, t) = \sigma(x)E(x, t)
\]

Maxwell’s equations for this situation

\[
\begin{align*}
\nabla D(x, t) &= \rho(x, t) \\
\nabla B(x, t) &= 0 \\
\nabla \times E(x, t) &= -\frac{\partial B(x, t)}{\partial t} \\
\nabla \times H(x, t) &= \frac{\partial D(x, t)}{\partial t} + j_e(x, t) + \sigma(x)E(x, t)
\end{align*}
\]

The current \( j_e(x, t) \) is an ‘externally impressed’ current, which is related to the 'externally impressed' charge density \( \rho_e \) by

\[
\nabla j_e(x, t) = -\frac{\partial \rho_e(x, t)}{\partial t}
\]

If we assume that this impressed current is only changing slowly we can neglect Faraday’s law and approximate

\[
\nabla \times E(x, t) \approx 0 \\
E(x, t) = -\nabla \varphi(x, t)
\]

and we can then write the electric field as the gradient of a potential, \( \varphi \), by taking the divergence of the last equation …

\[
\nabla \left[ \varepsilon(x) \nabla \frac{\partial \varphi(x, t)}{\partial t} + \sigma(x) \nabla \varphi(x, t) \right] = -\frac{\partial \rho_e(x, t)}{\partial t} 
\]

\( \rightarrow \) the same equation as on the previous slide
Performing the Laplace Transform of the quasi-static equation

\[ \nabla \left[ \varepsilon(x) \nabla \frac{\partial \varphi(x,t)}{\partial t} + \sigma(x) \nabla \varphi(x,t) \right] = - \frac{\partial \rho_e(x,t)}{\partial t} \]

we find

\[ \nabla \left[ \varepsilon(x) \nabla s \varphi(x,s) + \sigma(x) \nabla \varphi(x,s) \right] = - s \rho_e(x,s) \]

\[ \nabla \left[ \left( \varepsilon(x) + \sigma(x)/s \right) \nabla \varphi(x,s) \right] = - \rho_e(x,s) \]

So we can write this equation as

\[ \nabla [\varepsilon_{eff}(x) \nabla \varphi(x,s)] = - \rho_e(x,s) \]

\[ \varepsilon_{eff}(x) = \varepsilon(x) + \sigma(x)/s \]

\[ \rho(x,s) = - \nabla [\varepsilon(x) \nabla \varphi(x,s)] \]

\[ \rho_e(x,s) = - \nabla [\varepsilon_{eff}(x) \nabla \varphi(x,s)] \]

This is the Poisson equation with an effective permittivity !!

→ We can therefore find the time dependent solutions for a medium with a given conductivity by solving the electrostatic Poisson equation in the Laplace domain !

→ Knowing the electrostatic solution for a given permittivity \( \varepsilon(x) \) we just have to replace \( \varepsilon(x) \) by \( \varepsilon(x) + \sigma(x)/s \) and perform the inverse Laplace transform !
SIGNAL EVALUATION IN MULTIELECTRODE RADIATION DETECTORS BY MEANS OF A TIME DEPENDENT WEIGHTING VECTOR * 

E. GATTI, G. PADOVINI
Instituto di Fisica del Politecnico di Milano, Milano, Italy

and

V. RADEKA
Brookhaven National Laboratory, Upton, New York 11973, U.S.A.

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A time variant weighting field is introduced in order to calculate in ionization chambers or semi-conductor detectors the signal at the sensing electrode when this latter or other electrodes are not grounded.

\[ V_n^{\text{ind}}(t) = -\frac{q}{Q_0} \int_0^t \vec{K}_n \left[ \vec{x}_1(t', t' - t) \right] \vec{x}_1(t') dt' + \frac{q}{Q_0} \int_0^t \vec{K}_n \left[ \vec{x}_2(t', t' - t) \right] \vec{x}_2(t') dt' \]

Keywords: Signal theory. Particle detectors. RPCs. Silicon detectors.

Abstract

Most particle detectors are based on the principle that charged particles have a trail of ionization in the detector and that the movement of these charges in an electric field induces signals on the detector electrodes. Assuming detector elements that are insulating and electrodes with infinite conductivity one can calculate the signals with an electostatic approximation using the so-called ‘Ramo theorem’. This is the standard way for the calculation of signals, e.g., in wire chambers and silicon detectors. In case the detectors contain reactive elements, which is, e.g., the case in resistive plate chambers or underdepleted silicon detectors, the time dependence of the signals is not only given by the movement of the charges but also by the time-dependent reaction of the detector materials. Using the quasistatic approximation of Maxwell’s equations we present an extended formalism that allows the calculation of induced signals for detectors with general materials by time dependent weighting fields. As examples, we will discuss the signals in resistive plate chambers and underdepleted silicon detectors.

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PACS: Signal theory. Particle detectors. RPCs. Silicon detectors.
Charge spreading

\[ T_0 = \varepsilon_0 Rg \quad T = g/v \]

In some detectors, a resistive layer is applied on top of the readout strips to spread out the charge and therefore ‘increase’ the pad response function.

This example shows a thin resistive layer on top of the readout strips. The layer is insulated from the strips.

- The solid line shows the situation for different time constants
- The dashed line shows the situation for infinite resistivity
Detectors where the finite propagation time of EM waves as well as radiation effects cannot be neglected
Signals induced on electrodes by moving charges, a general theorem for Maxwell’s equations based on Lorentz-reciprocity

W. Riegler\textsuperscript{a}, P. Windischhofer\textsuperscript{b}

\textsuperscript{a}CERN
\textsuperscript{b}University of Oxford

Abstract

This report discusses a signal theorem for charged particle detectors where the finite propagation time of the electromagnetic waves produced by a moving charge cannot be neglected. While the original Ramo-Shockley theorem and related extensions are all based on electrostatic or quasi-electrostatic approximations, the theorem presented in this report is based on the full extent of Maxwell’s equations and does account for all electrodynamic effects. It is therefore applicable to all devices that detect fields and radiation from charged particles.
Lorentz reciprocity theorem

We assume the most general form of Maxwell’s equations for a linear anisotropic material of position- and frequency-dependent permittivity matrix $\varepsilon(x, \omega)$, permeability matrix $\mu(x, \omega)$ and conductivity matrix $\sigma(x, \omega)$. These $3 \times 3$ matrices relate the vector fields

$$
\mathbf{D} = \varepsilon \mathbf{E} \quad \mathbf{B} = \mu \mathbf{H} \quad \mathbf{J} = \sigma \mathbf{E}
$$

(1)

The source of the fields is an externally impressed current density $\mathbf{J}^e(x, \omega)$. In the Fourier domain, Maxwell’s equations then read as

$$
\nabla \cdot \varepsilon \mathbf{E} = \rho \quad \nabla \cdot \mu \mathbf{H} = 0
$$

(2)

$$
\nabla \times \mathbf{E} = -i\omega \mu \mathbf{H} \quad \nabla \times \mathbf{H} = \mathbf{J}^e + \sigma \mathbf{E} + i\omega \varepsilon \mathbf{E}
$$

(3)

Let us now look at the situation where two different externally impressed current densities $\mathbf{J}^e$ and $\mathbf{J}'^e$ are placed on the same material distribution, as shown in Fig. 1. The current density $\mathbf{J}^e$ will cause fields $\mathbf{E}$ and $\mathbf{H}$ and the current density $\mathbf{J}'^e$ will cause fields $\mathbf{E}$ and $\mathbf{H}$. We can relate these quantities using the

$$
\int_V \overline{\mathbf{E}}(x, \omega) \mathbf{J}^e(x, \omega) dV = \int_V \mathbf{E}(x, \omega) \overline{\mathbf{J}}^e(x, \omega) dV
$$

Figure 1: Two different current densities in the same geometry with the same material properties.
Figure 4: a) A moving point charge is creating an electric field and therefore a "potential difference" between the points $x_1$ to $x_2$. b) A line current $I_0 = Q_0 \delta(t)$ producing an electric field $E_w(x, t)$, the so-called "weighting field".

$$V^{\text{ind}}(t) = \frac{q}{Q_0} \int_{-\infty}^{\infty} E_w(x_0(t'), t - t') \dot{x}_0(t') dt'$$
Transmission Line

(a) \( V_{\text{ind}}(t) \)

(b) \( I(t) = Q_0 \delta(t) \)

\( V_\text{ind}(t) \)

\( \mathbf{v} \cdot \mathbf{q} \)

\( R \)

\( Z_p \)

\( Z_T \)

\( Z_0 \)

\( E_w(r, z, t) \)
Electric Dipole Antenna

\[ E_\theta^w(r, \theta) = \frac{Q_0 ds \sin \theta}{4\pi\varepsilon_0} \frac{r^3}{r^3} \left[ \Theta \left( t - \frac{r}{c} \right) + \frac{r}{c} \delta \left( t - \frac{r}{c} \right) + \left( \frac{r}{c} \right)^2 \delta' \left( t - \frac{r}{c} \right) \right] \]

\[ E_r^w(r, \theta) = 2 \frac{Q_0 ds \cos \theta}{4\pi\varepsilon_0} \frac{r^3}{r^3} \left[ \Theta \left( t - \frac{r}{c} \right) + \frac{r}{c} \delta \left( t - \frac{r}{c} \right) \right] \]
Electric Dipole Antenna

\[
x(t) = \begin{pmatrix} -vt \\ A \sin(\omega_0 t) \\ A \cos(\omega_0 t) \end{pmatrix}, \quad \dot{x}(t) = \begin{pmatrix} -v \\ A\omega_0 \cos(\omega_0 t) \\ -A\omega_0 \sin(\omega_0 t) \end{pmatrix}, \quad r(t) \approx | - vt |
\]

\[
\omega_0 = e_0 B/m_e
\]

with \( \omega = \omega_0/(1 - \beta) \). For galactic magnetic fields on the order of 1 nT, the frequency \( \omega_0 \) is only 176 Hz. For electrons with a kinetic energy \( E \) the observed frequency is larger by a factor \( 1/(1 - \beta) \approx 2(E/m_e c^2)^2 \). For an electron with a kinetic energy of 5 GeV, the frequency is increased by a factor \( 2 \times 10^6 \) so we measure radio waves of around 352 MHz.

The same expression describes the synchrotron radiation emitted by an electron beam that is passed through an undulator with a wavelength \( \lambda_0 \) and where the emitted radiation has a wavelength of \( \lambda = \lambda_0(1 - \beta) \).
Magnetic Dipole Antenna

\[ E_{\phi}^w = -\frac{Q_0 \mu_0}{4\pi} \frac{\sin \theta}{r^2} \left[ \delta'(t - \frac{r}{c}) + \frac{r}{c} \delta''(t - \frac{r}{c}) \right] \]

\[ V_{\text{ind}}(t) = \frac{q \mu_0 dA}{4\pi} \int \frac{x\dot{y} - y\dot{x}}{r^3} \left[ -\delta'(t' - t + \frac{r}{c}) + \frac{r}{c} \delta''(t' - t + \frac{r}{c}) \right] dt' \]

\[ x(t) = x_0 \quad \dot{x}(t) = 0 \quad y(t) = vt \quad \dot{y}(t) = v \quad r(t) = \sqrt{x_0^2 + v^2t^2} \]

\[ V_{\text{ind}}(t) = -\frac{qdA}{4\pi \epsilon_0} \frac{3x_0 c t \beta^3 (1 - \beta^2)}{[x_0^2 (1 - \beta^2) + \beta^2 c^2 t^2]^{5/2}} \]

\[ t_{\text{peak}} = \pm \frac{x_0}{2\beta c} \sqrt{1 - \beta^2} = \pm \frac{x_0}{2c \beta \gamma} \]

\[ V_{\text{peak}} = \pm \frac{qdA}{4\pi \epsilon_0} \frac{48\beta^2}{25 \sqrt{5} (1 - \beta^2) x_0^3} = \pm \frac{qdA}{4\pi \epsilon_0} \frac{48\beta^2 \gamma^2}{25 \sqrt{5} x_0^3} \]

\[ \frac{V_{\text{ind}}(t)}{V_{\text{peak}}} = \frac{25\sqrt{5}}{32} \left[ \frac{t}{t_{\text{peak}}} \right] \left[ 1 + \left( \frac{t}{2t_{\text{peak}}} \right)^2 \right]^{5/2} \]
Beam Current Transformer

How do you simulate the response of a fast BCT to a single particle, or single particle bunch?

Would it be possible/useful to simulate the signal directly and with the theorem to see how they compare?

Could that simulation be simplified by using the reciprocity theorem?