

Energy dependent three-pion scattering from QCD

Maxwell T. Hansen

September 9th, 2021

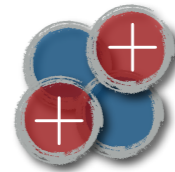


THE UNIVERSITY
of EDINBURGH

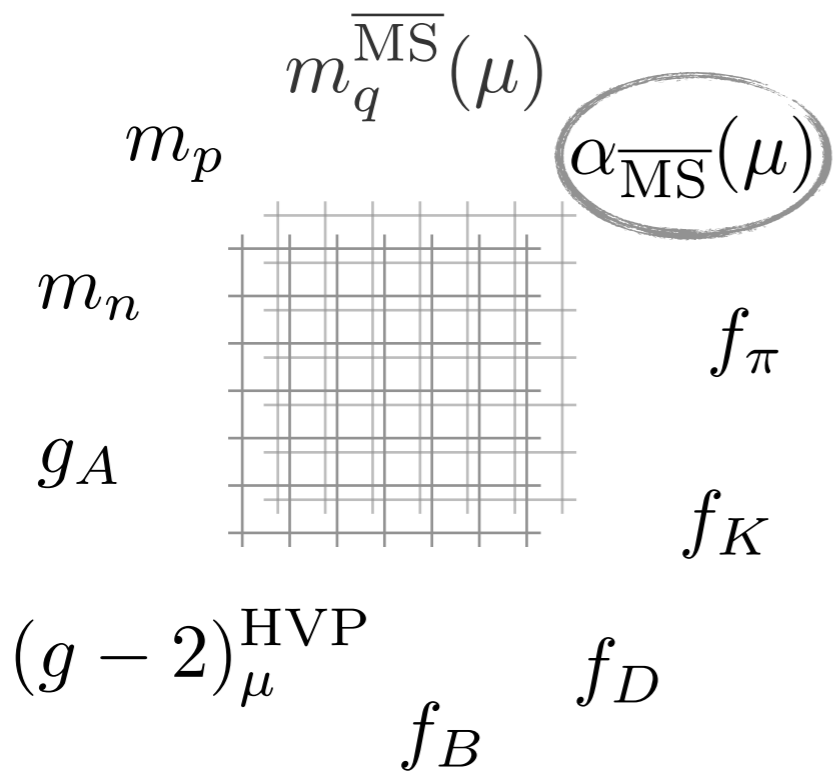
Recipe for strong force predictions

1. Lagrangian defining QCD +
2. Formal / numerical machinery (lattice QCD) +
3. A few experimental inputs (e.g. M_π, M_K, M_Ω) =

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\Psi}_f (i\not{D} - m_f) \Psi_f - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}$$



Wide range of precision pre-/post-dictions



$$\alpha_{\overline{\text{MS}}}^{(5)}(M_Z) = 0.1182(8)$$

lattice average

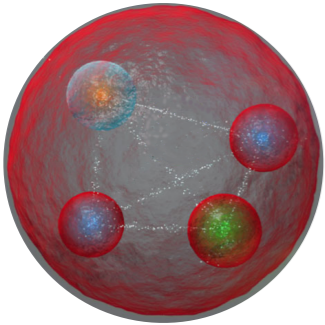
$$\alpha_{\overline{\text{MS}}}^{(5)}(M_Z) = 0.1174(16)$$

PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ → Tool for new physics searches

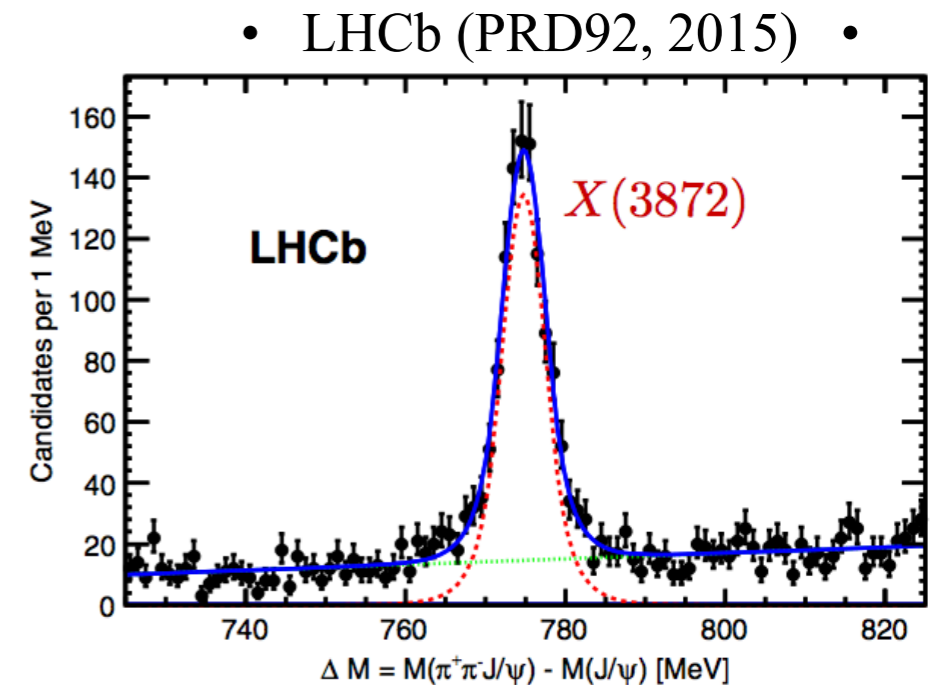
Multi-hadron observables

- Exotics, XYZs, tetra- and penta-quarks, H dibaryon



e.g. $X(3872)$

$$\sim |D^0 \bar{D}^{*0} + \bar{D}^0 D^{*0}\rangle?$$



- Electroweak, CP violation, resonant enhancement

CP violation in charm

$$D \rightarrow \pi\pi, K\bar{K}$$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

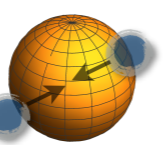
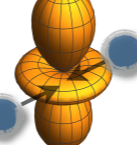
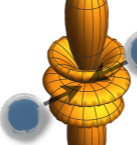

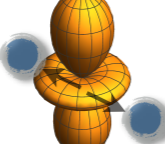
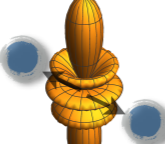
Resonant B decays

$$B \rightarrow K^* \ell\ell \rightarrow K\pi \ell\ell$$

$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \notin \text{QCD Fock space}$

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d, K \sim \bar{s}u, p \sim uud$
- Overlaps of multi-hadron *asymptotic states* \rightarrow S matrix

		$ \pi\pi, \text{in}\rangle$		
				
		$e^{2i\delta_0(s)}$	0	0
$S(s) \equiv \langle \pi\pi, \text{out} $		0	$e^{2i\delta_1(s)}$	0
		0	0	$e^{2i\delta_2(s)}$

depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

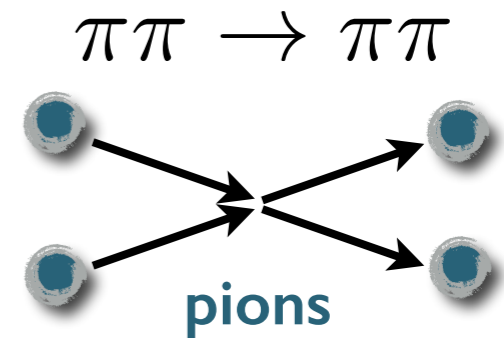
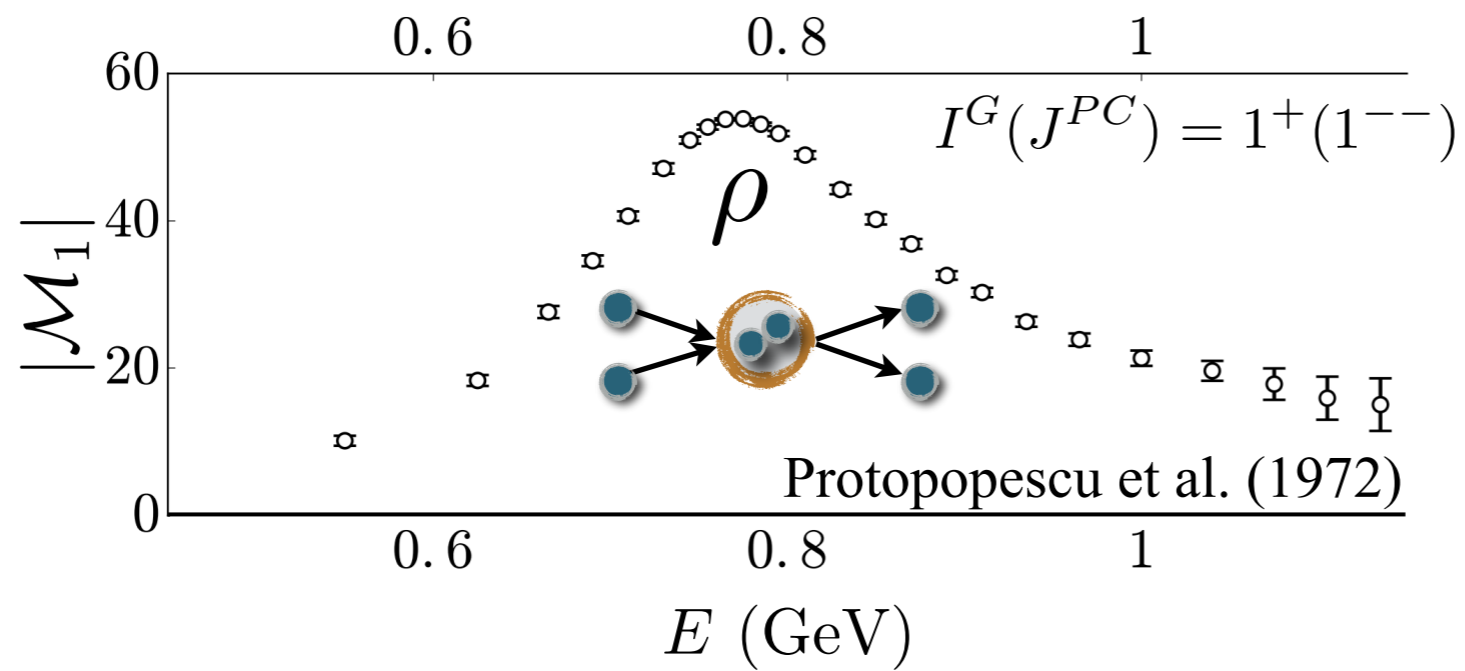
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

- An enormous space of information

$|\pi\pi\pi\pi, \text{in}\rangle \quad |K\bar{K}, \text{in}\rangle \quad \dots$

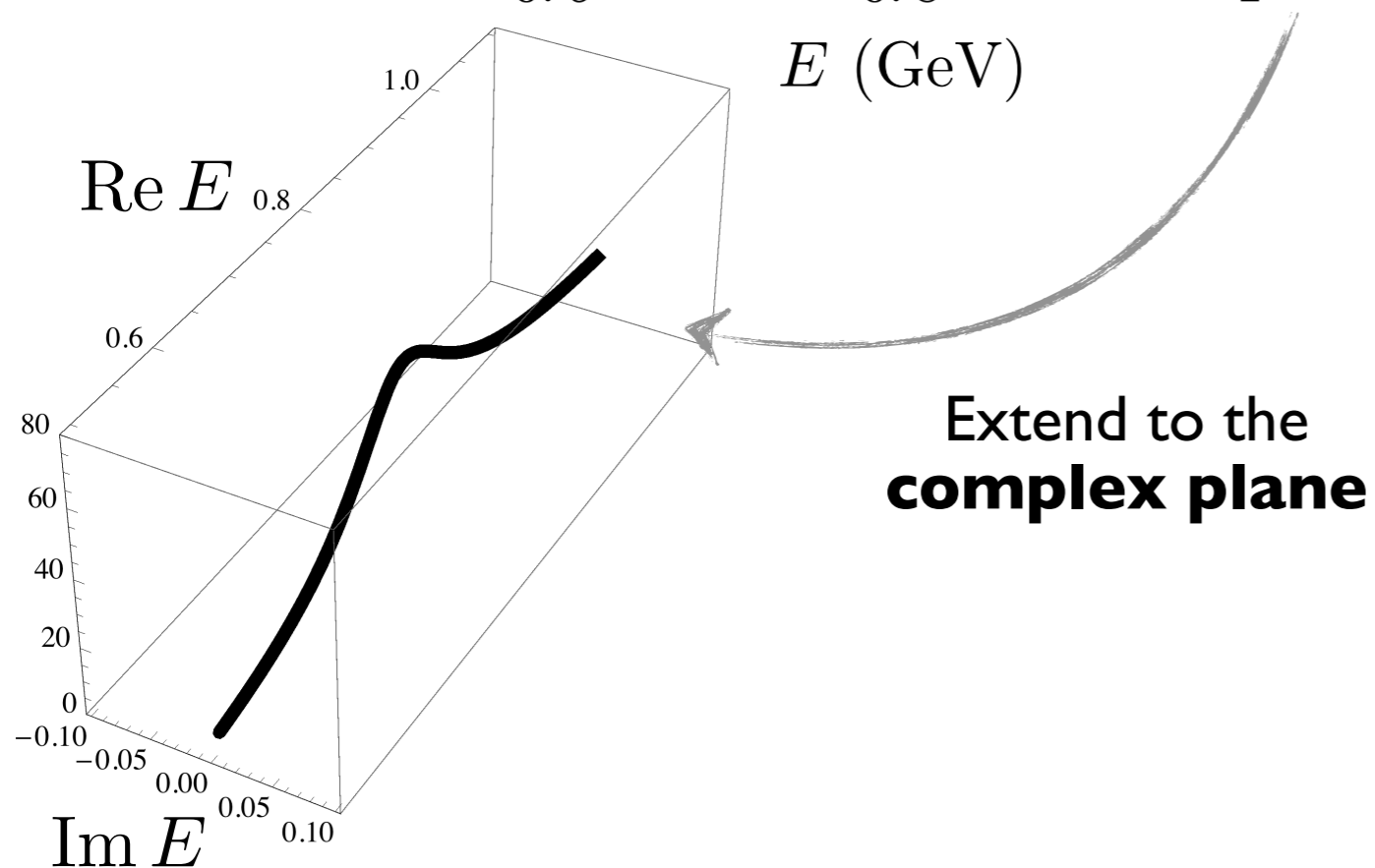
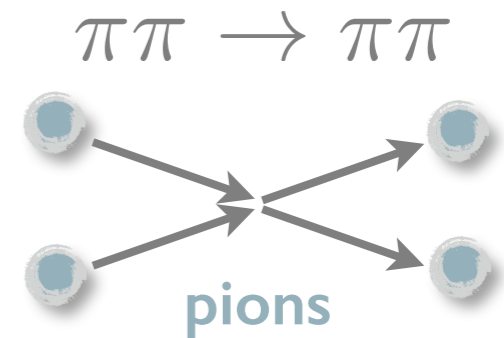
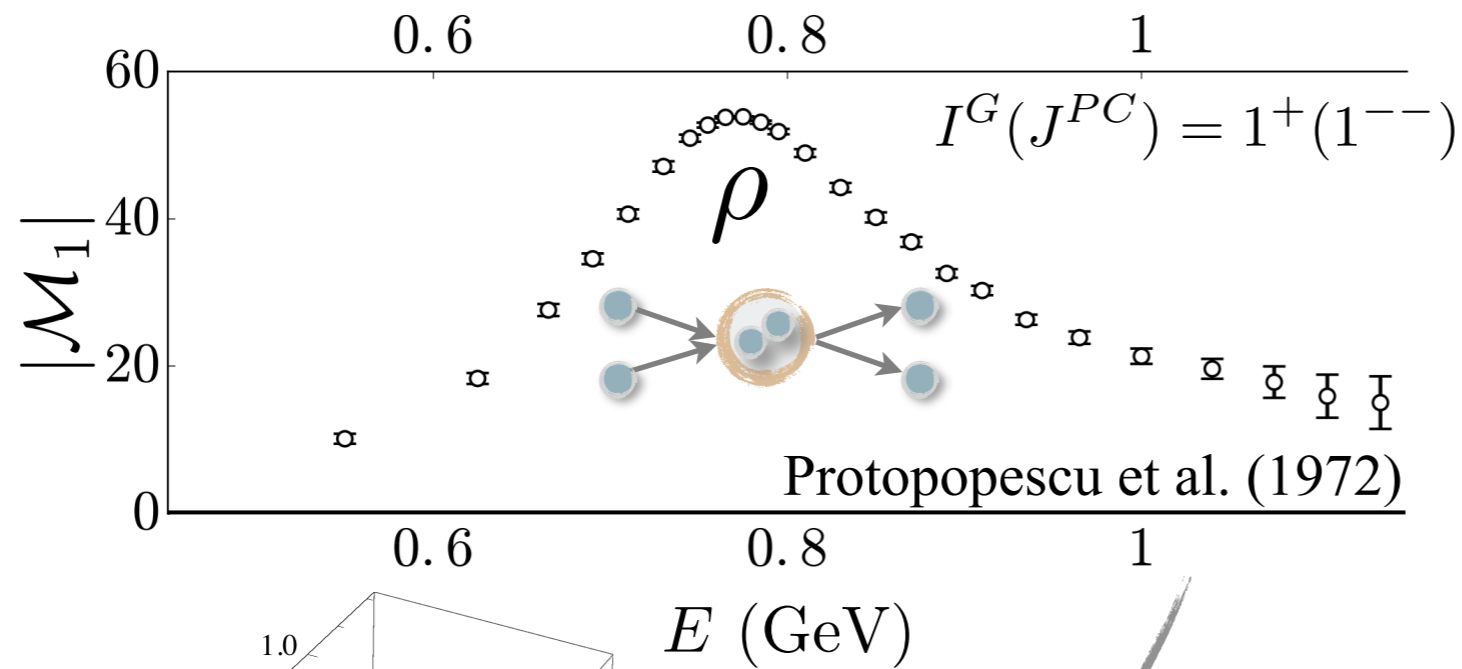
QCD resonances

□ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$
 scattering rate



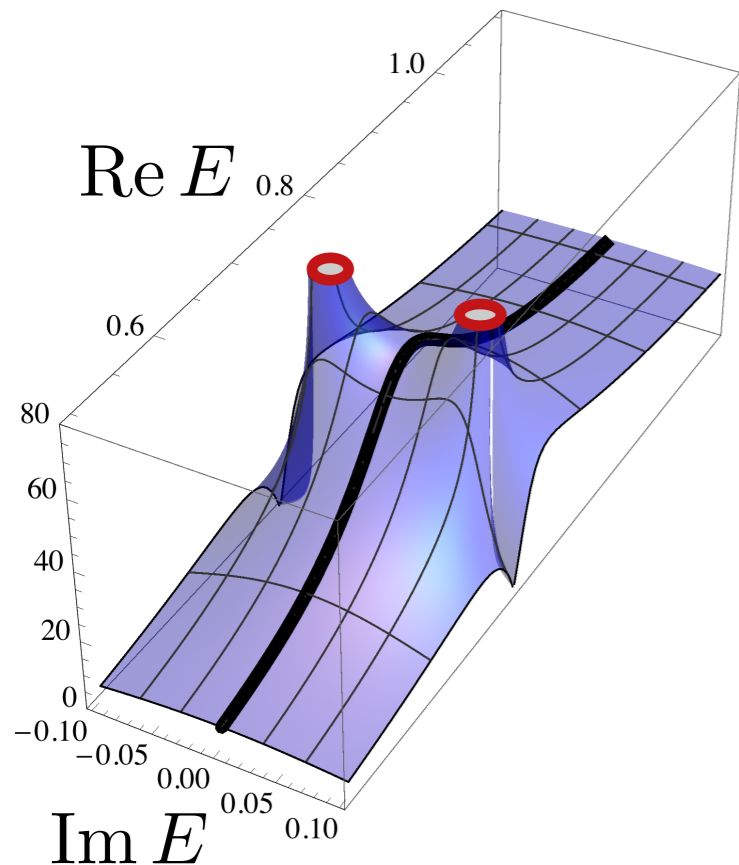
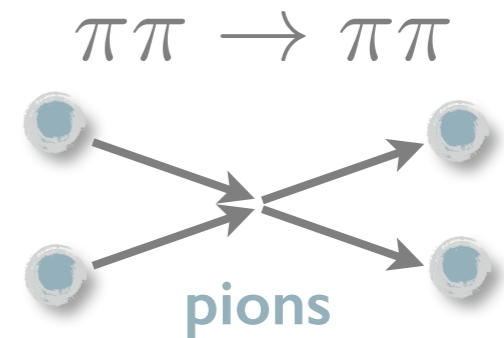
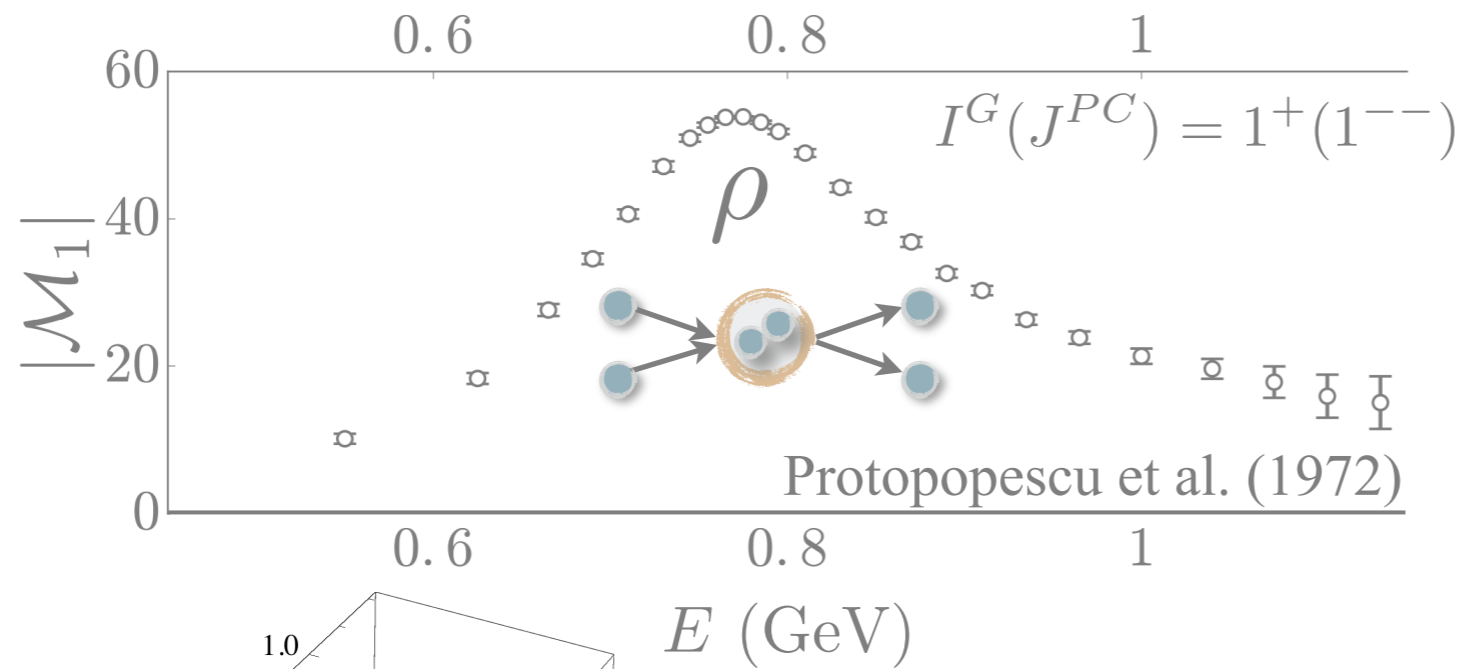
QCD resonances

□ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$
 scattering rate

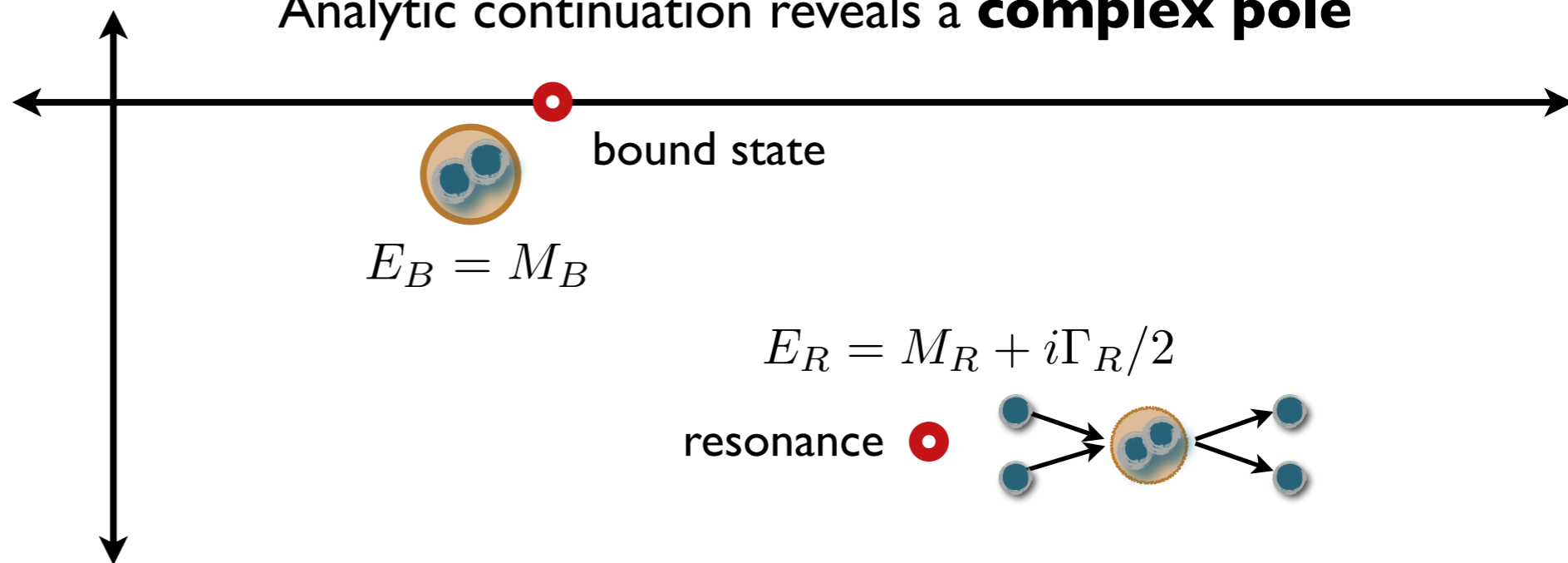


QCD resonances

□ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$
 scattering rate



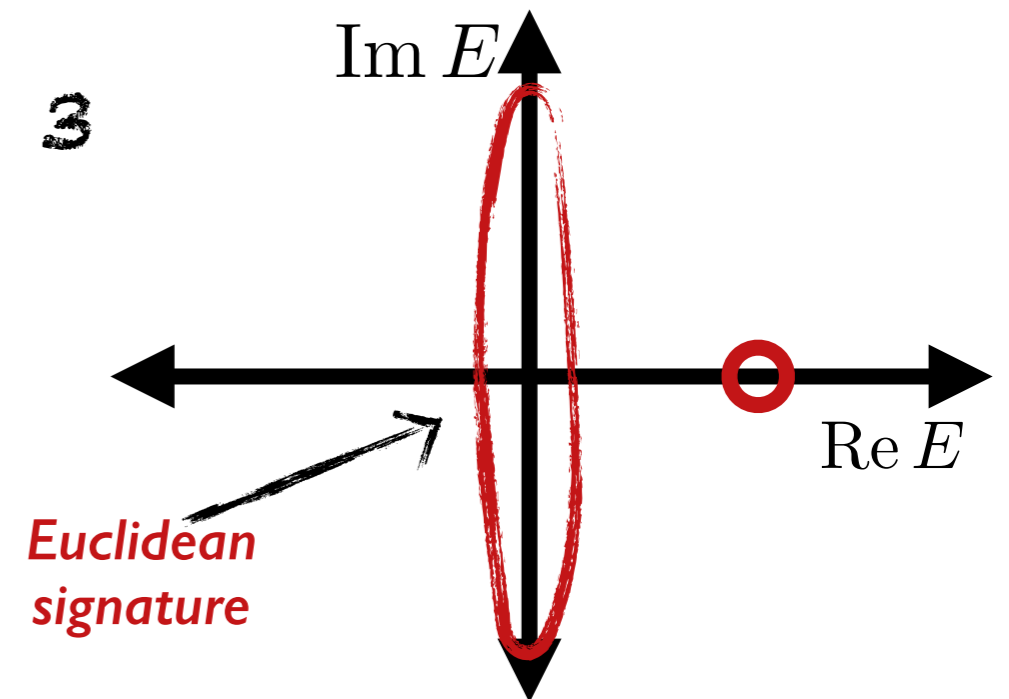
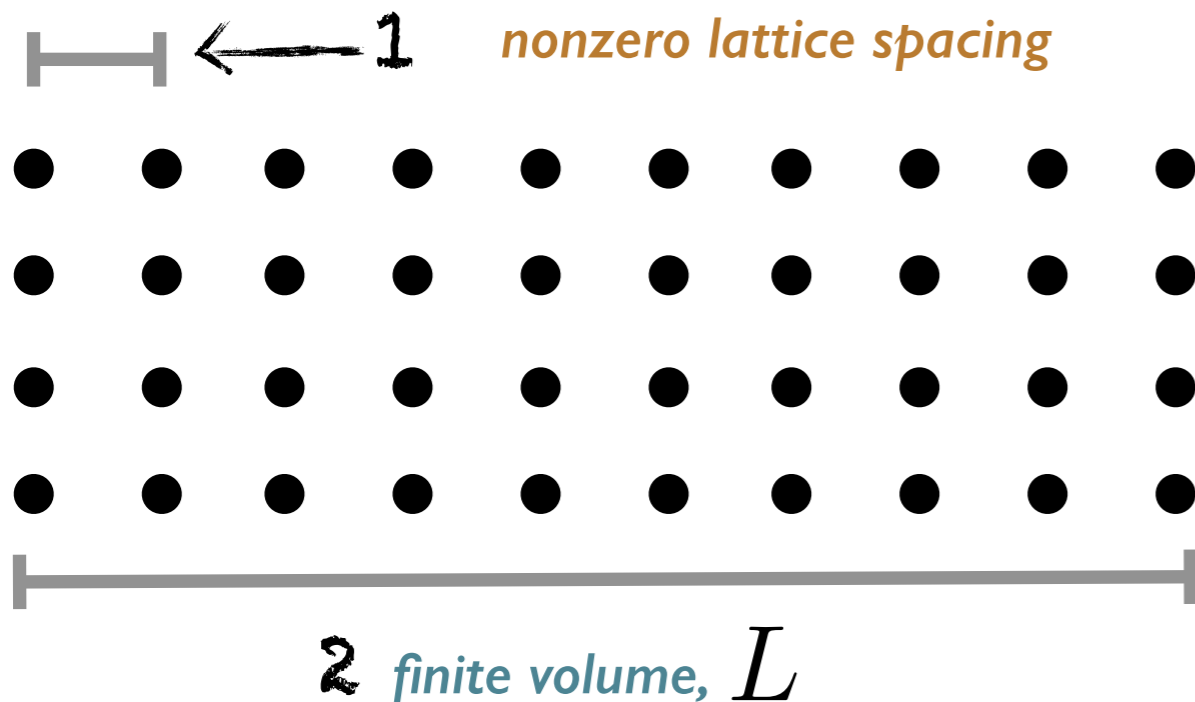
Analytic continuation reveals a **complex pole**



Lattice QCD

$$\text{observable?} = \int d^N \phi e^{-S} \left[\begin{array}{l} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

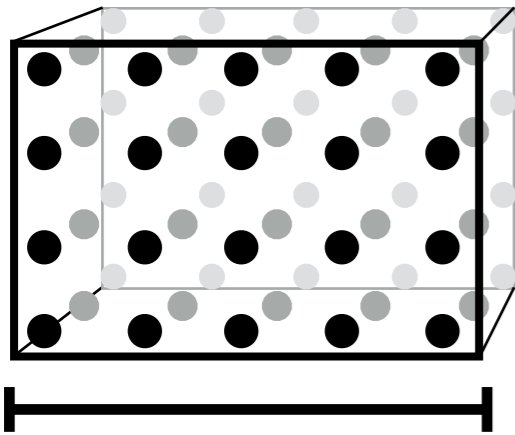
To proceed we have to make *three modifications*



Also... $M_{\pi, \text{lattice}} > M_{\pi, \text{our universe}}$
(but physical masses \rightarrow increasingly common)



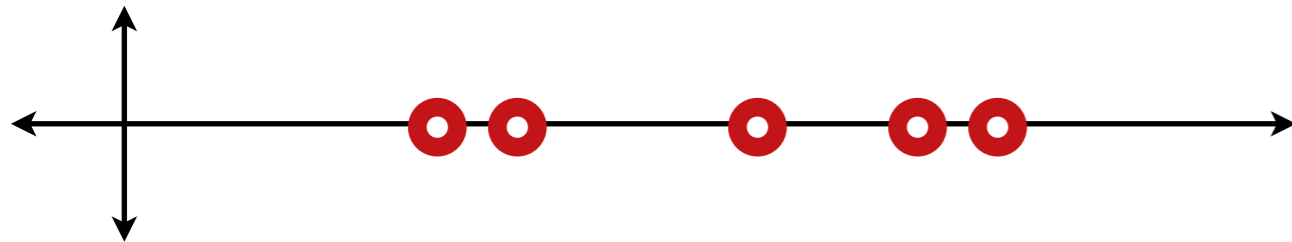
Difficulties for multi-hadron observables



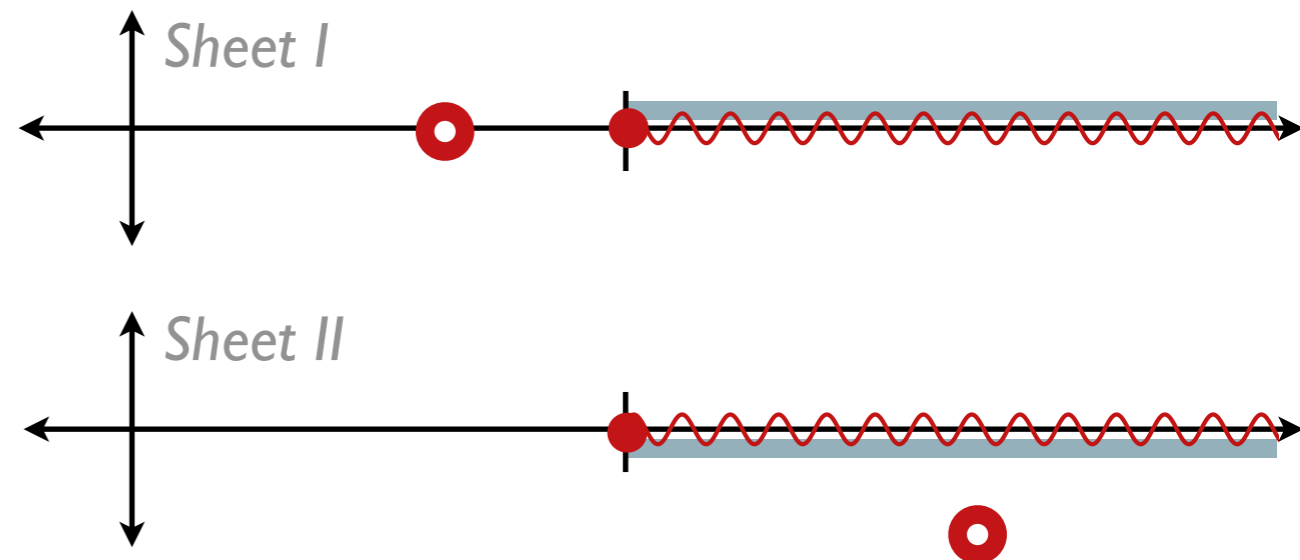
□ The *finite volume*...

- *Discretizes* the spectrum
- *Eliminates* the branch cuts and extra sheets
- *Hides* the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



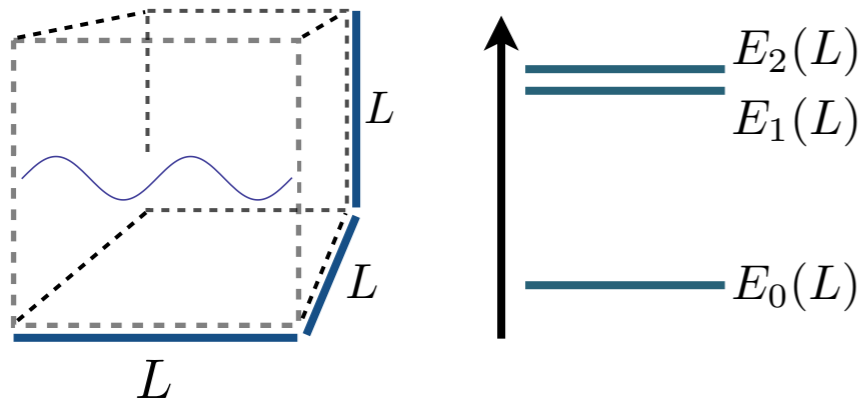
□ LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau) \mathcal{O}_i^\dagger(0) \rangle = \sum_n \langle 0 | \mathcal{O}_j(\tau) | E_n \rangle \langle E_n | \mathcal{O}_i^\dagger(0) | 0 \rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

□ Our task is relate $E_n(L)$ and $\langle E_{m'} | \mathcal{J}(0) | E_m \rangle$ to **experimental observables**

The finite-volume as a tool

Finite-volume set-up



□ **cubic**, spatial volume (extent L)

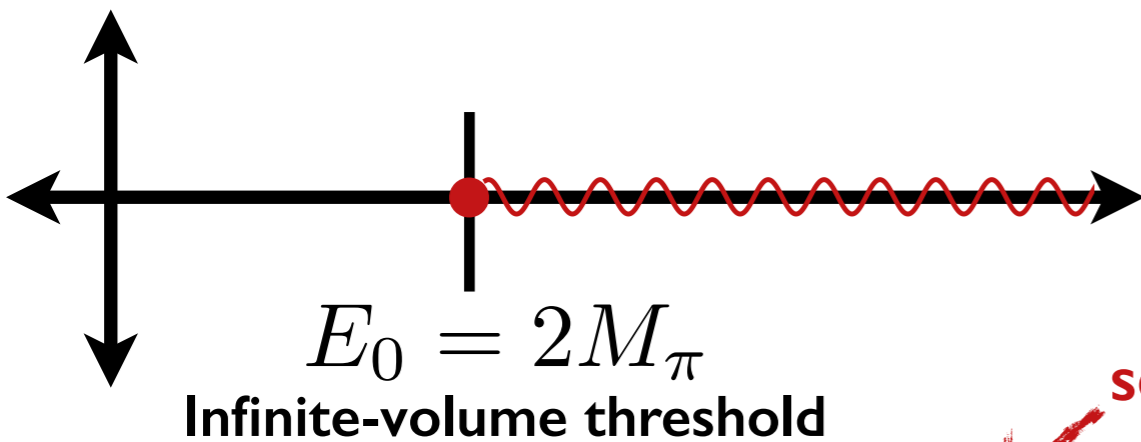
□ **periodic**

$$\vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3$$

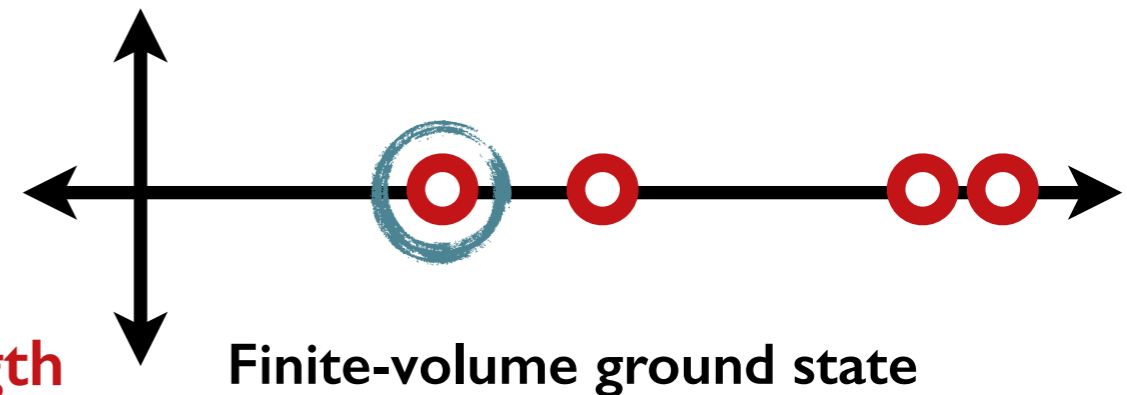
□ L is large enough to neglect $e^{-M_\pi L}$

□ T and lattice also negligible

□ Scattering leaves an *imprint* on finite-volume quantities



$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

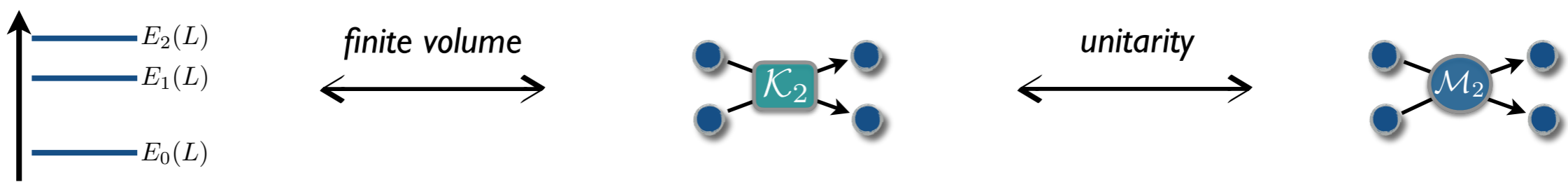


$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

General method

$$\det[\mathcal{K}^{-1}(s) + F(P, L)] = 0 \qquad F(P, L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$ Neglects e^{-mL}

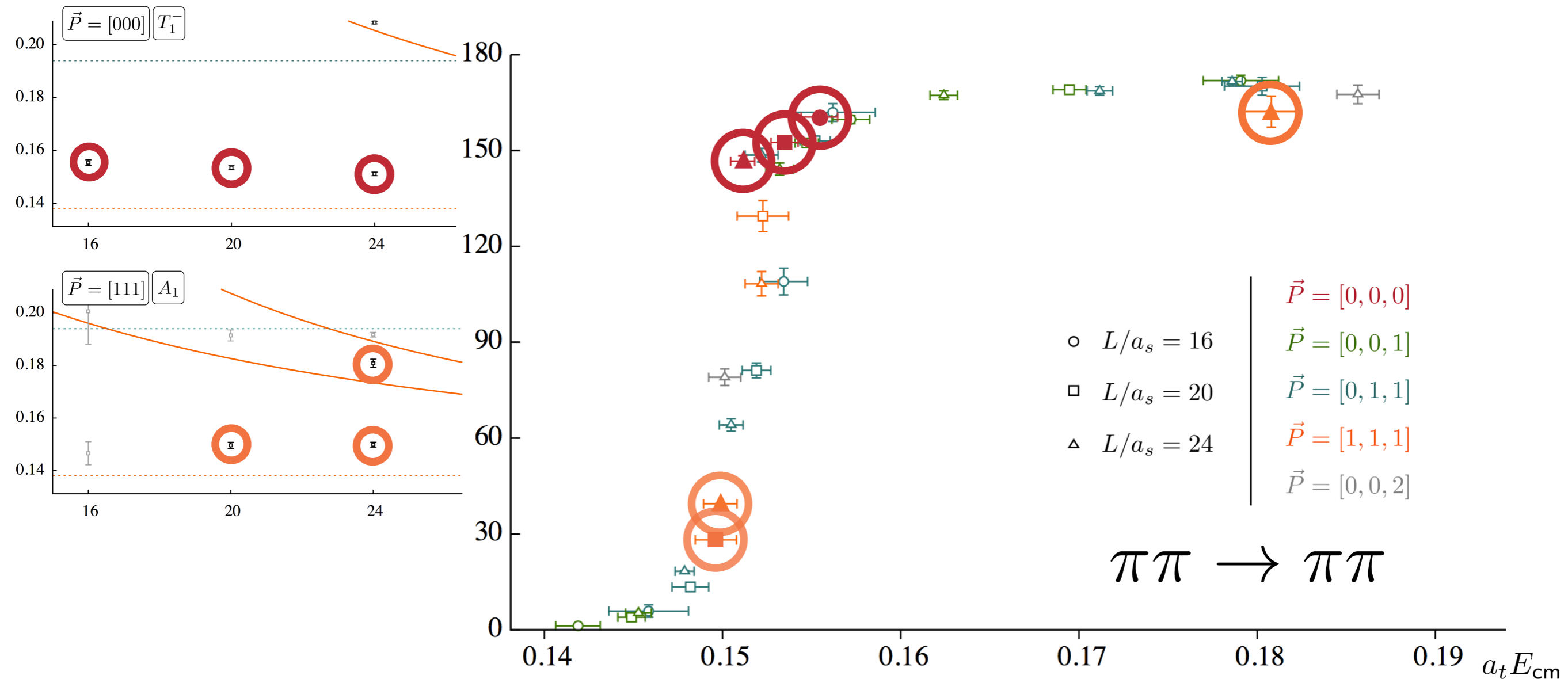
Generalized to *non-degenerate masses, multiple channels, spinning particles*
Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
 Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
 Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
 Li, Liu (2013) • Briceño (2014)

Using the result

□ Single-channel case (*pions in a p-wave*)

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$

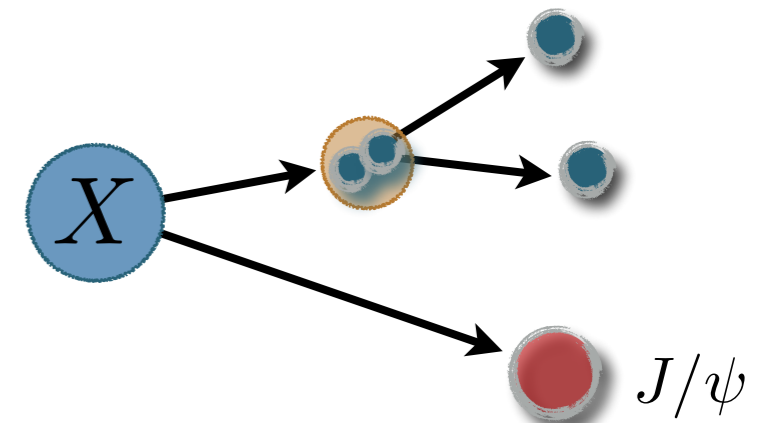


- Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

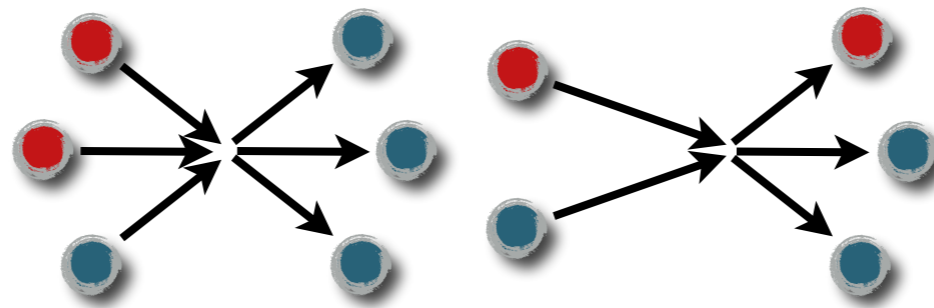
3-particle amplitudes

2-to-2 only samples J^P 0^+ 1^- 2^+ ...

many interesting resonances have significant 3-body decays



Goal: *finite-volume + unitarity formalism* for generic two- and three-particle systems



Applications...

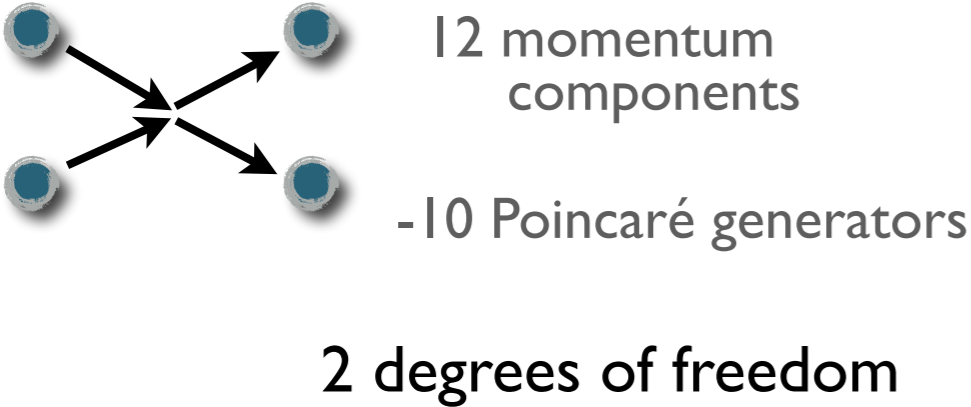
exotic resonance pole positions, couplings, quantum numbers

$\omega(782), a_1(1420) \rightarrow \pi\pi\pi$ $X(3872) \rightarrow J/\psi\pi\pi$ $X(3915)[Y(3940)] \rightarrow J/\psi\pi\pi$

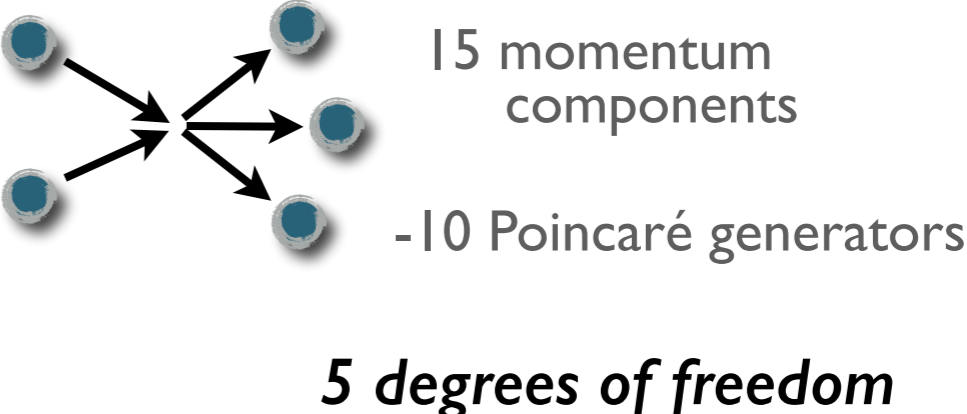
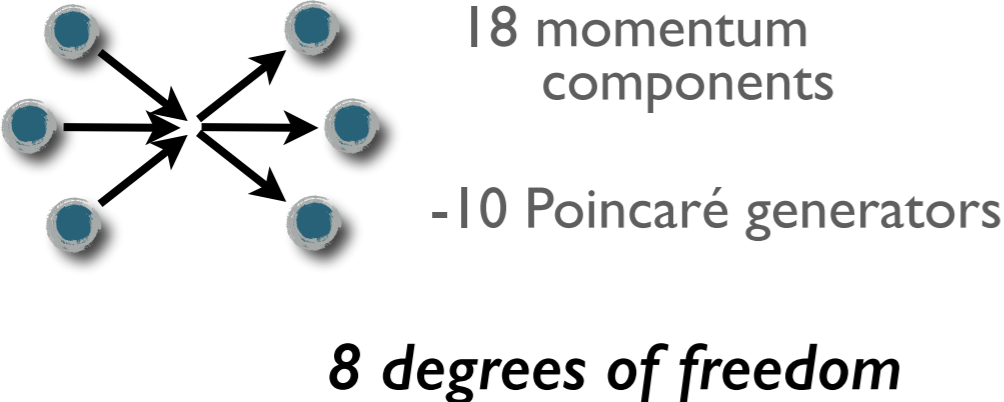
form factors and transitions

and much more!... (3-body forces, weak transitions, gluons content)

Complication: degrees of freedom



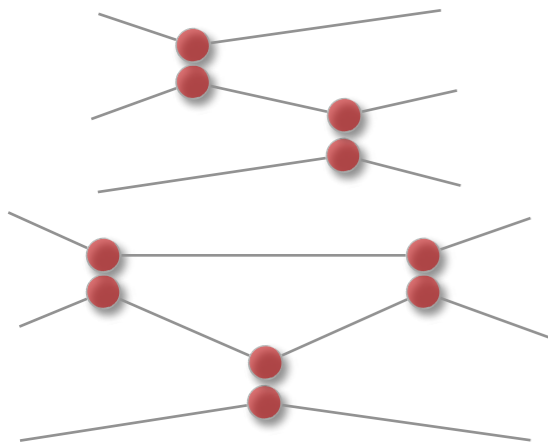
$$\vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_3 + \vec{p}_4 \longrightarrow \text{Mandelstam } s, t$$



Complication: on-shell states

□ Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to 3 binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin

AND

ROBERT SUGAR
Physics Department, Columbia University, New York, New York

AND

GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received 31 January 1966)

$$b = \frac{(m_1 + m_3)(m_2 + m_3)}{m_1 m_2}$$

It follows that if

$$b^{n-2}(b-1) > 1, \quad (\text{IV.18})$$

then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \epsilon$:
4 collisions possible

$$\pi\pi K$$

$$b < 2$$

5 collisions possible

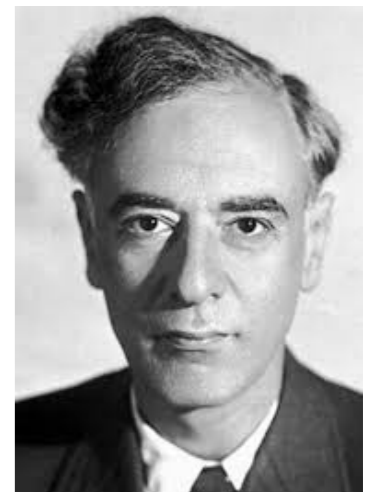
$$\pi K K$$

□ Correspond to Landau singularities

$$i\mathcal{M}_{3 \rightarrow 3} \equiv \text{fully connected correlator} = \text{diagram 1} + \text{diagram 2} + \dots$$

complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles



Two key observations

- Intermediate $K_{\text{df},3}$ removes singularities

$$\mathcal{K}_{\text{df},3} \equiv \text{fully connected diagrams w/ PV pole prescription} \quad - \quad \text{diagram 1} \quad + \quad \text{diagram 2} \quad + \dots$$

same degrees of freedom as M_3

smooth real function

relation to $M_3 = \text{known}$

- $K_{\text{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\text{df},3}(p_3, p_2, p_1; k_3, k_2, k_1) = \mathcal{K}_{\text{df},3}^{\text{iso},0} + \mathcal{K}_{\text{df},3}^{\text{iso},1} \Delta + \dots \quad \Delta = \frac{s - (3m)^2}{(3m)^2}$$

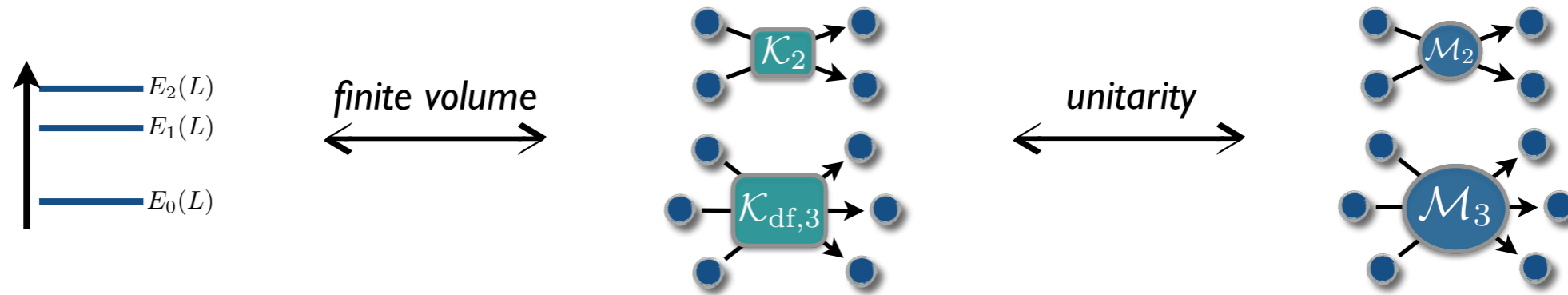
smooth real function

analogous to effective range expansion

$$p \cot \delta = -\frac{1}{a} + \frac{1}{2} r p^2 + \mathcal{O}(p^4)$$

gives handle on many degrees of freedom
(DOFs enter order by order)

Status...



Identical spin-zero, no 2-to-3, no K2 poles • MTH, Sharpe (2014, 2015) •

as above... but including 2-to-3 • Briceño, MTH, Sharpe (2017) •

as above... but including K2 poles • Briceño, MTH, Sharpe (2018) •

Non-identical, non-degenerate spin-zero $\pi\pi\pi \rightarrow \rho\pi \rightarrow \omega \rightarrow \rho\pi \rightarrow \pi\pi\pi$
 • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020, 2021)

Multiple three-particle channels... Spin!

Related work

Finite-volume unitarity method

Döring, Mai (2016,2017)

Gives connection to unitarity relations

Non-relativistic EFT method

Hammer, Pang, Rusetsky (2017)

Simplified derivation + integral equations

Do not yet include non-degenerate, 2-to-3

Review articles

MTH and Sharpe, 1901.00483 • Rusetsky, 1911.01253 • Mai, Döring, Rusetsky, 2103.00577

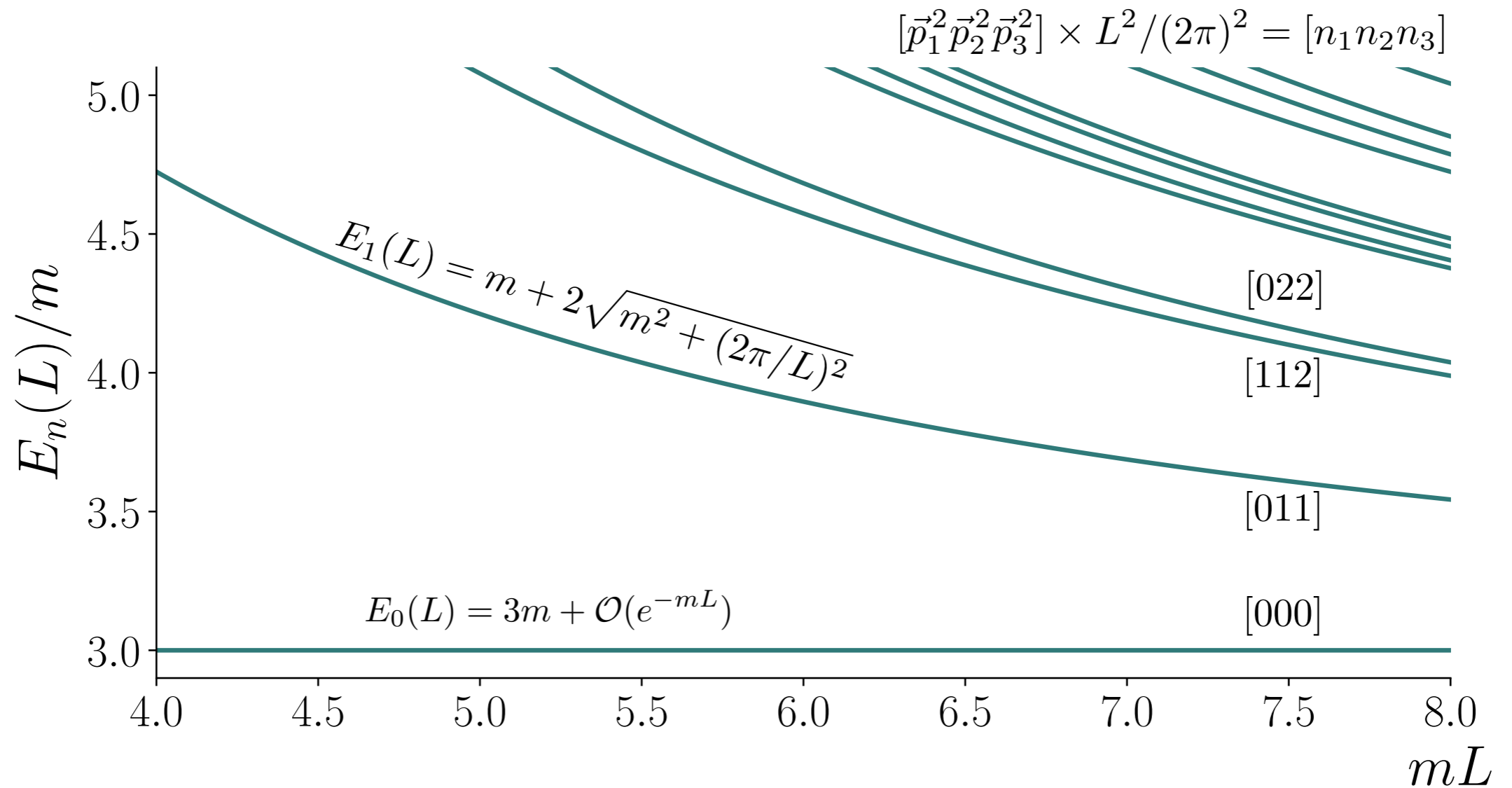
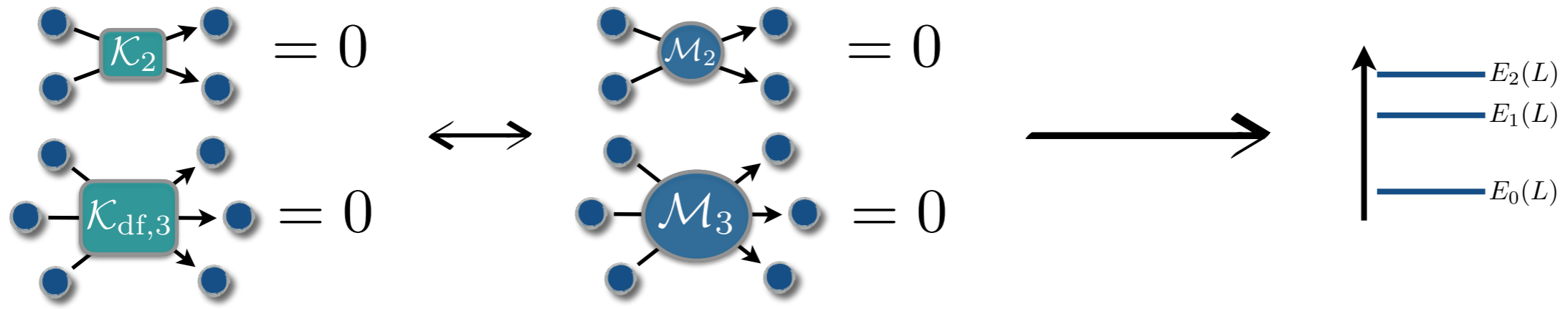
All methods

Rely on intermediate, scheme-dependent quantity

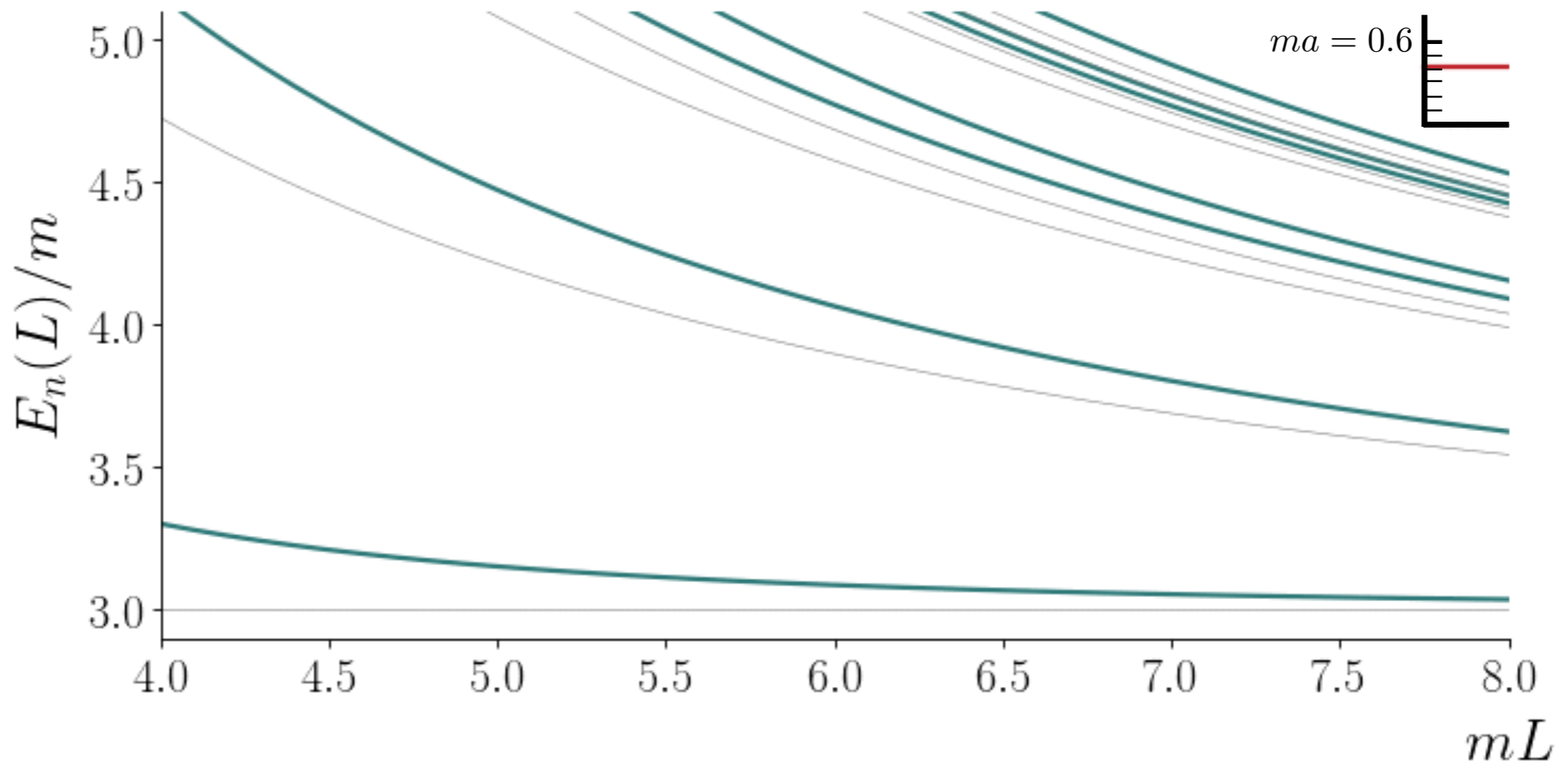
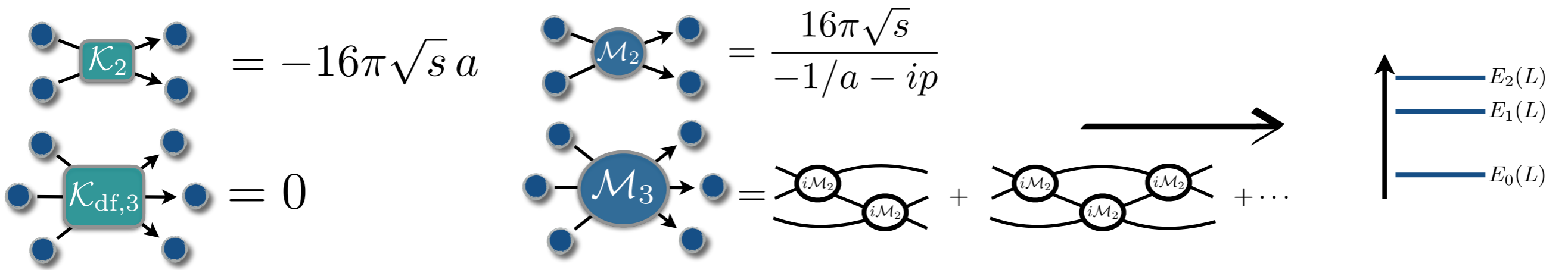
Hold up to e^{-mL} and for $E_3^* < 5m_\pi$

Equivalent where comparable

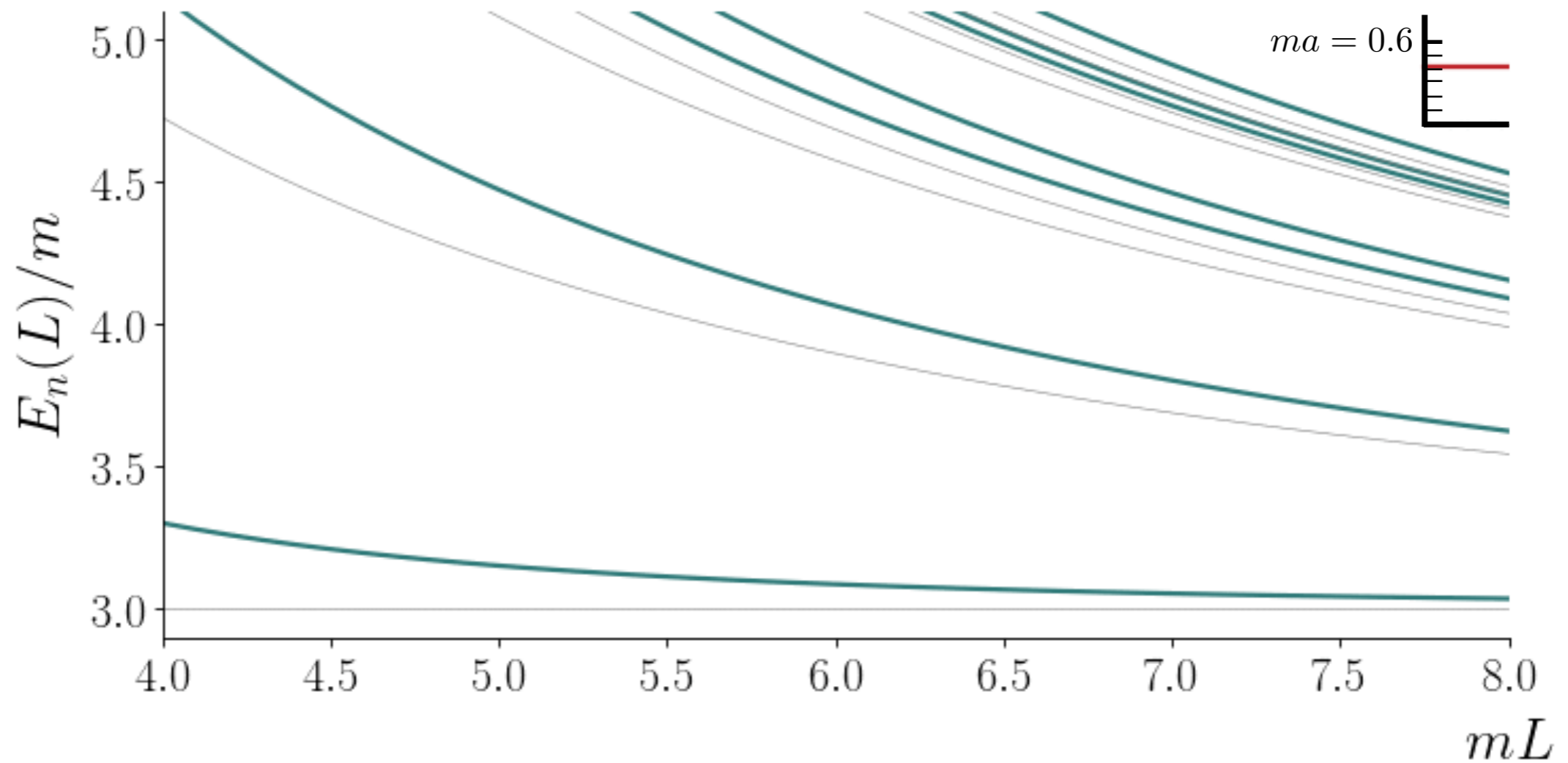
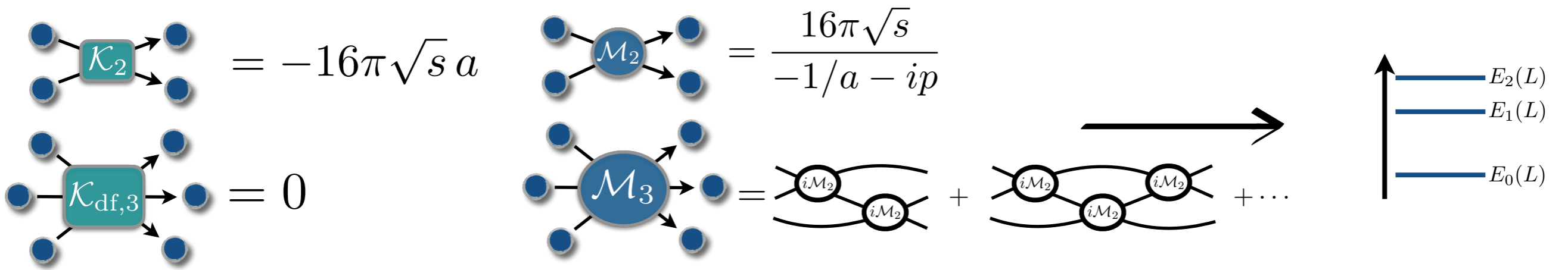
Non-interacting energies



Two-particle interactions

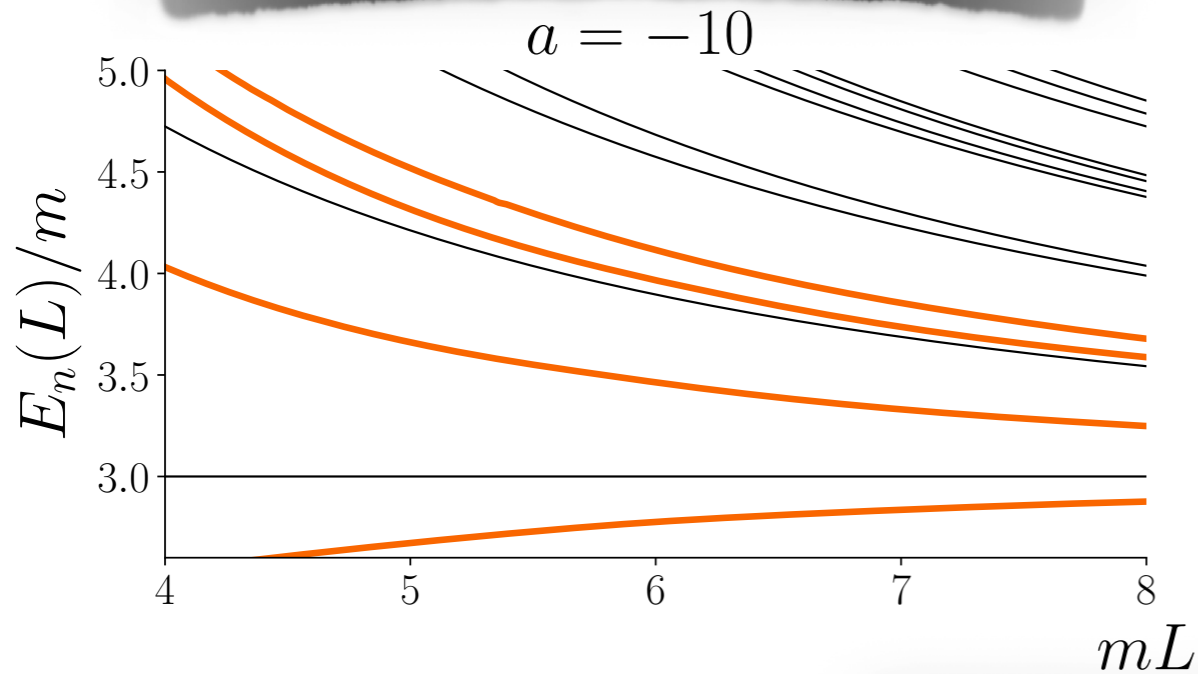


Two-particle interactions

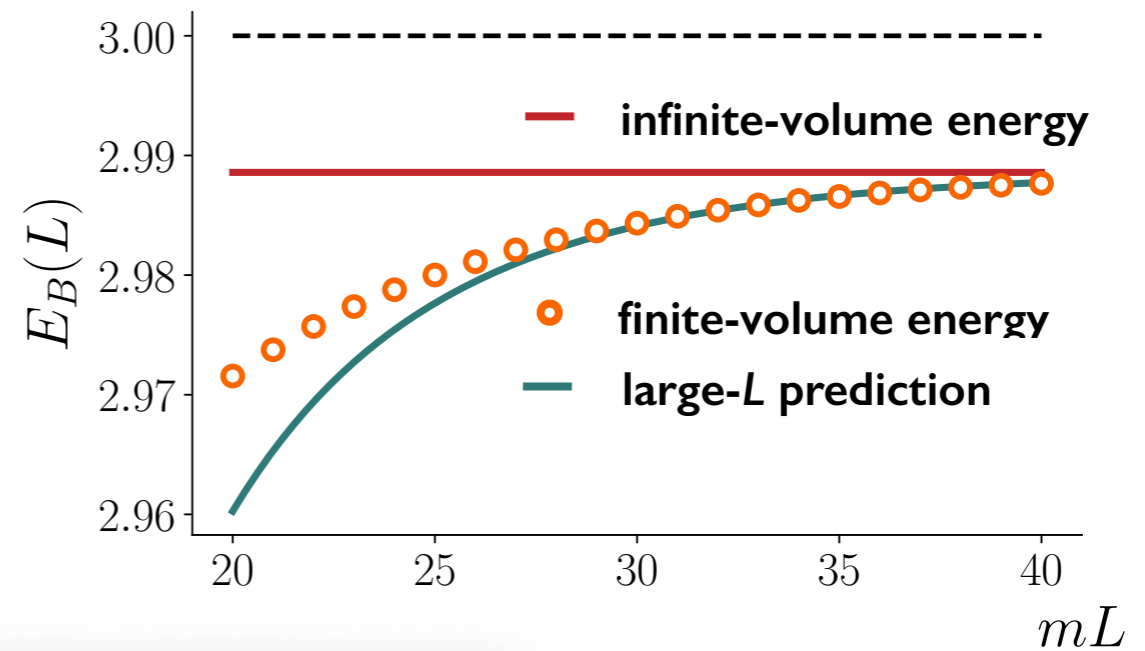


Many toy results

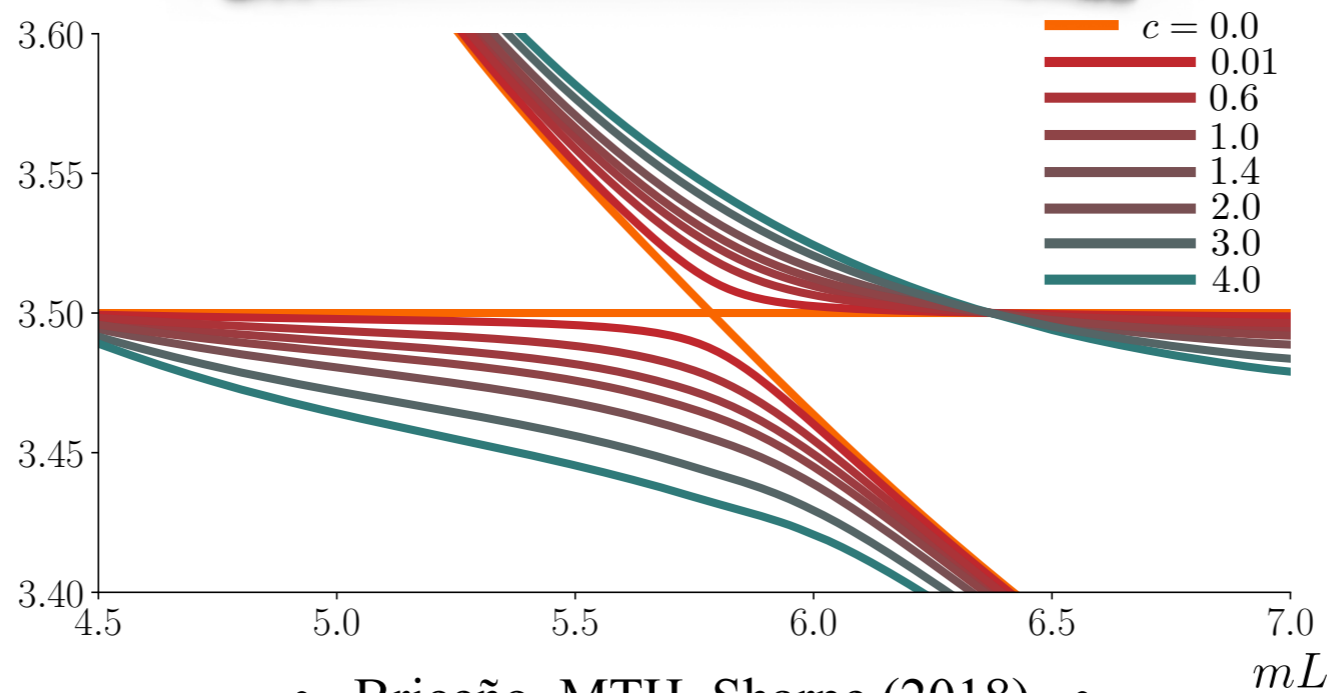
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state







Model of a 3-particle resonance

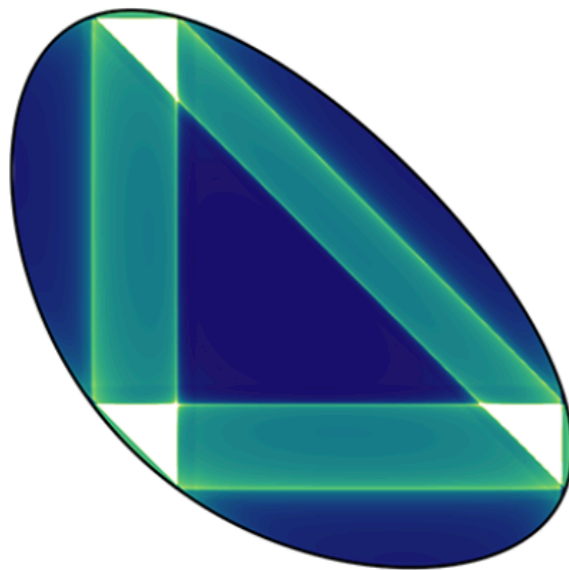


• Briceño, MTH, Sharpe (2018) •

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen ^{1,2,*} Raul A. Briceño,^{3,4,†} Robert G. Edwards ^{3,‡}
Christopher E. Thomas ^{5,§} and David J. Wilson ^{5,||}

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*

Phys. Rev. Lett. **126**, 012001 (2021)

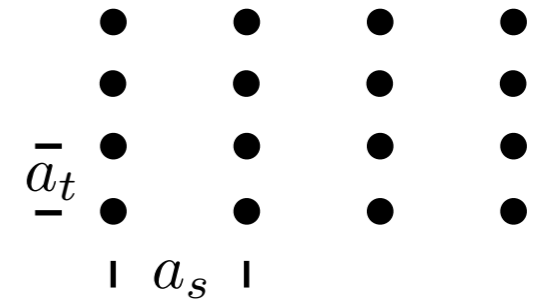
$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

lattice details

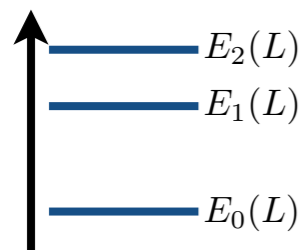
$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6)$$

$$m_\pi \approx 400\text{MeV} \quad a_s \approx 0.12\text{fm}$$

$$L_s/a_s = 20, 24$$

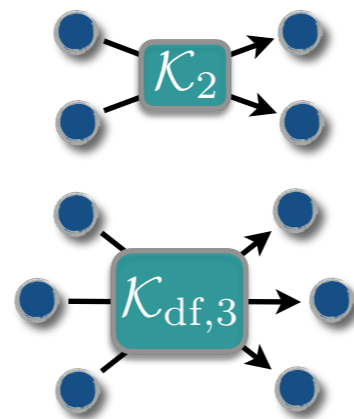


□ Workflow outline



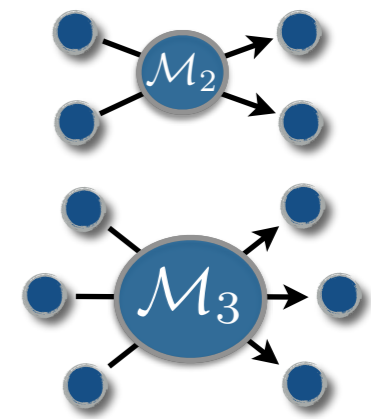
finite volume

↔



unitarity

↔



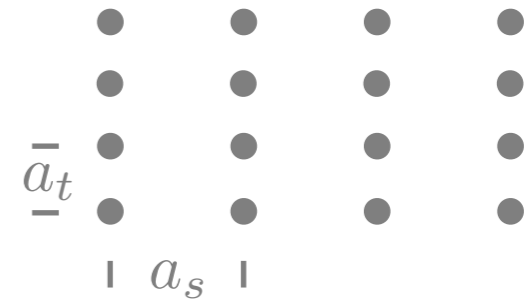
$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

lattice details

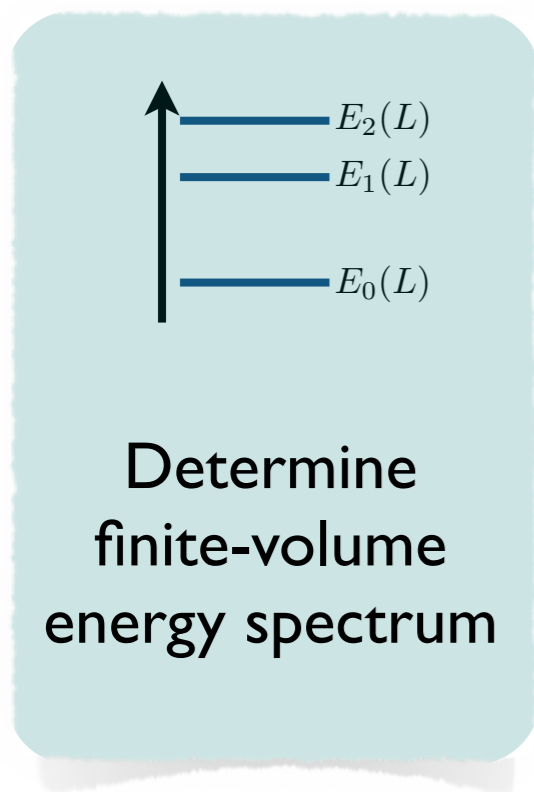
$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6)$$

$$m_\pi \approx 400\text{MeV} \quad a_s \approx 0.12\text{fm}$$

$$L_s/a_s = 20, 24$$

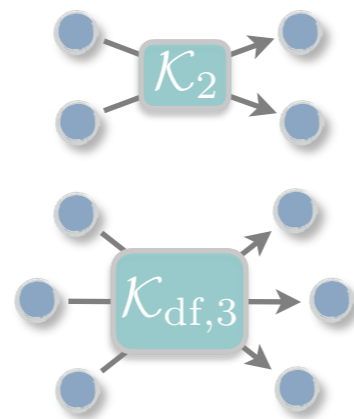


□ Workflow outline



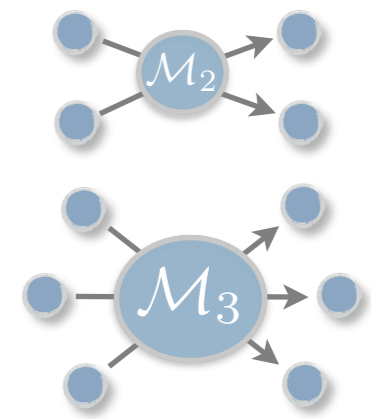
finite volume

↔



unitarity

↔



Operators

□ Variational method → *optimized single hadron* $\pi = c_1 \bar{q}\Gamma q + c_2 \bar{q}\Gamma Dq + \dots$

□ Build two-pion operators in the usual way...

$$\pi\pi(\mathbf{p}_{\pi\pi}, \Lambda_{\pi\pi}) = \sum \text{CG } \pi(\mathbf{p}_1) \pi(\mathbf{p}_2)$$

... three-pion operators from there

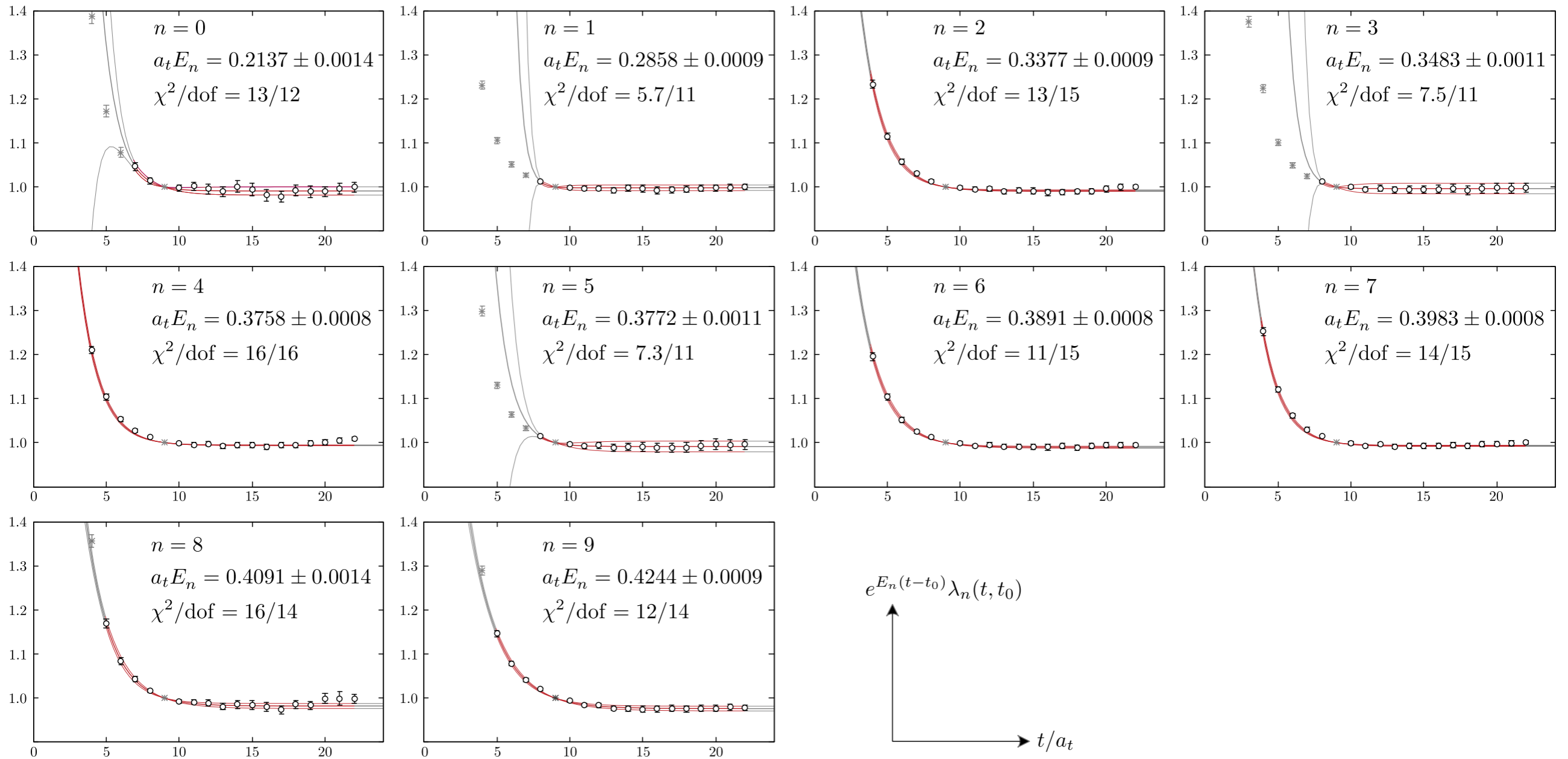
$$(\pi\pi\pi)(\mathbf{P}, \Lambda) = \sum \text{CG } \pi(\mathbf{p}_3) \pi\pi(\mathbf{p}_{\pi\pi}, \Lambda_{\pi\pi})$$

□ **Note: No optimization of $\pi\pi(\mathbf{p}_{\pi\pi}, \Lambda)$ operators**

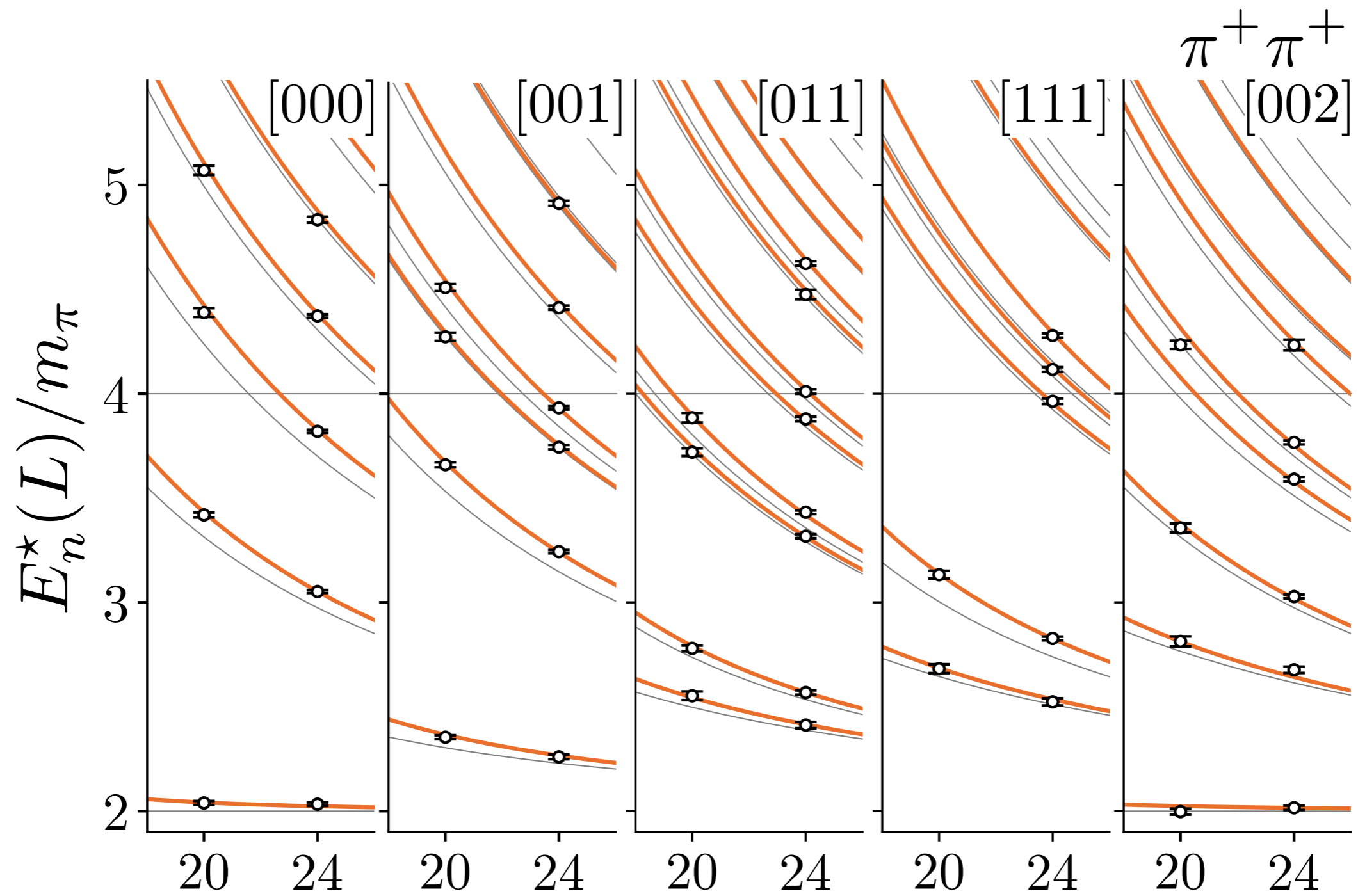
□ Analogous to...

$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} = (0 \oplus 1) \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$$

$$I = 3 (\pi^+ \pi^+ \pi^+), \quad \mathbf{P} = [000], \quad \Lambda = A_1^-, \quad L/a_s = 24$$

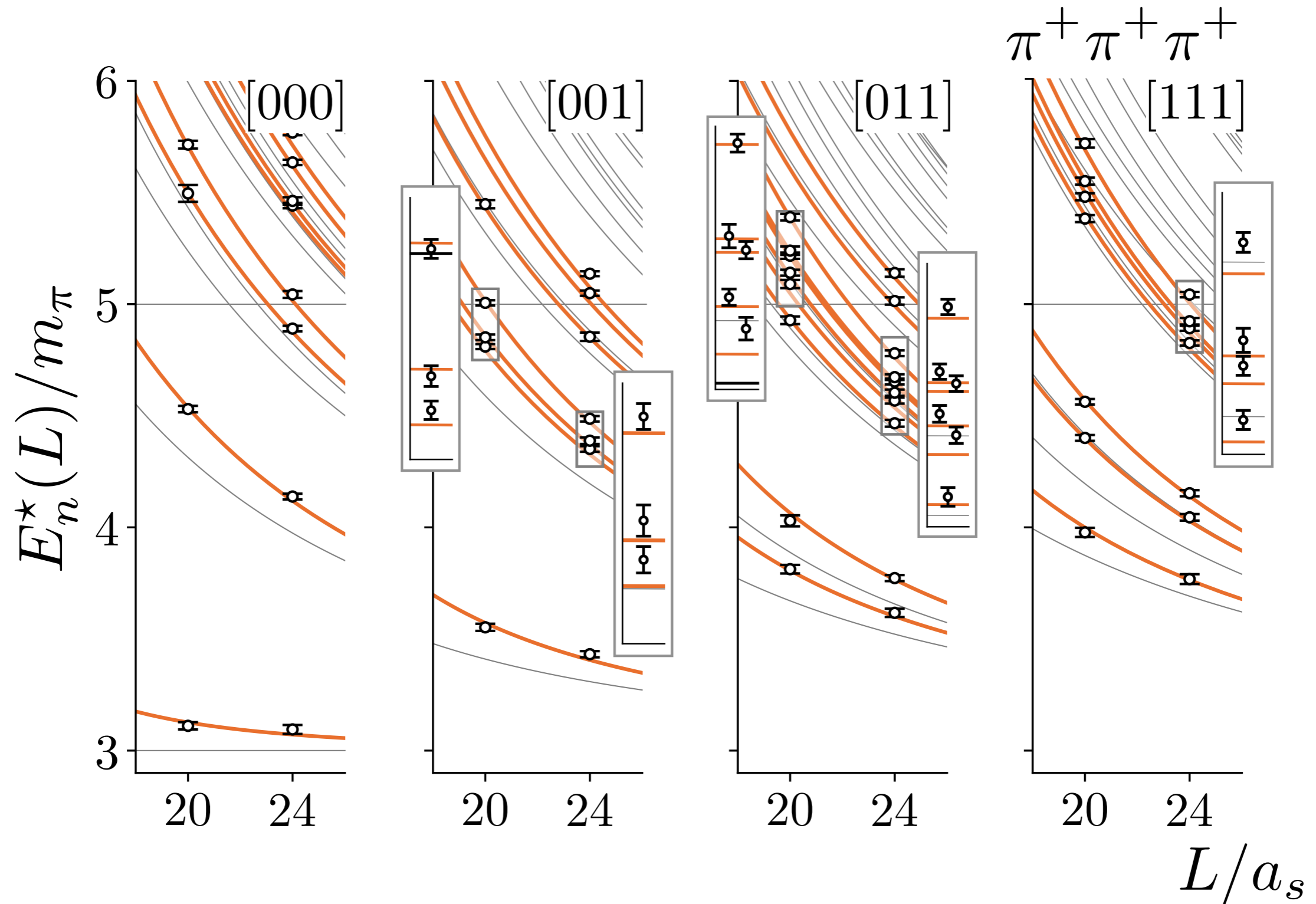


$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+\pi^+$ energies



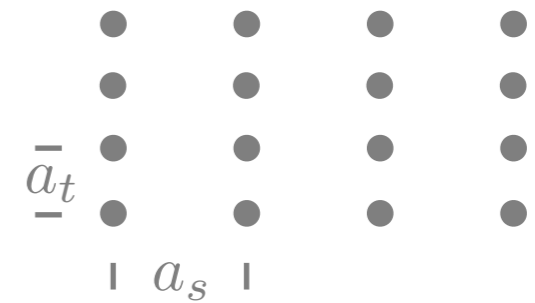
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

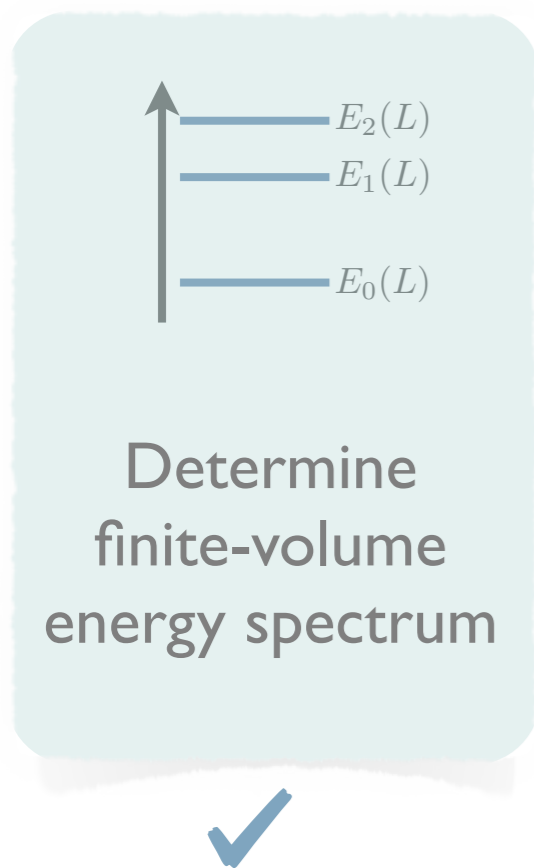
lattice details

$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6) \quad L_s/a_s = 20, 24$$

$$m_\pi \approx 400\text{MeV} \quad a_s \approx 0.12\text{fm}$$

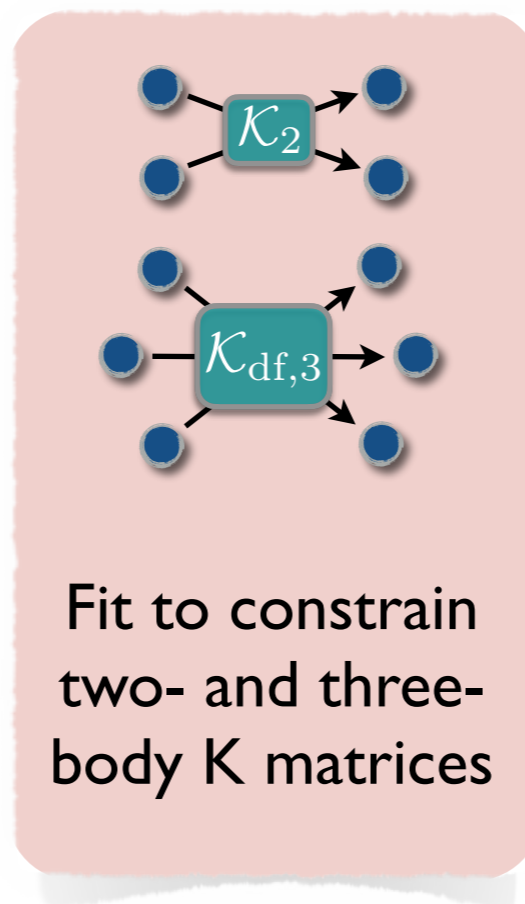


□ Workflow outline



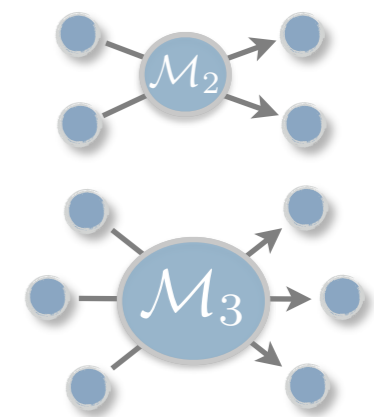
finite volume

↔

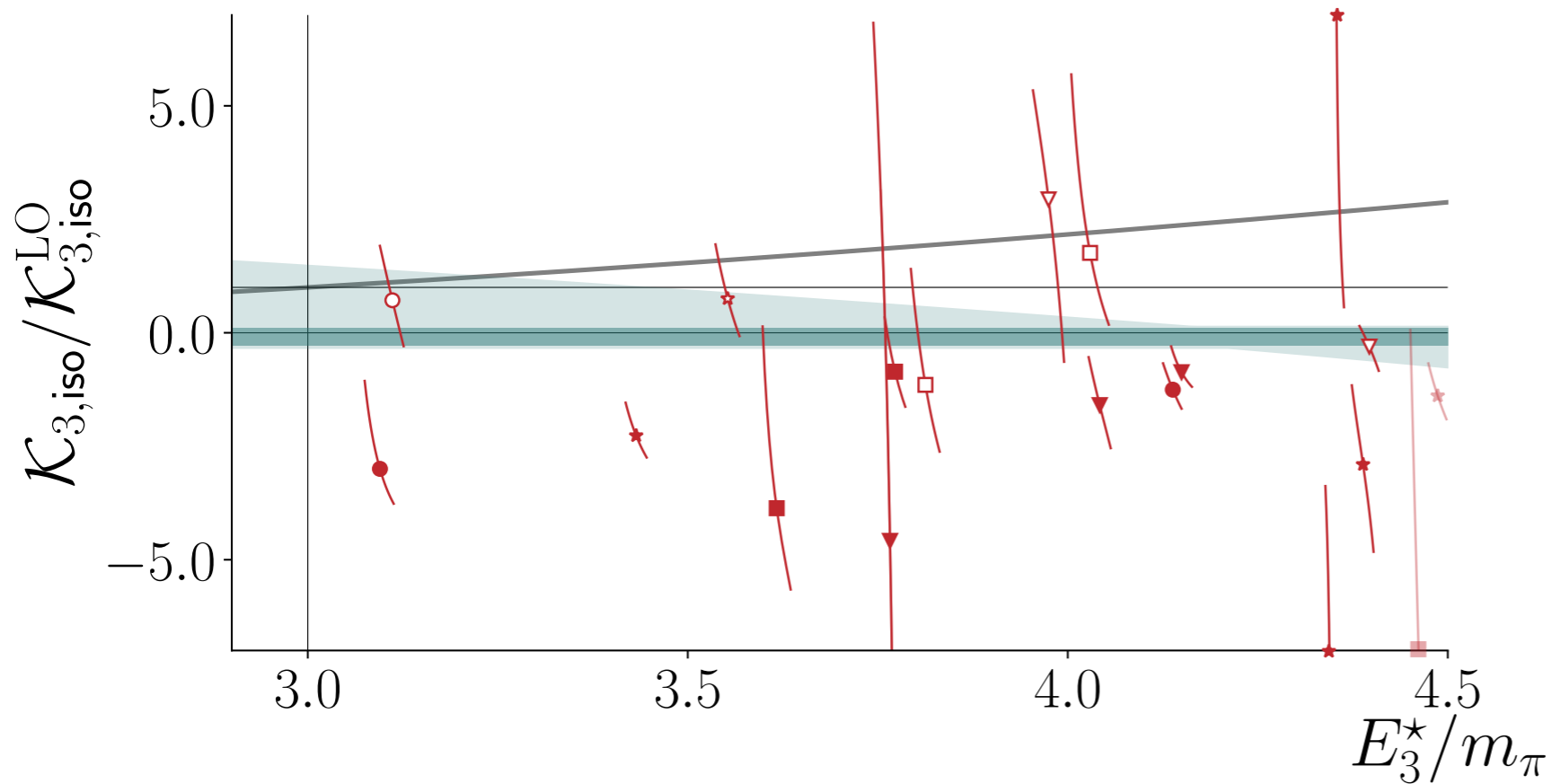
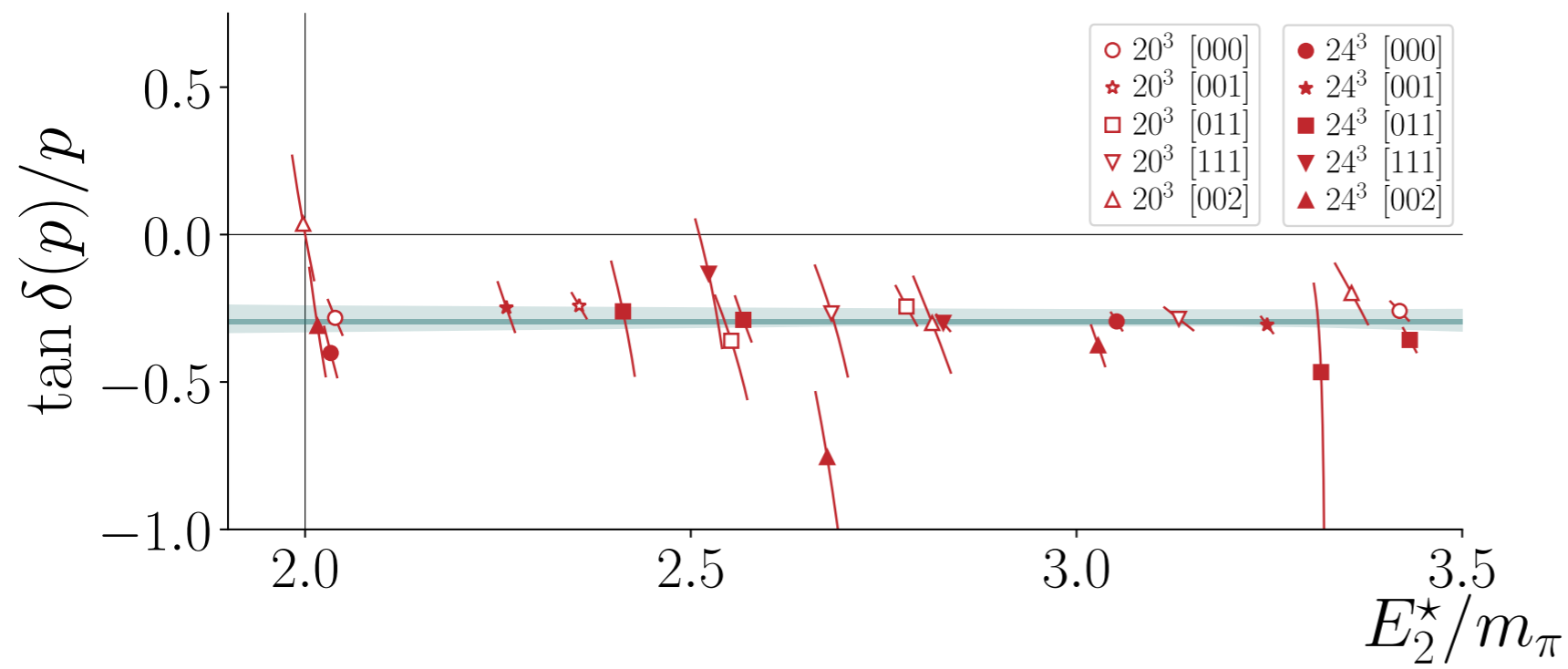


unitarity

↔



K matrix fits



Finite-volume formalism
relates energies to K matrices

One-to-one for $K_{\text{df},3}$
depending only on $E_{\text{cm}} = E^*$

Fit both two and three-body
K to various polynomials

Cut on the CM
energy in the fits

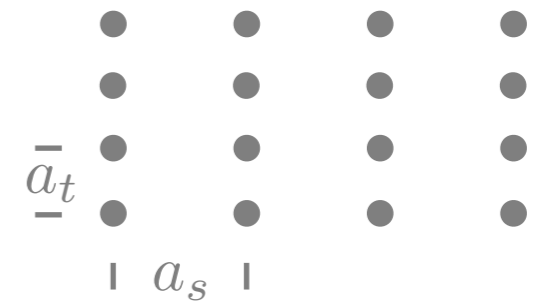
$K_{\text{df},3}$ is scheme
dependent (removed
upon converting to \mathcal{M}_3)

$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

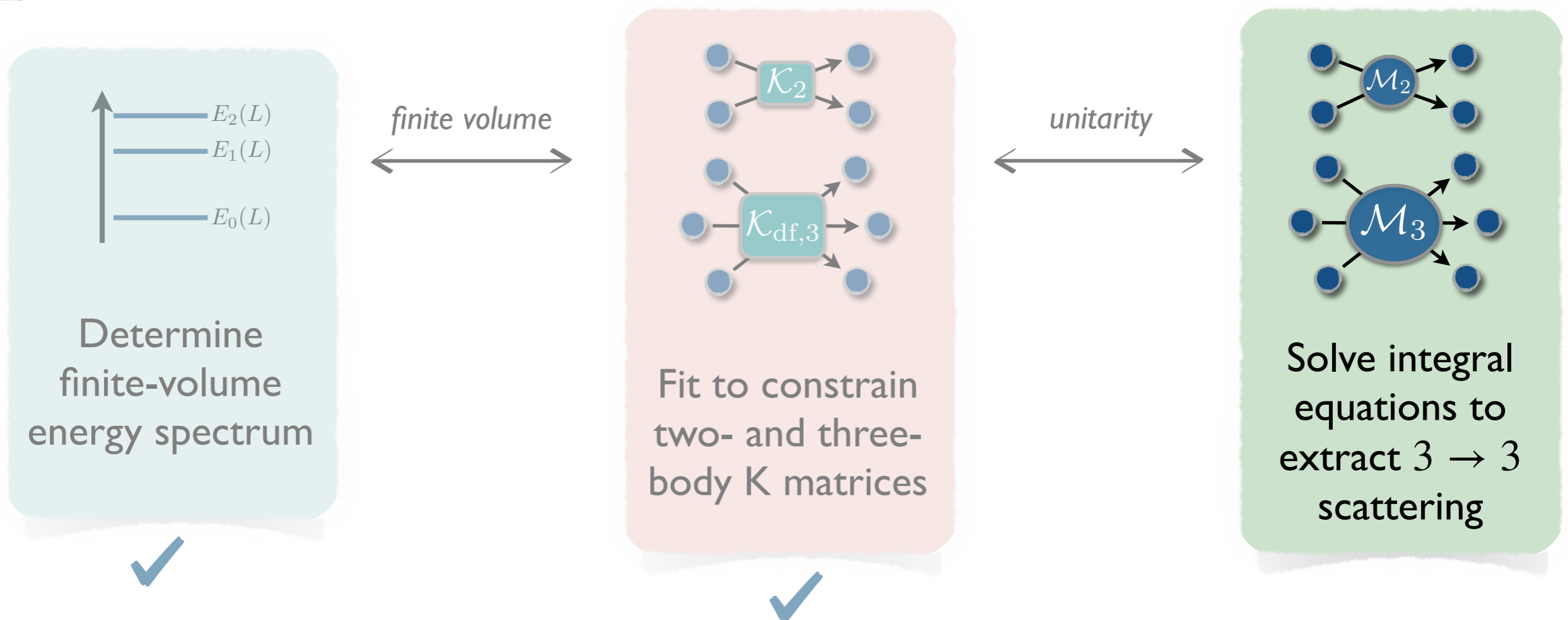
lattice details

$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6) \quad L_s/a_s = 20, 24$$

$$m_\pi \approx 400\text{MeV} \quad a_s \approx 0.12\text{fm}$$

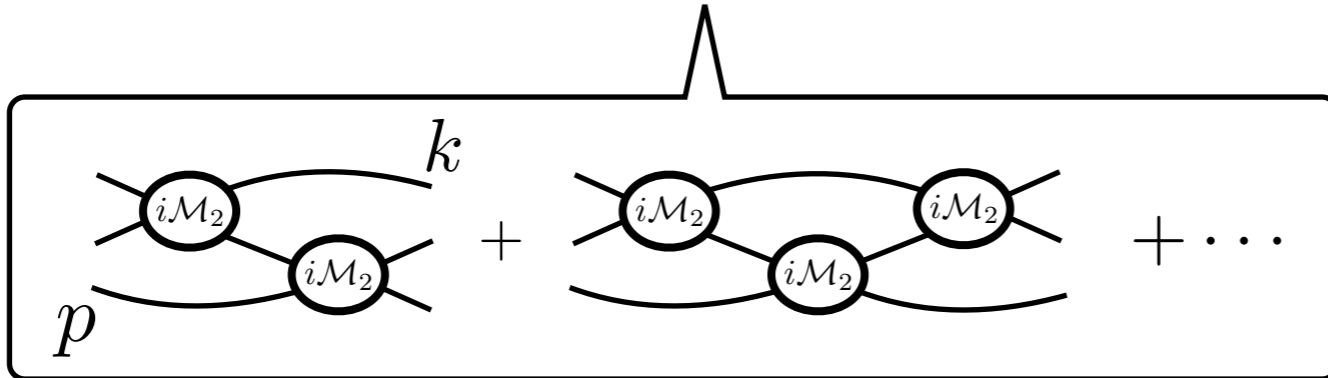


□ Workflow outline



Integral equation

$$\mathcal{M}_3^{\text{un}}(E_3^*, \mathbf{p}, \mathbf{k}) = \mathcal{D}^{\text{un}}(E_3^*, \mathbf{p}, \mathbf{k}) + \mathcal{E}^{\text{un}}(E_3^*, \mathbf{p}) \mathcal{T}(E_3^*) \mathcal{E}^{\text{un}}(E_3^*, \mathbf{k})$$

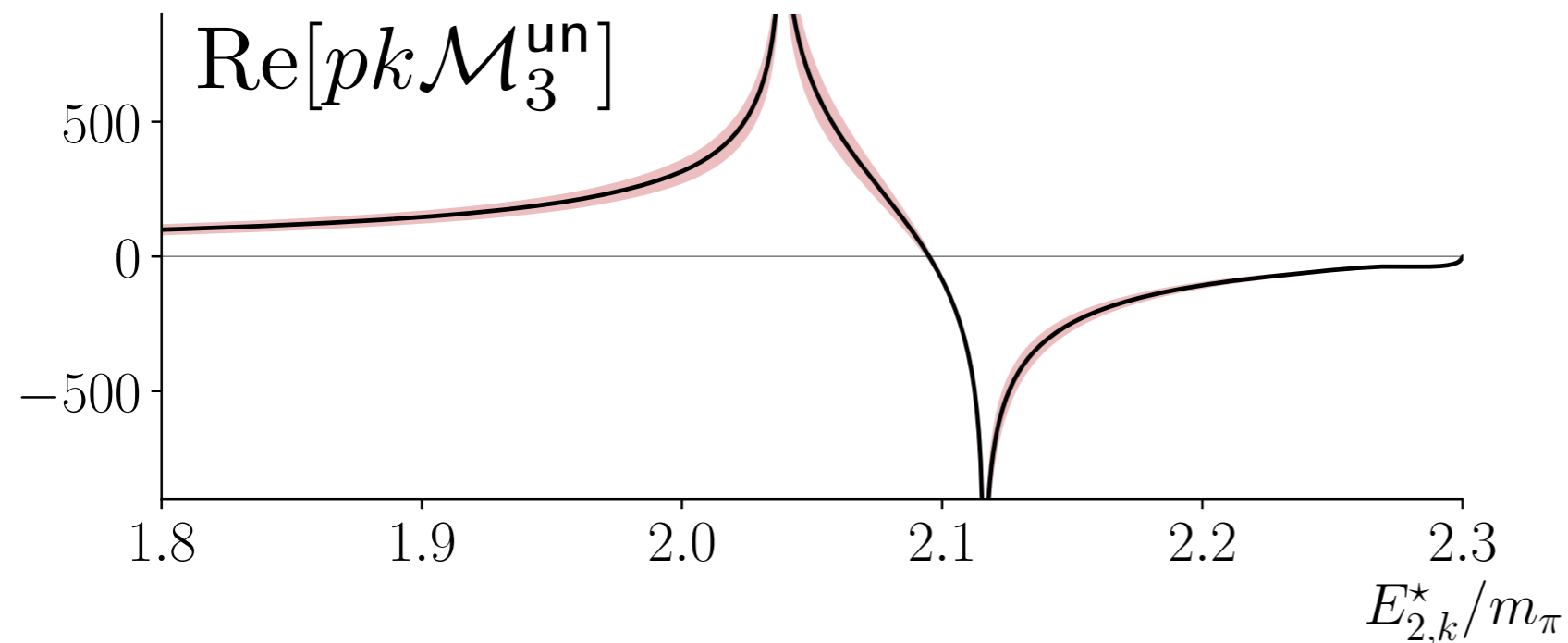


Vanishes for $K_{\text{df},3} = 0$

$$\mathbf{D}(N, \epsilon) = -\mathcal{M} \cdot \mathbf{G}(\epsilon) \cdot \mathcal{M} - \mathcal{M} \cdot \mathbf{G}(\epsilon) \cdot \mathbf{P} \cdot \mathbf{D}(N, \epsilon)$$

$$\mathcal{D}^{\text{un}}(E_3^*, \mathbf{p}, \mathbf{k}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbf{D}_{pk}(N, \epsilon)$$

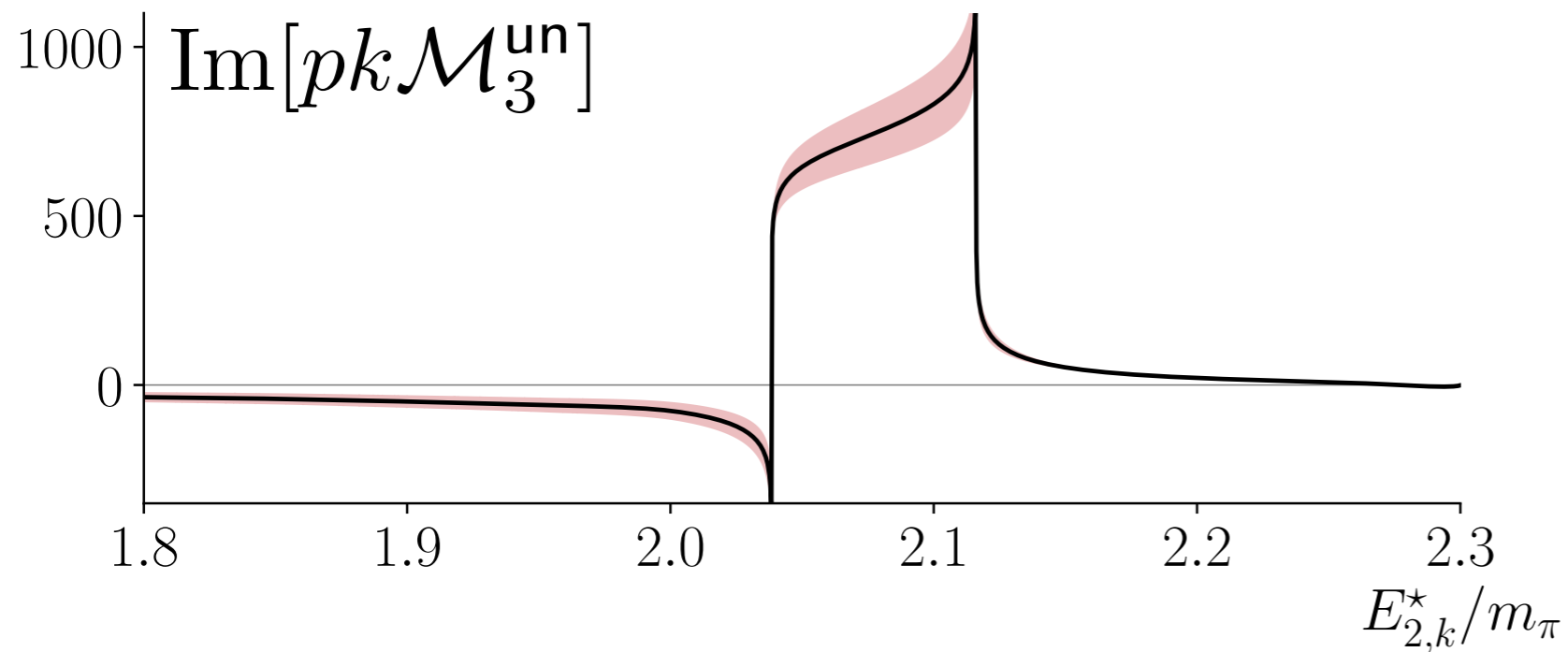
Integral equation



Total angular momentum = 0

Two-particle sub-system
angular momentum = 0

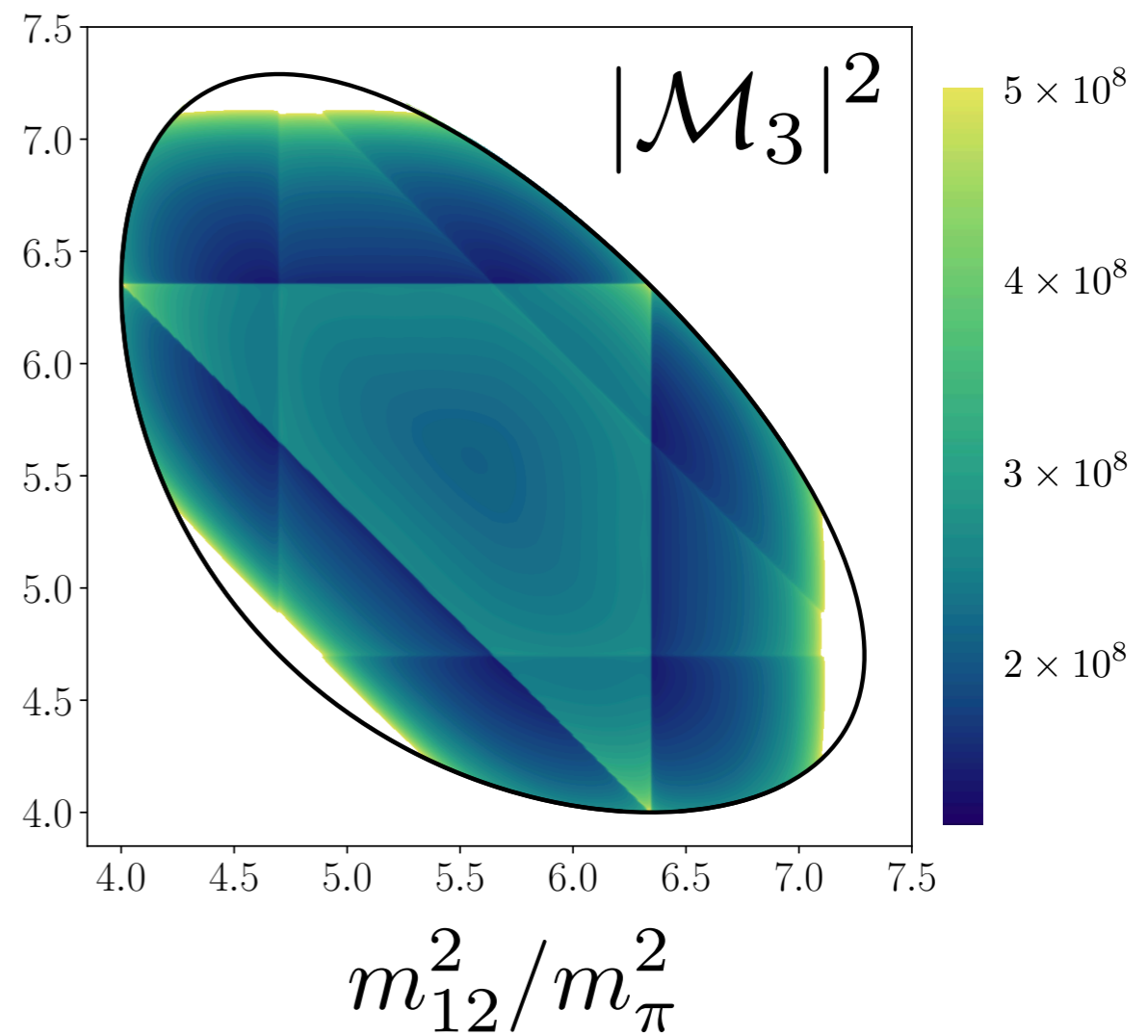
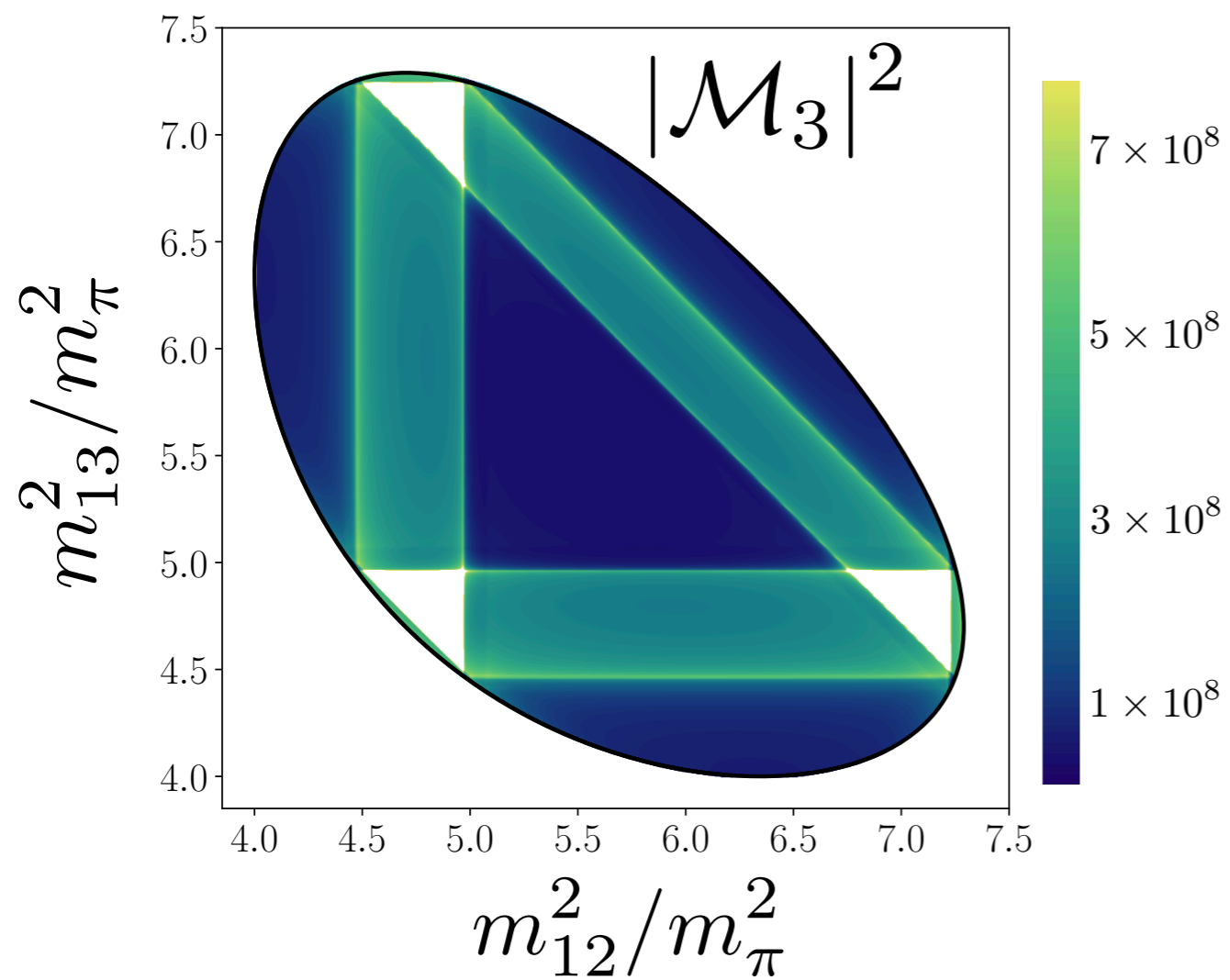
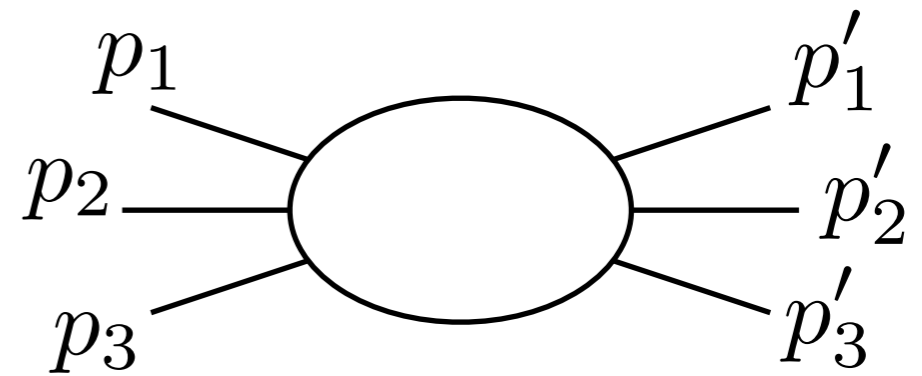
Plot at fixed E_3^* and p



Both two- and three-body
uncertainties estimated

Still need to symmetrize

$$\mathcal{M}_3 = \sum_{i,j \in \{1,2,3\}} \mathcal{M}_3^{\text{un}}(p'_i, p_j)$$

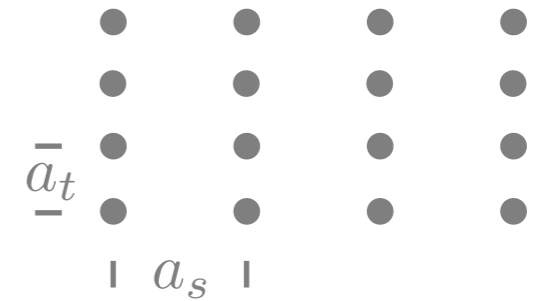


$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

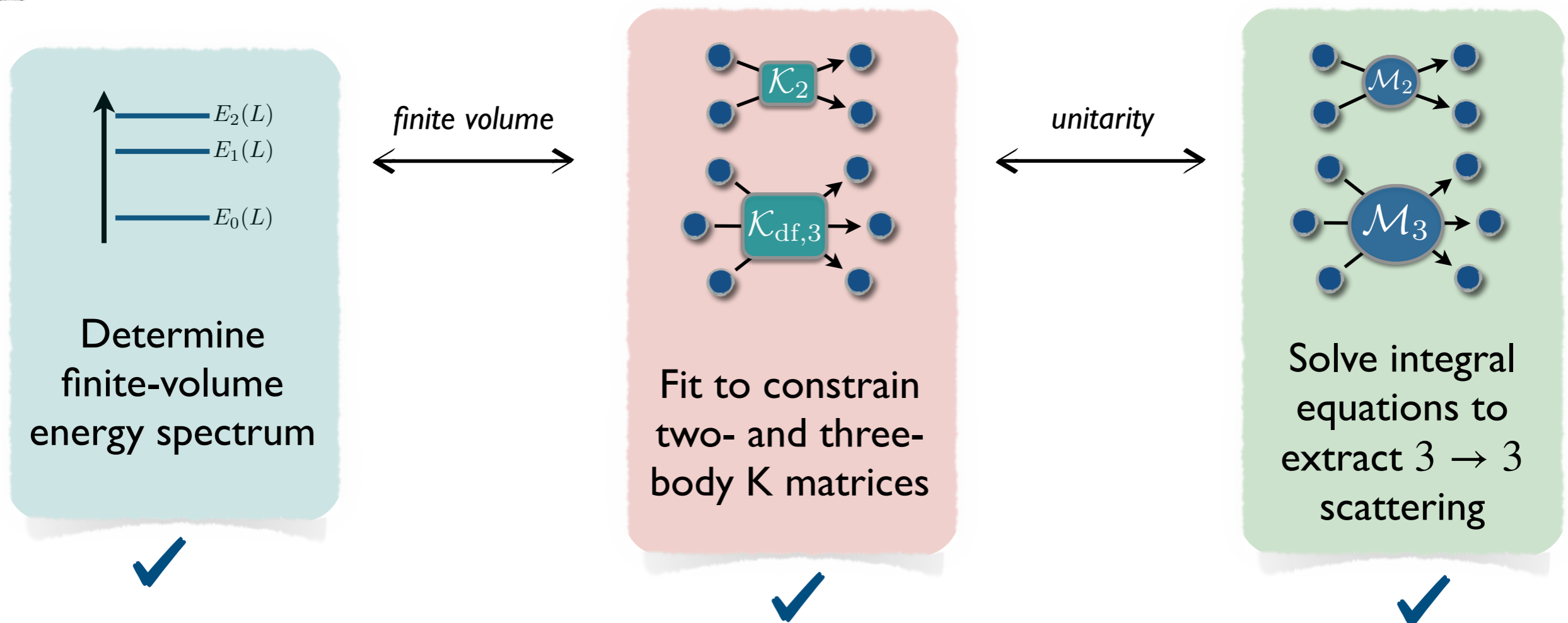
lattice details

$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6) \quad L_s/a_s = 20, 24$$

$$m_\pi \approx 400\text{MeV} \quad a_s \approx 0.12\text{fm}$$

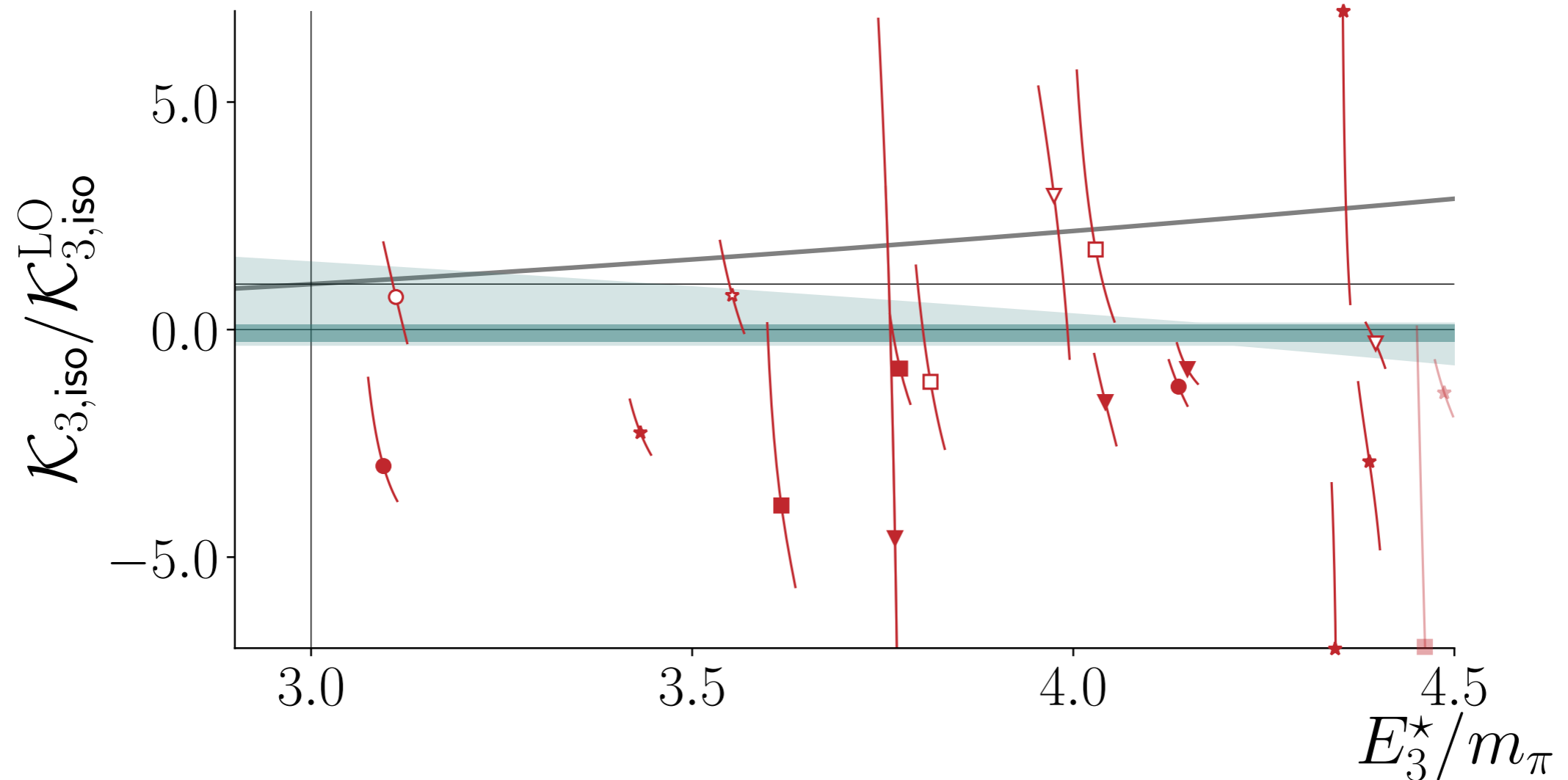


□ Workflow outline



Problems and difficulties (1)

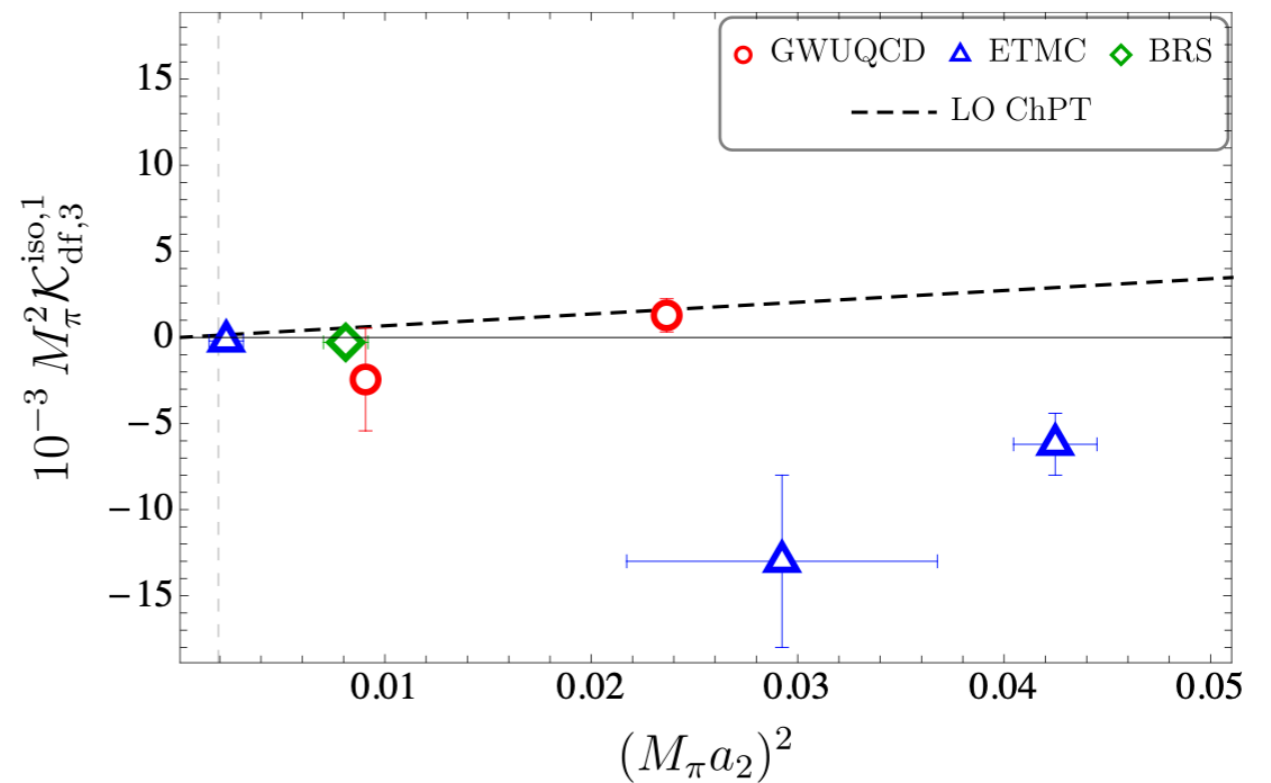
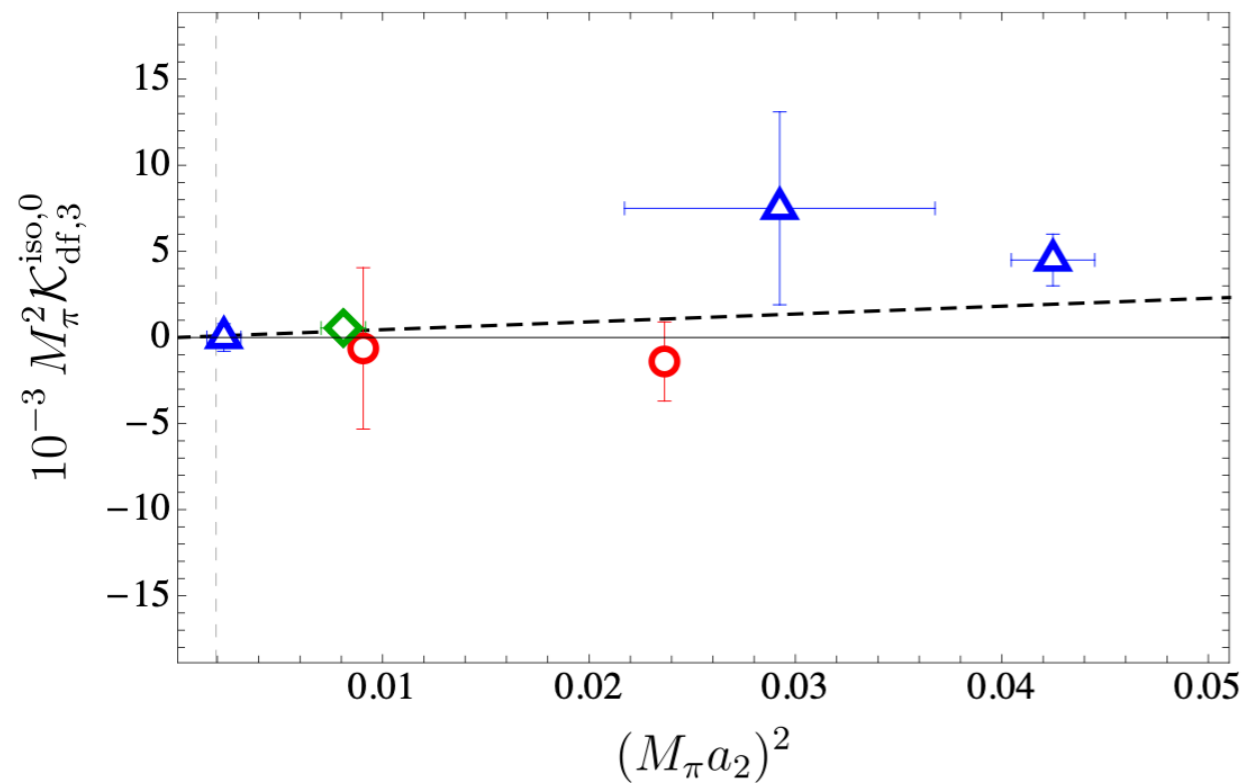
- For weakly interacting systems signal for $\mathcal{K}_{\text{df},3}$ is $1/(m_\pi L)^6$ suppressed



Problems and difficulties (2)

- $\mathcal{K}_{df,3}$ is scheme dependent, comparison to other schemes requires some effort

This result is based in a leading-order conversion, can be made more exact

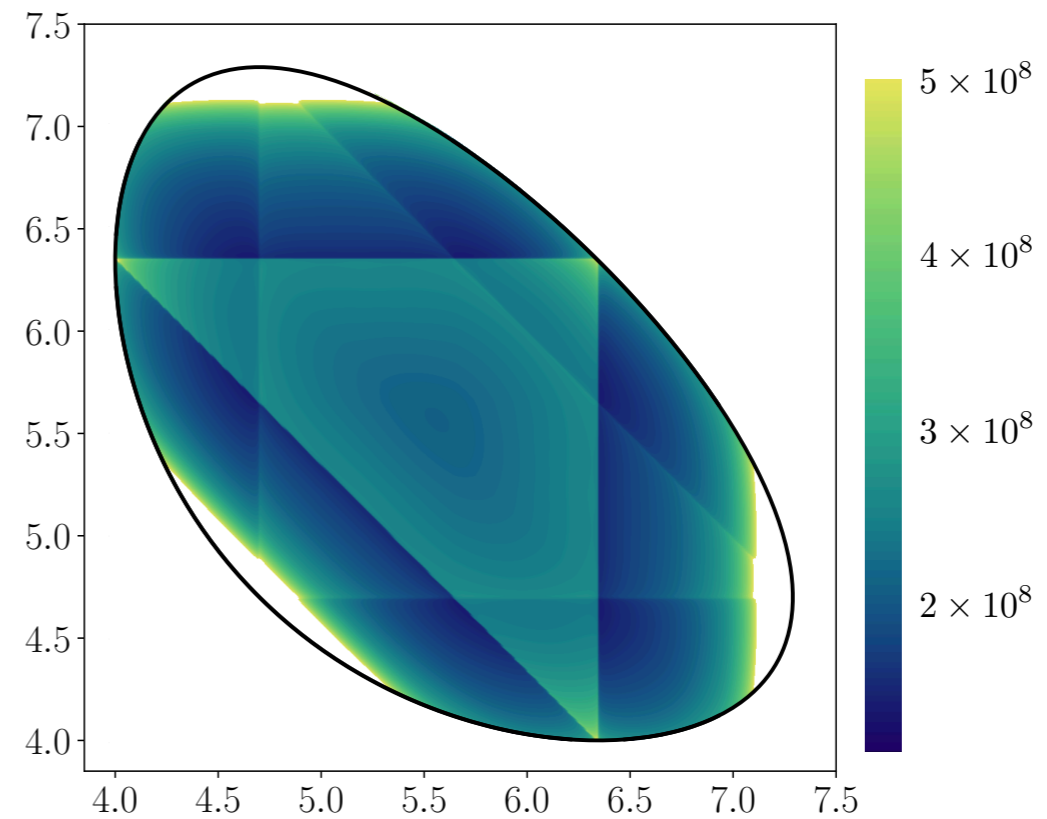
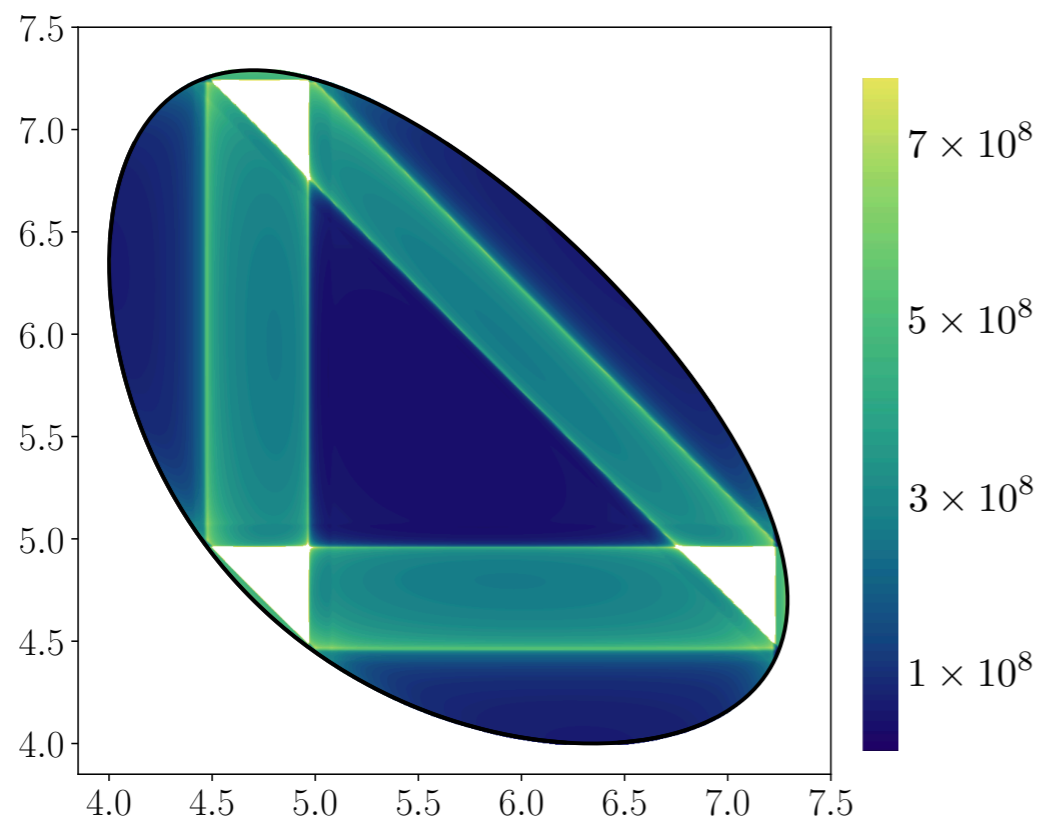


- Other issues for comparing calculations?

Mai, Döring, Rusetsky, 2103.00577, (2021)

Problems and difficulties (3)

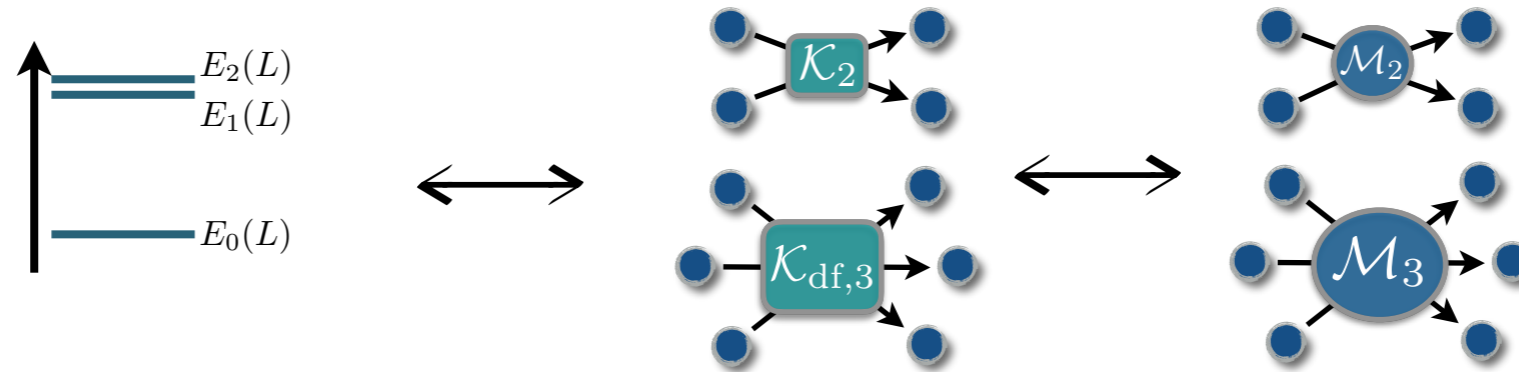
- ❑ More work is needed to build the bridge to the amplitudes community
- ❑ Analytic continuation of \mathcal{M}_3 to the complex plane requires better understanding



- ❑ Moving to more than three particles?

Conclusions

- The finite-volume = *a useful tool*



- First lattice QCD applications in 2019... *much more to come*

- Challenges and progress

formal analysis was technical → *ground work is now set*

3-body amplitude is highly singular → *intermediate K matrix overcomes this*

many-body demands high precision excited states → *advanced algorithms make this possible*

- Many groups are now involved... this is great! *Thanks for listening!*