Stability Criteria in $f(R)$ Gravity from Thermodynamics Analogy

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Synopsis

In this work we analyze the stability criteria in $f(R)$ theories of gravity in the **metric formalism** under the approach of a thermodynamics analogy.

We **starting in the Einstein frame** using ϕ^2 and double well inflationary potentials, and obtain a parametric form of $f(R)$ in the corresponding Jordan frame. Such approach yields plenty of new pieces of information, namely a self-terminating inflationary solution with a linear Lagrangian, and a robust criterion for stability of such theories.

Switching frames: **From Jordan to Einstein Frame**

Jordan $$S_{met} = \frac{1}{2k}\int_V d^4x\, \mathbf{f(R)}\,\sqrt{-g},$$

We can rewrite the function $f(R)$ using a Legendre transformation

$$\hat{L}(g,p) \equiv \sqrt{-g}\,(p\,R(g) - H(p))\,. \tag{1}$$

Now, replacing p by the new auxiliary field $\phi(p)$ defined by $p = e^{\beta\phi}$ and using a conformal transformation

Einstein $$\hat{L} = \sqrt{-\hat{g}}\left(\hat{R} - \hat{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2V(\phi)\right), \tag{2}$$

$$\therefore \quad V(\phi) \equiv \frac{1}{2p^2}\Big\{p(\phi)R[p(\phi)] - f[R(p(\phi))]\Big\}, \tag{3}$$

The Inverse Problem: **From Einstein to Jordan frame**

Parametric set for the $f(R)$ function that depends completely on the potential $V_E(\phi)$.[1]

$$f(\phi) = \mathrm{e}^{2\beta\,\phi}\left[2V_E(\phi) + 2\beta^{-1}\frac{dV_E(\phi)}{d\phi}\right]$$

$$R(\phi) = \mathrm{e}^{\beta\,\phi}\left[4V_E(\phi) + 2\beta^{-1}\frac{dV_E(\phi)}{d\phi}\right]$$

[1] Guido Magnano and Leszek M. Sokolowski. Phys. Rev.D50 (1994), pp. 5039–5059. arXiv:gr-qc/9312008 [gr-qc].[1]

Toy model: ϕ^2 Potential

We choose the inflationary quadratic potential to include a shift in the ϕ-vacuum value and a cosmological constant

$$V_{sqm}(\phi) = \frac{1}{2} m_\phi^2 \left(\phi - a\right)^2 + \Lambda, \tag{4}$$

We solved the full equation system of movement for the scalar field in an universe with a Friedmann-Lemaître-Robertson-Walker metric, with initial conditions set from the analytic slow-roll, namely:

$$\phi_{sqm}(0) \approx a - 15.5 M_P, \quad \dot{\phi}_{sqm}(0) = \sqrt{\frac{2}{3}}\, m_\phi \approx 0.81\, m_\phi. \tag{5}$$

Numerical solutions

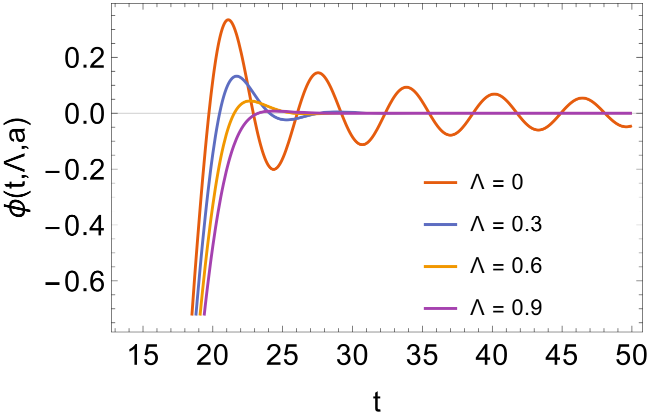
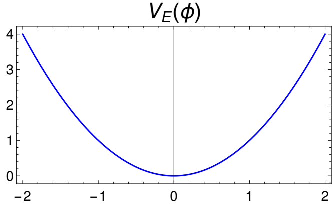
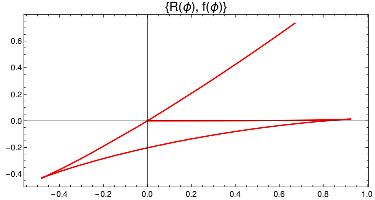


Figure: Numerical solutions potential (4), with $N = 60$ efolds and $a = 0$, using $m_\phi = 1$ and $M_P = 1$. Note that the curves are smoothed as Λ increases. Also, the curves shift up if $a > 0$ or shift down if $a < 0$.

The Inverse Problem: **From Einstein to Jordan frame**





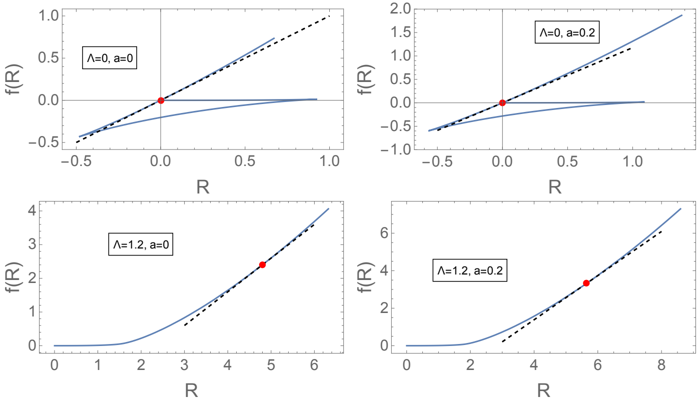
So, substituting the quadratic potential

$$V_E(\phi) = \frac{1}{2} m_\phi^2 (\phi - a)^2 + \Lambda$$

We then obtain the corresponding parametric form of $f(R)$:

$$f(\phi) = e^{2\beta\phi} \left[m_\phi^2 (a - \phi) \left(a - \phi - \frac{2}{\beta} \right) + 2\Lambda \right]$$

$$R(\phi) = 2e^{\beta\phi} \left[m_\phi^2 (a - \phi) \left(a - \phi - \frac{1}{\beta} \right) + 2\Lambda \right]$$



If $\Lambda < \Lambda_c$ (to be defined later on), the curve features a 3-branch structure. In particular, on the final branch, one recovers GR only if $f' = \exp(\beta a) = 1$, i.e, if $a = 0$. Regardless of a, the system does reach a de Sitter state, the Lagrangian in Jordan frame can be written as the linear function $f(R) = \exp(\beta a)R - 2\Lambda_J$.

Thermodynamics Interpretation

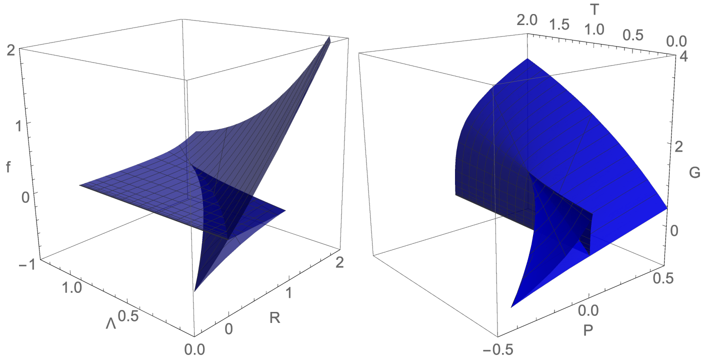


Figure: Plots of $f(R,\Lambda)$, and $G(P,T)$, given by Eqs. (7) and (8) with $\beta = \sqrt{2/3}$ and $a = 0$.

The Stability Criteria from Thermodynamics Analogy

For now, let us associate the Cosmological Constant Λ to an effective temperature $T \equiv \Lambda$. We define a new pair of coordinates $\{-G, P\}$ as a rotation of the original one $\{f, R\}$:

$$\begin{pmatrix} -G \\ P \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f \\ R \end{pmatrix}. \tag{6}$$

In order to define the exact correspondence, $\theta \equiv \pi/2$

$$G = 2e^{\beta\phi}\left(\beta^{-1}(\phi - a) + (\phi - a)^2 + 2T\right) = R \tag{7}$$

$$P = e^{2\beta\phi}\left(2\beta^{-1}(\phi - a) + (\phi - a)^2 + 2T\right) = f \tag{8}$$

The effective volume V is the variable "canonically conjugated" to the effective pressure P, i.e, since

$$dG(P,T) = V \cdot dP - S \cdot dT, \tag{9}$$

one can define an effective volume

$$V \equiv \left.\frac{\partial G}{\partial P}\right|_T = \left.\frac{\partial G/\partial\phi}{\partial P/\partial\phi}\right|_T = \exp(-\beta\phi) \tag{10}$$

which can be inverted and yield

$$\phi = -\frac{1}{\beta}\log(V). \tag{11}$$

Equations (8) and (11) yield the equation of state for our **vdW-like "efective gas"**, i.e, an expression that relates P, V and T:

$$P = \frac{\beta\left(a^2\beta - 2a + 2\beta T\right) + \left(2a\beta - 2 + \log V\right)\log V}{\beta^2 V^2}. \quad (12)$$

Here one obtains $P \propto TV^{-2}$ in the high-temperature limit, instead of the standard ideal-gas behavior $P \propto TV^{-1}$.

We can define the **binodal** and **spinodal** curves, that indicate, respectively, the regions of **metastability** and **instability** of the system — see Fig. 13. The *critical point* $\{P_c, T_c, V_c\}$, defined at the crossing of those curves, indicates the end of the coexistence line.

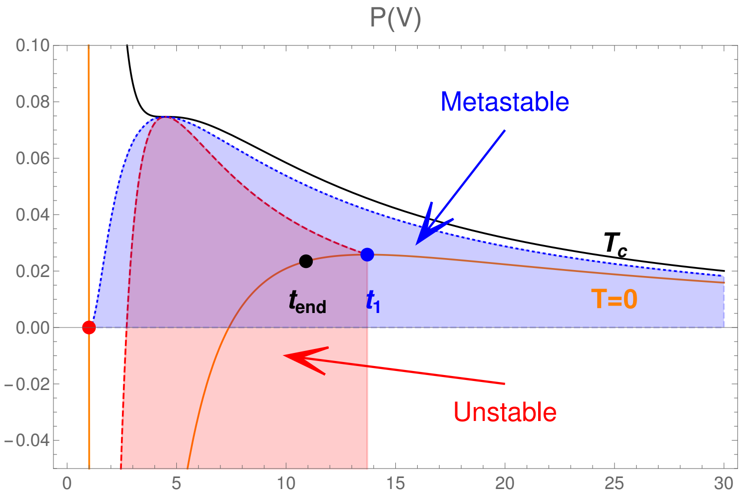
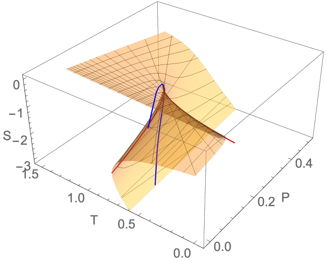
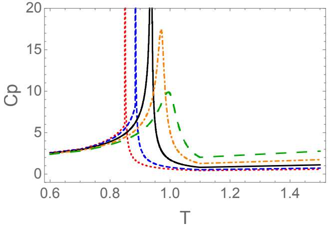


Figure: P vs V, $T = T_c \equiv 15/16$ (solid thick black). The **binodal** curve is plotted in dotted blue. The **spinodal** curve is plotted in dashed red.





Helmholtz energy

$$F(T,V) \equiv G - P\cdot V$$
$$= \frac{1}{V}\left[\left(a + \frac{1}{\beta}\log V\right)^2 + 2T\right]$$

Entropy

$$S(T,V) \equiv -\left.\frac{\partial F}{\partial T}\right|_V = -\frac{2}{V}$$

Specific Heat

$$C_V \equiv T\cdot \partial S/\partial T|_V = 0\,\forall T$$

Internal energy

$$U \equiv G - P\cdot V + T\cdot S$$
$$= \frac{1}{V}\left(a + \frac{1}{\beta}\log V\right)^2$$

vdW fluid

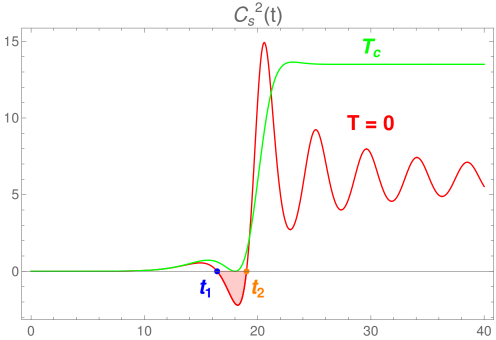


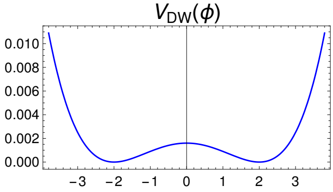
Figure: **The sound speed squared:** defined as $c^2_{\rm vdW} \equiv \dot{P}/\dot{\rho} = -(V^2/\kappa)\dot{P}/\dot{V}$ (where we define $\kappa > 0$ by $\rho =: \kappa/V$). We can see that $c^2_{\rm vdW} < 0$ *only* in the second branch, when $f'' < 0$, as expected from the usual *perturbative* argument on stability of $f(R)$ theories. Obviously, for $T > T_c$, the second branch is suppressed and one obtains $c^2_{\rm vdW} > 0\, \forall t$.

With an **imaginary sound speed**, **fluctuations grow exponentially fast**, but, during the spinodal decomposition process, only a given range of wavelength do so [2]. This is similar to a feature that has already been proposed in the preheating scenario [3].

Double Well Potential

Now, using a double well potential

$$V_{DW}(\phi) = m^4\,(\phi^2 - a^2)^2 + \Lambda$$



We obtain the parametric form of $f(R)$:

$$f(\phi) = 2e^{\beta\phi}\left[\frac{\phi\,(\phi^2 - a^2)}{2500\beta} + 2\left(\frac{(\phi^2 - a^2)^2}{10000} + \Lambda\right)\right]$$

$$R(\phi) = 2e^{2\beta\phi}\left[\frac{\phi\,(\phi^2 - a^2)}{2500\beta} + 2\left(\frac{(\phi^2 - a^2)^2}{10000} + \Lambda\right)\right]$$

$$P = \frac{2}{V^2}\left[\left(a^2 - \frac{\log^2(V)}{\beta^2}\right)^2 + \frac{4a^2\beta^2\log(V) - 4\log^3(V)}{\beta^4} + T\right].$$

Double Well Potential

We identifiying two cases:

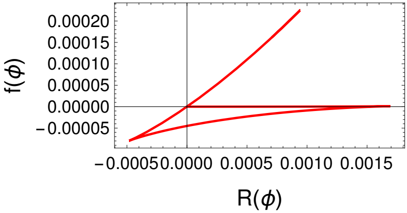


Figure: For $a < a_c \approx -0.81$

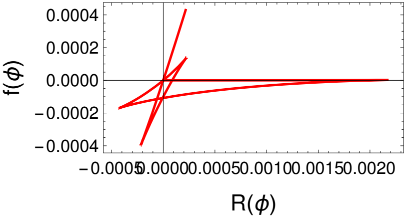
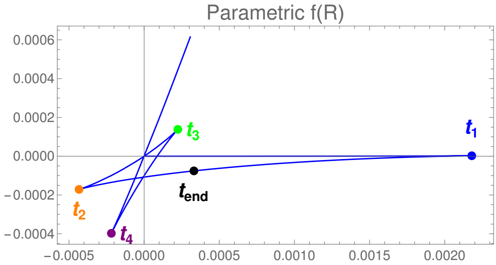
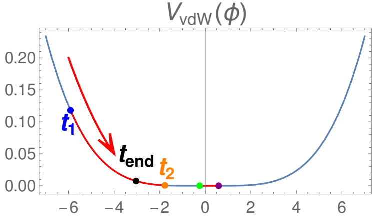
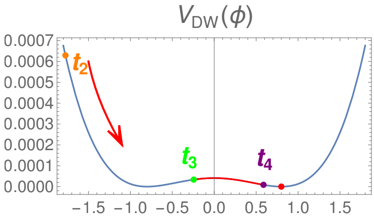


Figure: For $a_c < a < 0$

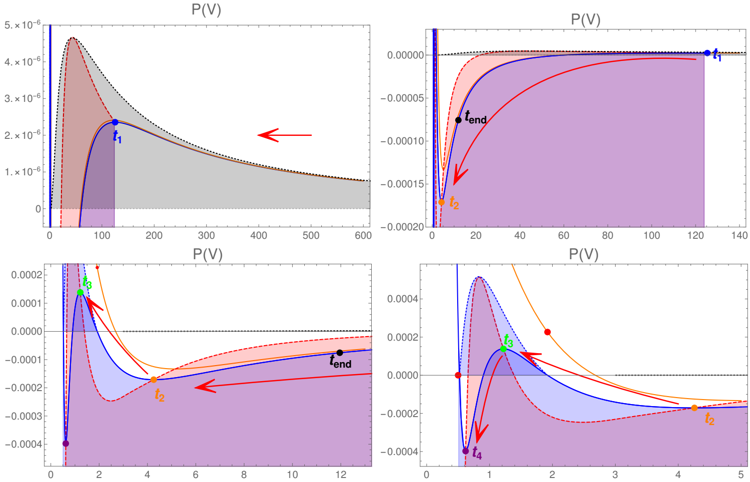
Double Well Potential: Jordan vs Einstein Frame





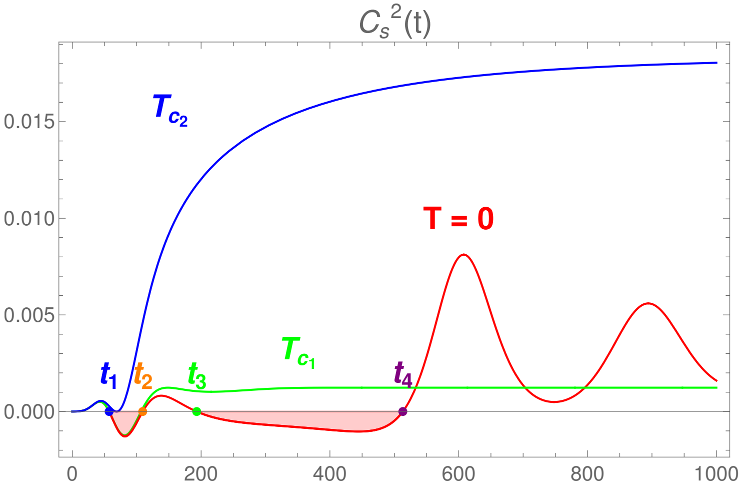


Double Well Potential: Binodal and Spinodal curves

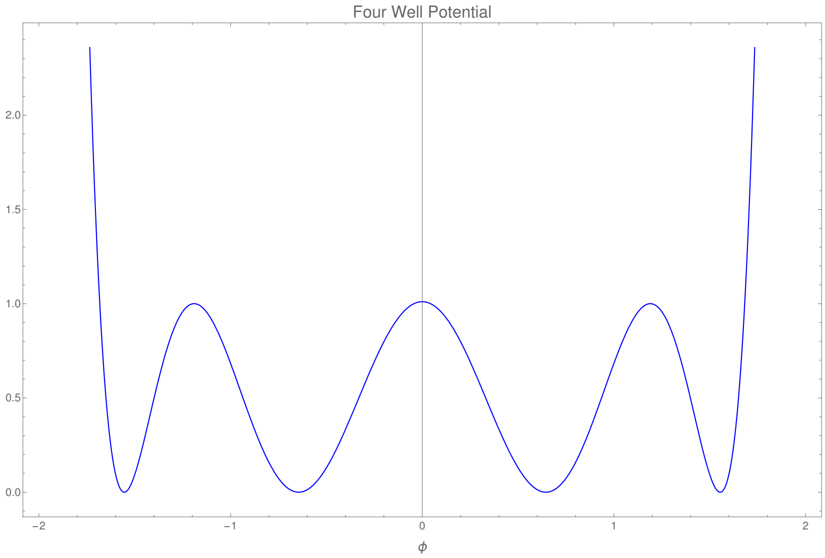


Two binodal curves (in Black-dashed) and a Spinodal curve (in red dashed).

Double Well Potential: Sound Velocity Square



Four Well Potential: $\left(\left(\phi^2 - b^2\right)^2 - a^2\right)^2 + \Lambda$



Conclusions

We provide this beautiful analogy that presents a great wealth of details, tools and very interesting interpretations.

Advantages

- Perhaps for every potential V_E in Einstein frame there is an analogous thermodynamic system in the Jordan frame.
- There is a huge family of scalar potentials to be explored.
- We can determine regions of metastability that in the Jordan frame are not easy to obtain.
- We can get a description of the phase transitions of the gravitational system by their thermodynamic counterparts.
- Instabilities are not bad (as long as it is temporary!).

Disadvantages

- We still do not know the equivalent gravitational meaning of all thermodynamic quantities.

Outlooks

- We are currently investigating the spinodal decomposition process, for the toy model, in order to obtain the corresponding range of wavelength that is exponentially amplified.
- Growth of curvature perturbations of this models.
- Couple Curvature-Matter

[1] Guido Magnano and Leszek M. Sokolowski. "On physical equivalence between nonlinear gravity theories and a general relativistic selfgravitating scalar field". In: *Phys. Rev.* D50 (1994), pp. 5039–5059. DOI: 10.1103/PhysRevD.50.5039. arXiv: gr-qc/9312008 [gr-qc].

[2] Philippe Chomaz, Maria Colonna, and Jørgen Randrup. "Nuclear spinodal fragmentation". In: *Physics Reports* 389.5 (2004), pp. 263–440. ISSN: 0370-1573. DOI: https://doi.org/10.1016/j.physrep.2003.09.006. URL: http://www.sciencedirect.com/science/article/pii/S037015730300393

[3] Rouzbeh Allahverdi et al. "Reheating in Inflationary Cosmology: Theory and Applications". In: *Annual Review of Nuclear and Particle Science* 60.1 (2010), pp. 27–51.