EFT of Black Hole Perturbations with Time-like Scalar Profile

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Outline

- Introduction
- Formulation of the EFT
- Static Spherically symmetric case
- Application: odd-parity perturbation
- Conclusions/Future Directions

Introduction: Covariant approach

- Scalar-Tensor Theories: 2 tensor + 1 scalar dofs

$$X = g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$$

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$$\mathscr{L} = R - \frac{1}{2}X - V(\Phi)$$ Quintessence

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$$\mathscr{L} = f(\Phi)R - \frac{1}{2}X - V(\Phi) \qquad \text{Brans-Dicke}$$

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$$\mathscr{L} = R - P(X) - V(\Phi)$$ k-essence

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Introduction: Covariant approach

- The most general scalar-tensor theories without the Ostrogradski ghost
  : (Beyond) Horndeski Theories

$$L_2 = G_2(\Phi, X) \qquad L_3 = G_3(\Phi, X) \Box \Phi$$

$$L_4 = G_4(\Phi, X)R - 2G_{4X}(\Phi, X)[(\Box\Phi)^2 - \Phi_{\mu\nu}\Phi^{\mu\nu}] - F_4(X, \Phi)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\Phi_\mu\Phi_{\mu'}\Phi_{\nu\nu'}\Phi_{\rho\rho'}$$

$$L_5 = G_5(\Phi, X)G_{\mu\nu}\Phi^{\mu\nu} + \frac{1}{3}G_{5X}(\Phi, X)[(\Box\Phi)^3 - 3(\Box\Phi)\Phi_{\mu\nu}\Phi^{\mu\nu} + 2\Phi_{\mu\nu}\Phi^{\sigma\mu}\Phi^{\nu}{}_{\sigma}]$$

$$-F_5(\Phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\Phi_\mu\Phi_{\mu'}\Phi_{\nu\nu'}\Phi_{\rho\rho'}\Phi_{\sigma\sigma'}$$

Horndeski 74, Deffayet et al. 11, Zumalacárregui and García-Bellido 14, Gleyzes et al. 14

$$\Phi_\mu \equiv \nabla_\mu \Phi$$

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$$\Phi_{\mu} \equiv \nabla_{\mu}\Phi$$

- DHOST Theories: Lagrangian contains second-order derivatives of a scalar field

Langlois and Noui 15, Langlois 17

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- Efficient way to study a perturbation around fixed background

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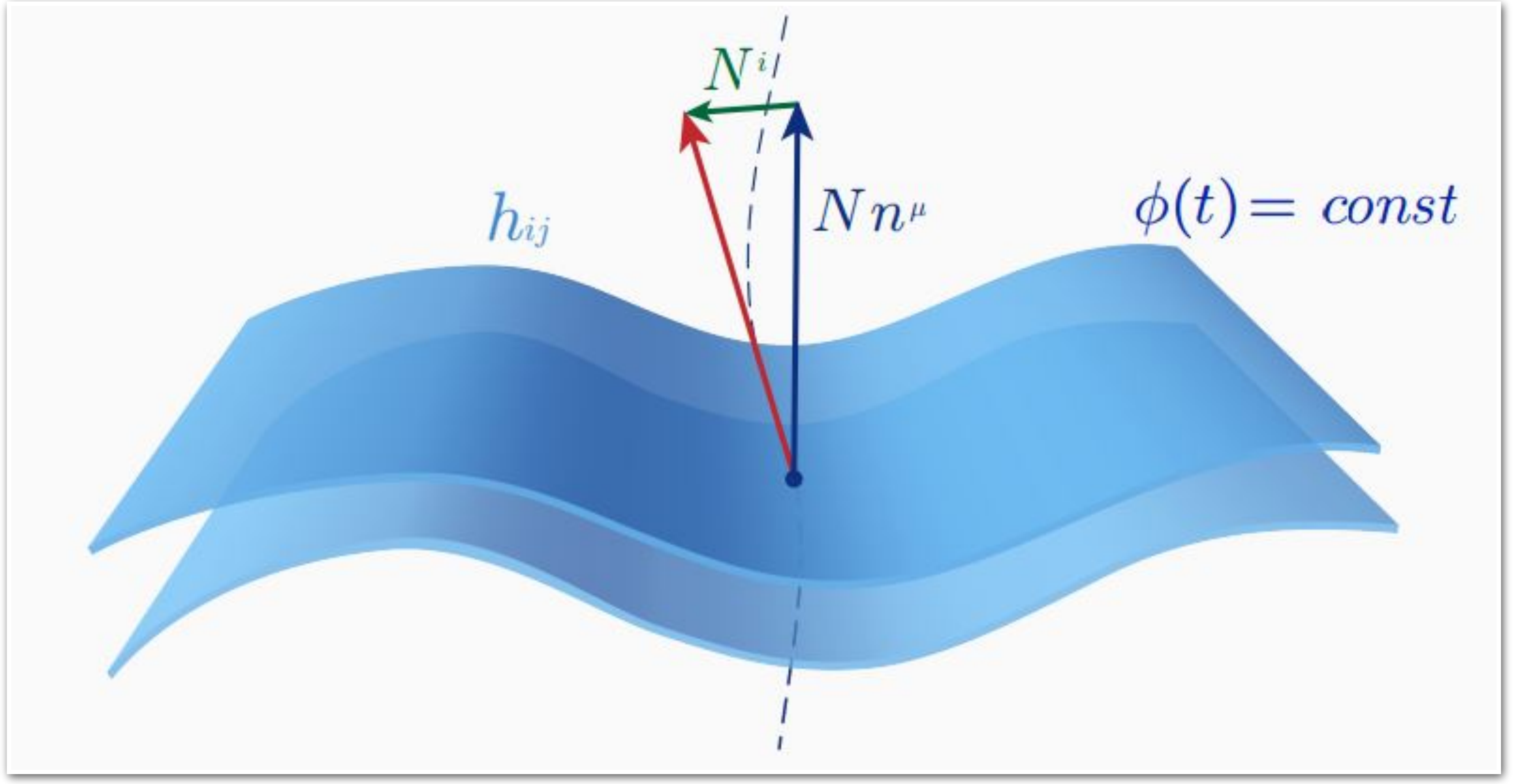

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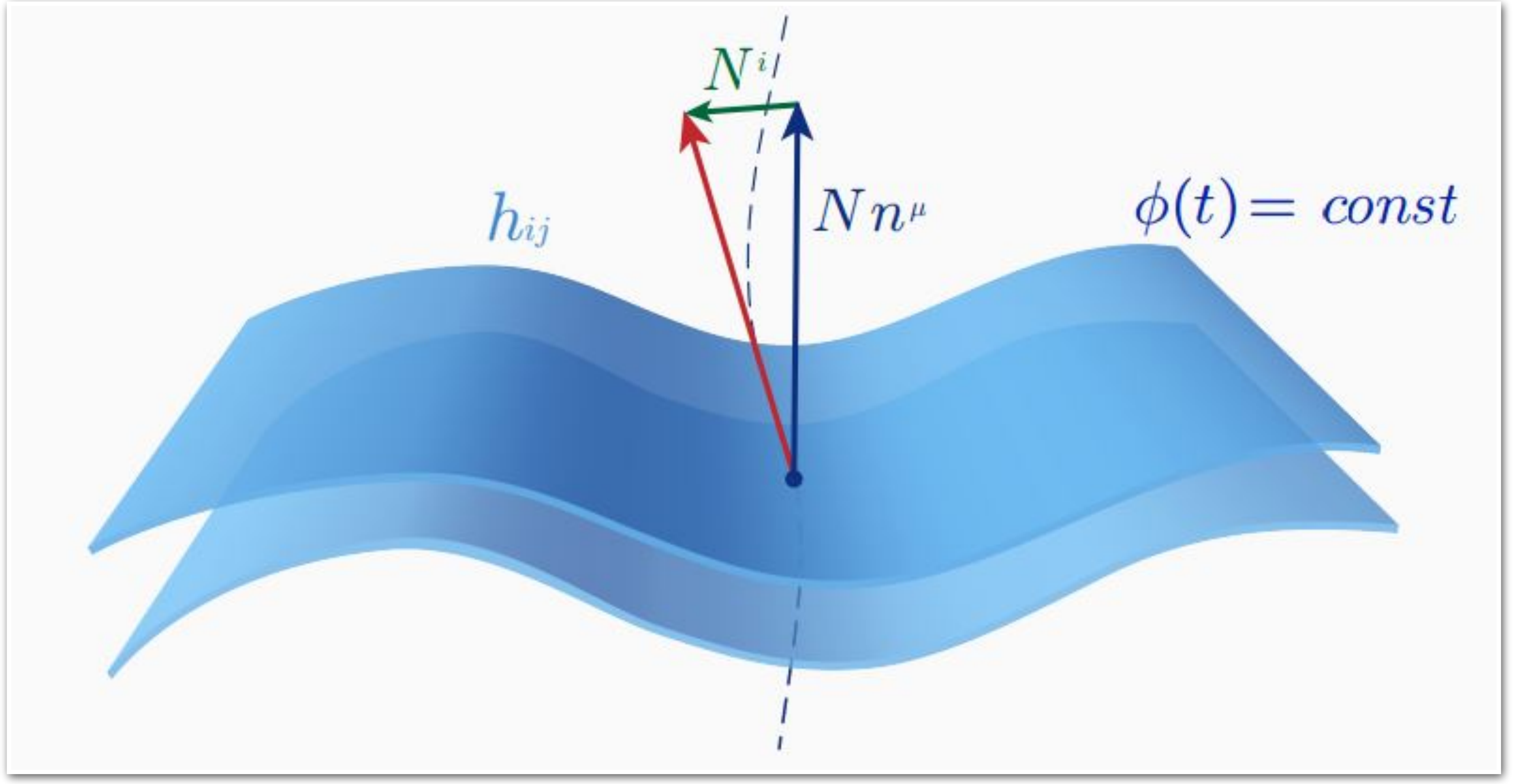

$$S = \int d^4x \sqrt{-g}\ L[t; N, K^i_j, {}^{(3)}R, \ldots]$$

The action is invariant under 3d diff.

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$

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No GW decay + No instabilities — Creminelli et al. 18, Creminelli, Tambalo, Vernizzi and VY 19 + 20

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Is it possible to formulate an EFT of perturbations on an inhomogeneous background with timelike scalar profile?

e.g. Schwarzschild—dS, Schwarzschild—FRW

A. Single EFT that works on both cosmological and black hole regimes: DE + BH
B. Accommodate the scordatura mechanism avoiding strong coupling problem around stealth solutions e.g. DHOST or Horndeski

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- The EFT of perturbations on a black hole background with spacelike scalar profile (Franciolini et al. 18, Hui et al. 2)

Introduction: Motivated example

- Timelike scalar solution on Schwarzschild background   Mukohyama 05

  Stealth solution i.e. non-trivial scalar profile on a metric of GR

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The solution is $\Phi(\tau) = \sqrt{-X_0}\;\tau$ with constant X_0

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$$S = \int d^4x\sqrt{-g}\; P(X) \qquad T^{\Phi}_{\mu\nu} = P(X_0)g_{\mu\nu} \qquad P'(X_0) = 0$$

The solution is $\Phi(\tau) = \sqrt{-X_0}\;\tau$ with constant X_0

$$ds^2 = -d\tau^2 + \frac{r_s}{r}d\rho^2 + r^2 d\Omega^2$$

$$r(\tau,\rho) = r_s^{1/3}\left[\frac{3}{2}(\rho-\tau)\right]^{2/3}$$

Lemaitre coordinates

Introduction: Motivated example

- Timelike scalar solution on Schwarzschild background

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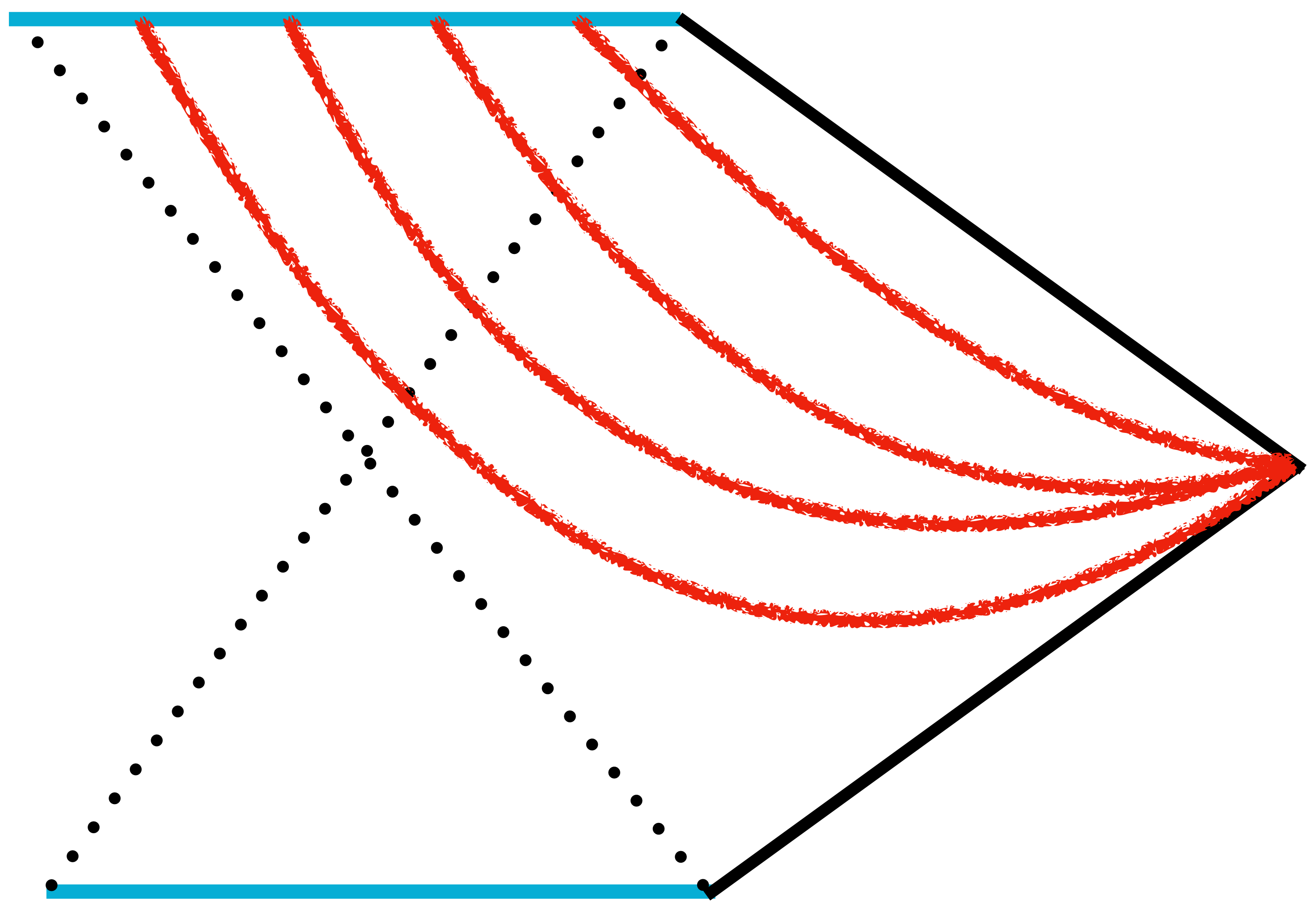

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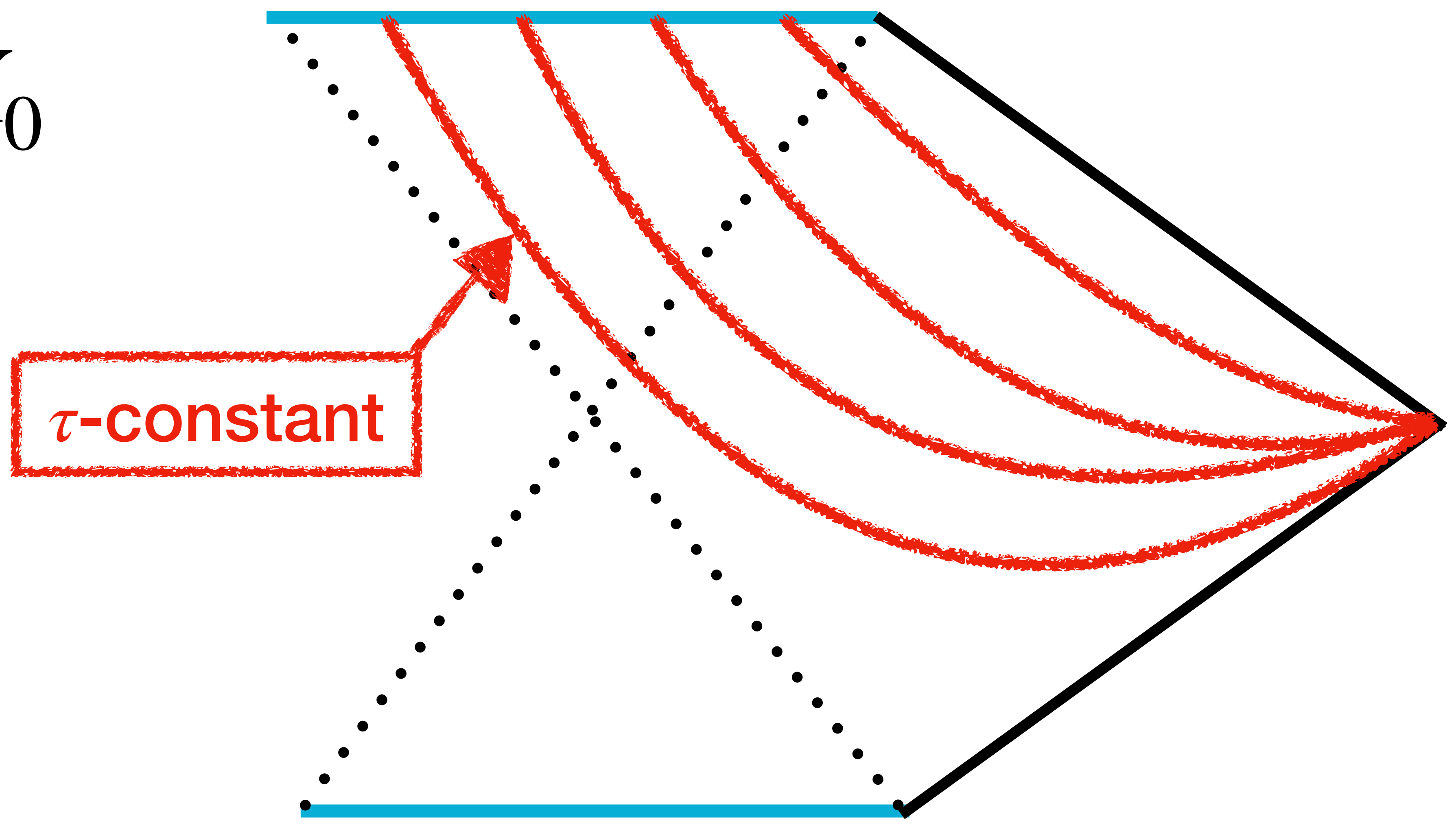

$u^\mu = -\partial^\mu \Phi$: timelike everywhere

Formulation of the EFT

- EFT action in unitary gauge: $\delta\Phi = 0$

$$n_\mu = -\frac{\partial_\mu \Phi}{\sqrt{-(\partial\Phi)^2}} \to -\frac{\dot{\Phi}\delta^\tau_\mu}{\sqrt{-g^{\tau\tau}\dot{\Phi}^2}}$$

$$ds^2 = -N^2 d\tau^2 + h_{ij}(dx^i + N^i d\tau)(dx^j + N^j d\tau)$$

$$g^{\tau\tau} = -1/N^2$$

$$K_{\mu\nu} \equiv h^\sigma_\mu \nabla_\sigma n_\nu$$

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- $\bar{\Phi}(\tau)$ breaks τ-diffeo. spontaneously

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This action is invariant under 3d diffeo and valid for generic background geometries

Formulation of the EFT

- On a generic background metric:

$$\delta g^{\tau\tau} \equiv g^{\tau\tau} - \bar{g}^{\tau\tau}(\tau, x_i) \qquad \delta K^\mu_\nu = K^\mu_\nu - \bar{K}^\mu_\nu(\tau, x_i) \qquad \delta^{(3)}R^\mu_\nu \equiv {}^{(3)}R^\mu_\nu - {}^{(3)}\bar{R}^\mu_\nu(\tau, x_i)$$

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$$S = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2} f(\tau)\, R + \Lambda(\tau) - c(\tau)\, g^{\tau\tau} + \frac{1}{2} m_2^4(\tau)(\delta g^{\tau\tau})^2 + \frac{1}{2} M_2^2(\tau)\delta K^2 + \frac{1}{2}\mu_1^2(\tau)\delta g^{\tau\tau}\delta^{(3)}R + \ldots\right]$$

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They do not transform as a scalar under spatial diffeo.

Formulation of the EFT

- The unitary gauge action (manifestly 3d diff invariant)

$$S = \int d^4x \sqrt{-g}\ F(R_{\mu\nu\alpha\beta}, g^{\tau\tau}, K_{\mu\nu}, \nabla_\nu, \tau)$$

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$$\sigma_{\mu\nu} \equiv K_{\mu\nu} - \frac{1}{3}Kh_{\mu\nu} \qquad r_{\mu\nu} \equiv {}^{(3)}R_{\mu\nu} - \frac{1}{3}{}^{(3)}Rh_{\mu\nu} \qquad \bar{F}_X \equiv \left(\frac{\partial F}{\partial X}\right)_{\mathrm{BG}}$$

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Each term generally depends on x_i

$$+\frac{1}{2}\bar{F}_{g^{\tau\tau}g^{\tau\tau}}(\delta g^{\tau\tau})^2 + \bar{F}_{g^{\tau\tau}K}\delta g^{\tau\tau}\delta K + \bar{F}_{g^{\tau\tau}\sigma^\mu_\nu}\delta\sigma^\mu_\nu\delta g^{\tau\tau} + \frac{1}{2}\bar{F}_{KK}\delta K^2 + \bar{F}_{K\sigma^\mu_\nu}\delta K\delta\sigma^\mu_\nu + \frac{1}{2}\bar{F}_{\sigma^2}\delta\sigma^\mu_\nu\delta\sigma^\nu_\mu + T(\delta^{(3)}R, \delta r) + \dots\Big]$$

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Apply the same to $\bar{F}_{g^{\tau\tau}}$, $\bar{F}_K$, $\bar{F}_{{}^{(3)}\!R}$ etc. $\Rightarrow$ Non-trivial relations among EFT parameters

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- Imposing these relations ensures 3d diff. invariance of the action

Formulation of the EFT

- EFT action up to second order: $y = \{\tau, x_i\}$

$$S = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}f(y)R - \Lambda(y) - c(y)g^{\tau\tau} - \beta(y)K - \alpha^\mu_\nu(y)\sigma^\nu_\mu - \gamma^\mu_\nu(y)r^\nu_\mu + \frac{1}{2}m_2^4(y)(\delta g^{\tau\tau})^2\right.$$

$$+\frac{1}{2}M_1^3(y)\delta g^{\tau\tau}\delta K + \frac{1}{2}M_2^2(y)\delta K^2 + \frac{1}{2}M_3^2(y)\delta K^\mu_\nu\delta K^\nu_\mu + \frac{1}{2}M_4(y)\delta K\delta^{(3)}R$$

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$$M_\star^2 f(y) = 2\bar{F}_{{}^{(3)}R} \qquad \Lambda(y) = -\bar{F} + \bar{F}_K\bar{K} + \bar{F}_{{}^{(3)}R}\left({}^{(3)}\bar{R} + \frac{2}{3}\bar{K}^2 - \bar{\sigma}^\mu_\nu\bar{\sigma}^\nu_\mu\right) + \bar{F}_{g^{\tau\tau}}\bar{g}^{\tau\tau} + \bar{F}_{\sigma^\mu_\nu}\bar{\sigma}^\mu_\nu + \bar{F}_{r^\mu_\nu}\bar{r}^\mu_\nu$$

$$c(y) = -\bar{F}_{g^{\tau\tau}} \qquad \beta(y) = -\bar{F}_K - \frac{4}{3}\bar{F}_{{}^{(3)}R}\bar{K} \qquad \alpha^\mu_\nu(y) = -\bar{F}_{\sigma^\nu_\mu} + 2\bar{F}_{{}^{(3)}R}\bar{\sigma}^\mu_\nu \qquad \gamma^\mu_\nu(y) = -\bar{F}_{r^\nu_\mu}$$

$$\vdots$$

Formulation of the EFT

- Consistency relations among EFT parameters:

$$\partial_i\Lambda + \bar{g}^{\tau\tau}\partial_i c - \frac{1}{2}M_\star^2 {}^{(3)}\!\bar{R}\partial_i f + \frac{1}{3}\bar{K}(M_\star^2\bar{K}\partial_i f + 3\partial_i\beta) - \frac{1}{2}\bar{\sigma}^\mu_\nu(M_\star^2\bar{\sigma}^\nu_\mu\partial_i f - 2\partial_i\alpha^\nu_\mu) + \bar{r}^\mu_\nu\partial_i\gamma^\nu_\mu \simeq 0$$

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Formulation of the EFT

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$$+\frac{1}{2}M_1^3(y)\delta g^{\tau\tau}\delta K + \frac{1}{2}\mu_1^2(y)\delta g^{\tau\tau}\delta^{(3)}\!R - \frac{1}{2}m_4^2(y)(\delta K^2 - \delta K^\mu_\nu\delta K^\nu_\mu) + \tilde{m}_4(\delta K\delta^{(3)}\!R/2 - \delta K^\mu_\nu\delta^{(3)}\!R^\nu_\mu)$$

$$\left.+\frac{1}{2}\lambda_1(y)^\nu_\mu\delta g^{\tau\tau}\delta K^\mu_\nu + \tilde{\lambda}(y)^\nu_\mu(\delta g^{\tau\tau}\delta^{(3)}\!R^\mu_\nu/2 - \delta K\delta K^\mu_\nu/3)\right]$$

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Dictionary w/ quadratic HOST theories
see S. Mukohyama, K. Takahashi and VY 22

Static Spherically Symmetric case

- Background metric: $ds^2 = -d\tau^2 + [1 - A(r)]d\rho^2 + r^2 d\Omega^2$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

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$$S_{\text{tadpole}} = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}R - \Lambda(y) - c(y)g^{\tau\tau} - \tilde{\beta}(y)K - \alpha(\tau)\bar{K}^\mu_\nu K^\nu_\mu\right]$$

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- Stealth Schwarzschild—dS solution in shift and Z_2 symmetric DHOST:

$A(r) = 1 - \frac{r_s}{r} - \frac{\Lambda_c}{3} r^2$ (Motohashi and Minamitsuji 19, Achour and Liu 19)

Application: Odd-parity sector

- Metric perturbations (no scalar): $\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$

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- Only one physical degree of freedom — (h_0, h_1) aren't physical

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- The odd EFT action up to second order:

$$S_{\text{odd}} = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}R - \Lambda(y) - c(y)g^{\tau\tau} - \tilde{\beta}(y)K - \alpha(\tau)\bar{K}^\mu_\nu K^\nu_\mu + \frac{1}{2}M_3^2(y)\delta K^\mu_\nu \delta K^\nu_\mu\right]$$

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$$\mathscr{L}_2 = p_1 h_0^2 + p_2 h_1^2 + p_3[(\dot{h}_1 - \partial_\rho h_0)^2 + 2p_4 h_1 \partial_\rho h_0]$$

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(Takahashi and Motohashi 21)

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- Integrate out (h_0, h_1) and introduce χ: $\mathscr{L}_2 = s_1\dot{\chi}^2 - s_2(\partial_\rho \chi)^2 - s_3\chi^2$

$$s_1 = -\frac{(j^2-2)p_3^2}{\tilde{p}_2} \qquad s_2 = \frac{(j^2-2)p_3^2}{p_1} \qquad s_3 = (j^2-2)p_3\left[1 + \frac{p_3p_4^2}{\tilde{p}_2} - \left(\frac{p_1+\tilde{p}_2}{p_1\tilde{p}_2}\dot{p}_3\right)^{\cdot} - p_3\left(\frac{p_4}{\tilde{p}_2}\right)^{\cdot}\right]$$

$$j^2 \equiv \ell(\ell+1) \qquad \tilde{p}_2 \equiv p_2 - (p_3p_4)^{\cdot} - p_3p_4^2$$

Application: Odd-parity sector

- Radial sound speed: $$c_\rho^2 = \frac{\bar{g}_{\rho\rho}}{|\bar{g}_{\tau\tau}|}\frac{s_2}{s_1} = \frac{M_\star^2}{M_\star^2 + M_3^2} + \frac{r^2 p_4^2}{j^2 - 2}$$

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New time coord. due to the crossed term in $\mathscr{L}_2$

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Application: Odd-parity sector

- Radial sound speed: $$c_\rho^2 = \frac{\bar{g}_{\rho\rho}}{|\bar{g}_{\tau\tau}|}\frac{s_2}{s_1} = \frac{M_\star^2}{M_\star^2 + M_3^2} + \frac{r^2 p_4^2}{j^2 - 2}$$

ℓ-dependent when $p_4 \neq 0$

- Angular sound speed: $$c_\theta^2 = \lim_{\ell\to\infty} \frac{r^2}{|\bar{g}_{\tau\tau}|}\frac{s_3}{j^2 s_1} = \frac{M_\star^2}{M_\star^2 + M_3^2}$$

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New time coord. due to the crossed term in $\mathscr{L}_2$

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Effective potential written in terms of our parameters

Conclusions/Future directions

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Future directions

- Determine a spectrum of quasi-normal modes in both sectors
- Study a dynamics of even-parity perturbations
- Generalize our EFT to a stationary and axisymmetric black hole
- Apply the same procedure to EFT of vector-tensor theories

(Work in progress w/ K. Aoki, M. A. Gorji, S. Mukohyama, & K. Takahashi)

Backup

Relevant scales of the EFT

Define an energy scale E in a region of interest

$$E \equiv \max\{ |{}^{(3)}\bar{R}|^{1/2}, |{}^{(3)}\bar{R}^{\mu}_{\nu}{}^{(3)}\bar{R}^{\nu}_{\mu}|^{1/4}, |\bar{K}|, |\bar{K}^{\mu}_{\nu}\bar{K}^{\nu}_{\mu}|^{1/2}, |{}^{(3)}\bar{R}^{\mu}_{\nu}\bar{K}^{\nu}_{\mu}|^{1/3} \}$$

It captures a relevant scale of background geometry e.g. $E \sim H$ for flat FRW case

Let $\Lambda_\star$ be a cutoff and μ be a Lorentz breaking scale e.g. $\Lambda_\star \sim \Lambda_3$ and $\mu \sim \Lambda_2$ for EFT of DE

We assume $\mu \gg \Lambda_\star$ and $E < \Lambda_\star$

$$f \sim \mathcal{O}(1)\,, \quad \Lambda \sim \mathcal{O}(M_\star^2 E^2)\,, \quad \beta \sim \alpha \sim \mathcal{O}(M_\star^2 E)\,, \quad \gamma \sim \mathcal{O}(M_\star^2)$$

$$m_2^4 \sim \mathcal{O}(M_\star^2 E^2)\,, \quad M_1^3 \sim \mathcal{O}(M_\star^2 E)\,, \quad M_3^2 \sim \mathcal{O}(M_\star^2)\,, \quad \mu_1^2 \sim \mathcal{O}(M_\star^2)$$

$$M_4 \sim M_5 \sim \mathcal{O}(M_\star^2 E \Lambda_\star^{-2})\,, \quad \mu_2 \sim \mu_3 \sim \mathcal{O}(M_\star^2 \Lambda_\star^{-2})$$

Decoupling limit action for π

Neglect the mixing with gravity $\Rightarrow$ no need to solve for N and N_i

$$S = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}f(y)R - \Lambda(y) - c(y)g^{\tau\tau} + \frac{1}{2}m_2^4(y)(\delta g^{\tau\tau})^2 + \frac{1}{3!}m_3^4(y)(\delta g^{\tau\tau})^3\right]$$

Use $g^{\tau\tau} \to g^{\tau\tau} + 2g^{\tau\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$

$$S_\pi = \int d\tau d^3\tilde{x}\sqrt{-g}\ c_s c\left[\left(\dot{\pi}^2 - h^{ij}\tilde{\partial}_i\pi\tilde{\partial}_j\pi\right) + c_s^2\left(\frac{1}{c_s^2} - 1\right)\dot{\pi}(\dot{\pi}^2 - h^{ij}\tilde{\partial}_i\pi\tilde{\partial}_j\pi) + \frac{8}{3}\frac{m_3^4c_s^2}{c}\dot{\pi}^3\right]$$

where $x_i = c_s\tilde{x}_i$ and $\dfrac{1}{c_s^2} \equiv 1 + \dfrac{2m_2^4}{c}$

$$\frac{\mathscr{L}_2}{\mathscr{L}_3} \sim 1 \quad \Rightarrow \quad E_{\text{Cubic}} \sim \frac{(c_sM_\star^2E^2)^{1/4}}{\sqrt{1-c_s^2}}$$

Strong coupling scale above which $\mathscr{L}_3$ becomes important compared to $\mathscr{L}_2$

Dictionary e.g. Quartic Horndeski

$$L_4 = G_4(\Phi, X)R - 2G_{4X}(\Phi, X)(\Box\Phi^2 - \nabla_\nu \nabla_\mu \Phi\, \nabla^\nu \nabla^\mu \Phi)$$

This can be rewritten as $L_4 = G_4{}^{(3)}\!R + (2XG_{4X} - G_4)\mathscr{K}_2 - 2\sqrt{-X}G_{4\Phi}K$ $\quad \mathscr{K}_2 \equiv K^2 - K_{\mu\nu}K^{\mu\nu} = 2K^2/3 - \sigma_{\mu\nu}\sigma^{\mu\nu}$

Determine $\bar{F}$ and its derivatives e.g. $\bar{F} = \bar{G}_4{}^{(3)}\!\bar{R} + (2\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X} - \bar{G}_4)\bar{\mathscr{K}}_2 - 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi}\,, \quad \bar{F}_{{}^{(3)}\!R} = \bar{G}_4$

Use the matching we obtain

$$M_\star^2 f = 2\bar{G}_4 \quad \Lambda = \bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}({}^{(3)}\!\bar{R} + 3\bar{\mathscr{K}}_2) + 2(-\bar{g}^{\tau\tau})^2\dot{\bar{\Phi}}^4\bar{G}_{4XX}\bar{\mathscr{K}}_2 - \sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi} + 2(-\bar{g}^{\tau\tau})^{3/2}\dot{\bar{\Phi}}^3\bar{K}\bar{G}_{4\Phi X}$$

$$c = -\dot{\bar{\Phi}}^2\bar{G}_{4X}{}^{(3)}\!\bar{R} - \mathscr{G}_4\bar{\mathscr{K}}_2 - \frac{1}{\sqrt{-\bar{g}^{\tau\tau}}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi} + 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}^3\bar{K}\bar{G}_{4\Phi X} \quad \beta = -\frac{8}{3}\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{K}\bar{G}_{4X} + 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}\bar{G}_{4\Phi}\,, \quad \alpha^\mu_\nu = 4\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}\bar{\sigma}^\mu_\nu$$

$$m_2^4 = \dot{\bar{\Phi}}^4\bar{G}_{4XX}{}^{(3)}\!\bar{R} + (3\dot{\bar{\Phi}}^4\bar{G}_{4XX} + 2\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^6\bar{G}_{4XXX})\bar{\mathscr{K}}_2 + \frac{1}{2(-\bar{g}^{\tau\tau})^{3/2}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi} + \frac{2}{\sqrt{-\bar{g}^{\tau\tau}}}\dot{\bar{\Phi}}^3\bar{K}\bar{G}_{4\Phi X} - 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}^5\bar{K}\bar{G}_{4\Phi XX}$$

$$M_1^3 = \frac{8}{3}\bar{K}\mathscr{G}_4 + \frac{2}{\sqrt{-\bar{g}^{\tau\tau}}}\dot{\bar{\Phi}}\bar{G}_{4\Phi} - 4\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}^3\bar{G}_{4\Phi X}\,, \quad M_2^2 = 4\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}\,, \quad M_3^2 = -4\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}\,, \quad \mu_1^2 = 2\dot{\bar{\Phi}}^2\bar{G}_{4X}\,, \quad \lambda_{1\ \nu}^{\ \mu} = -4\mathscr{G}_4\bar{\sigma}^\mu_\nu$$

Applicable for any solution of $\bar{\Phi}(\tau)$

Schwarzschild solution

$$ds^2 = -d\tau^2 + [1 - A(r)]d\rho^2 + r^2 d\Omega^2 \qquad A(r) = B(r) = 1 - \frac{r_s}{r}$$

$$r(\tau, \rho) = r_s^{1/3}\left[\frac{3}{2}(\rho - \tau)\right]^{2/3}$$

$$\bar{g}^{\tau\tau} = -1 \, , \quad \bar{K}(\tau, \rho) = -\frac{3}{2r(\tau, \rho)}\sqrt{\frac{r_s}{r(\tau, \rho)}} \, , \quad {}^{(3)}\bar{R} = 0$$

Gauge transformation

- Under $x^\mu \to x^\mu + \epsilon^\mu$ and $\epsilon^\tau = \epsilon^\rho = 0\,, \quad \epsilon^a = \sum_{\ell,m} \Xi_{\ell m}(\tau,\rho) E^{ab} \bar{\nabla}_b Y_{\ell m}(\theta,\phi)$

$$h_0 \to h_0 - \dot{\Xi}\,, \quad h_1 \to h_1 - \partial_\rho \Xi\,, \quad h_2 \to h_2 - 2\Xi$$

- Choose $\Xi = h_2/2 \;\Rightarrow h_2 = 0$

Parameters p's

- Quadratic Lagrangian from EFT with $A(r) = B(r)$ and Z_2 and shift symmetries:

$$\mathscr{L}_2 = p_1 h_0^2 + p_2 h_1^2 + p_3[(\dot{h}_1 - \partial_\rho h_0)^2 + 2p_4 h_1 \partial_\rho h_0]$$

$$p_1 \equiv \frac{1}{2}(j^2-2)r^2\sqrt{1-A}\,(M_\star^2 + M_3^2) \qquad p_2 \equiv -(j^2-2)\frac{r^2 M_\star^2}{2\sqrt{1-A}} + (p_3 p_4)^{\cdot}$$

$$p_3 \equiv \frac{(M_\star^2 f + M_3^2)r^4}{2\sqrt{1-A}} \qquad p_4 \equiv \sqrt{\frac{B}{A(1-A)}}\left(\frac{A'}{2} + \frac{1-A}{r}\right)\frac{\alpha + M_3^2}{M_\star^2 + M_3^2}$$

- Integrate out (h_0, h_1) : $\quad \mathscr{L}_2 = p_1 h_0^2 + \tilde{p}_2 h_1^2 + p_3[-\chi^2 + 2\chi(\dot{h}_1 - \partial_\rho h_0 - p_4 h_1)]$

$$\tilde{p}_2 \equiv p_2 - (p_3 p_4)^{\cdot} - p_3 p_4^2$$

$$h_0 = -\frac{\partial_\rho(p_3\chi)}{p_1} \qquad h_1 = \frac{(p_3\chi)^{\cdot} + p_3 p_4 \chi}{\tilde{p}_2}$$

Parameters s's

$$\mathscr{L}_2 = s_1\dot{\chi}^2 - s_2(\partial_\rho\chi)^2 - s_3\chi^2$$

$$s_1 = \frac{j^2-2}{2\sqrt{1-A}}\frac{(M_\star^2+M_3^2)^2 r^6}{(j^2-2)M_\star^2 + (M_\star^2+M_3^2)r^2p_4^2} \qquad s_2 = \frac{(M_\star^2+M_3^2)r^6}{2(1-A)^{3/2}}$$

$$s_3 = j^2\frac{(M_\star^2+M_3^2)r^4}{2\sqrt{1-A}} + \mathscr{O}(j^0)$$

- The absence of ghost and gradient instabilities require:

$$s_1 > 0\ , \quad c_\rho^2 > 0\ , \quad c_\theta^2 > 0$$

$$M_\star^2 + M_3^2 > 0\ , \quad M_\star^2 > 0$$

$\mathscr{L}_2$ in $(t, r)-$ and $(\tilde{t}, r)-$coordinates

- We use $dt = \frac{1}{A}d\tau - \frac{1-A}{A}d\rho$ and $dr = -\sqrt{\frac{B(1-A)}{A}}d\tau + \sqrt{\frac{B(1-A)}{A}}d\rho$

$$\mathscr{L}_2 = a_1(\partial_t\chi)^2 - a_2(\partial_r\chi)^2 + 2a_3(\partial_t\chi)(\partial_r\chi) - s_3\chi^2$$

$$a_1 = \frac{s_1 - (1-A)^2 s_2}{\sqrt{A^3B(1-A)}} \qquad a_2 = \sqrt{\frac{B(1-A)}{A}}(s_2 - s_1) \qquad a_3 = \frac{(1-A)s_2 - s_1}{A}$$

- Remove the crossed term by $\tilde{t} = t + \int \frac{a_3}{a_2}dr$ and $r \to r$

All the parameters are explicitly independent of t

$$\mathscr{L}_2 = b_1(\partial_{\tilde{t}}\chi)^2 - b_2(\partial_r\chi)^2 - s_3\chi^2 \quad \text{with} \quad b_1 = a_1 + \frac{a_3^2}{a_2} \quad \text{and} \quad b_2 = a_2$$

Dipole perturbations ($\ell = 1$)

- h_2 trivially vanishes by definition in terms of spherical harmonics
- Setting $\partial_\rho \Xi = h_1 \Rightarrow h_1 = 0$ is an incomplete gauge fixing since one is still free to choose Ξ up to an arbitrary function of τ
- Setting $h_1 = 0$ in $\mathscr{L}_2$ leads to a loss of indep. Eqns of motion
- Eqns of motion for h_0 and h_1:

$$\partial_\rho[p_3(\dot{h}_1 - \partial_\rho h_0)] + \partial_\rho(p_3 p_4 h_1) = 0 \qquad [p_3(\dot{h}_1 - \partial_\rho h_0)]^\cdot - (p_3 p_4)^\cdot h_1 - p_4 p_3 \partial_\rho h_0 = 0$$

- Setting $h_1 = 0$ gives $\quad \dot{C}_1(\tau) + p_4 C_1(\tau) = 0 \quad$ where $\quad p_3 \partial_\rho h_0 = C_1(\tau)$

Dipole perturbations $(\ell = 1)$

- For constant p_4: $$h_0 = 2C_2 e^{-p_4\tau} \int d\rho \, \frac{\partial_\rho r}{(M_\star^2 + M_3^2) r^4} \sqrt{\frac{A}{B}}$$

- For $A = B$: $$h_0 = -\frac{2C_2}{3} \frac{e^{-p_4\tau}}{(M_\star^2 + M_3^2) r^3}$$

Angular momentum of slowly rotating BH

- For $p_4 = 0$ (as a check): $$h_0 = -\frac{J}{4\pi(M_\star^2 + M_3^2) r^3}$$

Dipole perturbations correspond to the angular momentum of slowly rotating BH

e.g. Expanding Kerr metric up to linear order in J gives $\# J dt d\phi$

- For $A \neq B$ and/or $M_3^2 \neq const.$: a rotating BH doesn't belong to Kerr family, even at the linear level