Quantum-induced corrections to spacetime curvature

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Abstract: General relativity and quantum mechanics are not only fundamentally different theories giving explanation on how nature works but also genuinely incompatible description of reality. The violation of the principles of relativity and equivalence, at quantum scales, belongs to the main conceptual difficulties of reconciling principles of quantum mechanics with general relativity so that their scales of applicability are entirely distinct. When generalized noncommutative Heisenberg algebra accommodating impacts of finite gravitational fields as specified by quantum loop gravity, doubly-special relativity, and string theory, for instance, is thoughtfully applied on eight-dimensional Finsler manifold, the natural generalization of the pseudo-Riemannian manifold, in which the quadratic restriction on the length measure is relaxed, we have been able to define quantum-induced corrections to the fundamental tensor and thereby extending its applicability to quantum scales. By constructing the affine connections on four-dimensional pseudo--Riemannian manifold, we have determined the quantum-induced corrections to the Riemann curvature tensor and its contractions, the Ricci curvature tensor and scalar. Consequently, we have been able to construct the Einstein tensor, in which besides quantization additional geometric structures and curvatures are emerged. As in Einstein's classical theory of general relativity, we have proved that the covariant derivative of the modified Einstein tensor vanishes, as well. We conclude that the quantum-induced corrections establish quantum properties to the spacetime coordinates and momenta. Accordingly, the spacetime curvature endows additional curvature and geometrical structure as well as discretization which likely enable sensical predictions of Einstein's general relativity, at quantum scale.

Formalism (Relativistic Noncommutative Algebra):

We assume that the physical four-coordinates and four-momenta in terms of the auxiliary 4-vectors, which are canonically conjugate variables, are given as

$$\hat{x}^\mu = \hat{x}_0^\mu = (\hat{x}_0^0, \hat{x}_0^i), \qquad \mu,\nu \in \{0,1,2,3\} \qquad x_0^0 = ct \text{ and } p_0^0 = E/c$$

$$\hat{p}^\mu = \hat{p}_0^\mu \left(1+\beta \hat{p}_0^\rho \hat{p}_{0\rho}\right) = \left(\hat{p}_0^0,\, \hat{p}_0^i\right)\left(1+\beta \hat{p}_0^\rho \hat{p}_{0\rho}\right),$$

With isotropic property of spacetime and under Snyder algebra [10], the relativistic generalized noncommutative relation, RGUP, which is Lorentz covariant in Mankowski spacetime, for instance, is suggested as

$$[\hat{x}^\mu,\, \hat{p}^\nu] = i\hbar\left[(1+\beta\hat{p}^\rho\hat{p}_\rho)\,\eta^{\mu\nu} + 2\beta\hat{p}^\mu\hat{p}^\nu\right]$$

The uncertainties of the position and momentum measurements can be approximated as

$$\Delta x^\mu\, \Delta p^\nu \gtrsim \frac{\hbar}{2}\left[g^{\mu\nu} + \beta\,(\Delta p)^2 + \beta\langle p\rangle^2 - \beta\,(\Delta p^\mu)^2 - \beta\,(\Delta p^\nu)^2\right]$$

Thus, the uncertainty in the minimum measurable length sets limit on the coordinate measure

$$\Delta x^\mu_{\mathtt{min}} \gtrsim \pm\sqrt{-g^{\mu\nu}}\,\hbar\sqrt{\beta}$$

$$x^\mu = x_0^\mu - \Delta x^\mu_{\mathtt{min}} = \left(\left(x_0^0 - |\Delta x_{\mathtt{min}}|\right),\, \left(x_0^i - |\Delta x_{\mathtt{min}}|\right)\right)$$

Also, for momenta we can also determine corresponding measure uncertainties

$$\Delta p^\nu \lesssim \frac{2}{\hbar\,\beta}\,\Delta x^\mu \qquad \Delta x^\mu = \Delta x_0^\mu - 2\sqrt{\beta_0}\ell_{\mathtt{p}} \qquad \Delta p_0^\nu \lesssim \frac{2}{\hbar\,\beta}\frac{\Delta x_0^\mu - 2\sqrt{\beta_0}\ell_{\mathtt{p}}}{1+\beta\, g^{0\rho}\,\Delta|p_0|^2}$$

If these results are correct, they suggest quantum-mechanical corrections to the curved spacetime in GR, especially at the relativistic energy scale.

- The spacetime is no longer smooth or continuous. This is rather rough, where an inaccessible spacetime element whose volume is characterized by $\Delta x^\mu_{\mathtt{min}}$ likely exists.
- With coordinate and momentum uncertainties, events likely happen in jumps with nondeterministic outcomes.
- Besides coordinates and momenta, other physical quantities, such as time and energy, likely have noncommutation relations.
- The measurements in GR with quantum–induced corrections are no longer precise or noncoherent.

Formalism (Generalization of Riemannian Manifold):

Caianiello suggested that additional curvature in relativistic eight-dimensional spacetime tangent bundle TM_8 is conjectured to mimic quantization of the four–dimensional spacetime M_4, pseudo–Riemannian. For the eight–dimensional $TM_8=M_4 \otimes M_4$, we assume Finsler geometry, which is Riemannian without quadratic restriction on the length measure. The Finsler structure or the Minkowski smooth norm is given as

$$F^2(x,\dot{x}) = g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$$

With the homogeneity condition in *x-dot* and the minimal measurable length, which counts for quantum-mechanical corrects, e.g., sets limitations on GR coordinates and momenta and determines their measurement uncertainties.

$$F(x, (\sqrt{-|g|}\,\hbar\sqrt{\beta_0})\,\dot{x}) = (\sqrt{-|g|}\,\hbar\sqrt{\beta_0})\,F(x,\dot{x}), \qquad \forall(\sqrt{-|g|}\,\hbar\sqrt{\beta_0}) \in \mathbb{R}^+$$

On TM_8, the metric tensor given as $g_{AB}= g_\mu \otimes g_\mu$ could be determined by the Hessian in the (x,$\dot{x}$)–coordinates,

$$g_{AB} = \frac{1}{2}\frac{\partial^2}{\partial\dot{x}^\alpha\,\partial\dot{x}^\beta}F^2(x, (\sqrt{-|g|}\hbar\sqrt{\beta})\,\dot{x})$$

The indices $A,\, B \in \{0,1,\cdots,7\}$, while $\alpha,\, \beta \in \{0,1,2,3\}$.

The equality of the line element measure on TM_8 and M_4 based on property and tangent bundle that $d\tilde{s}^2 = g_{AB}\, dx^A\, dx^B$ and $d\tilde{s}^2 = \tilde{g}_{\mu\nu}\, d\zeta^\mu\, d\zeta^\nu$ assures that

$$\tilde{g}_{\mu\nu} = \left[1+(-|g|\,\hbar^2\,\beta)\,|\ddot{x}|^2\right] g_{\mu\nu} = \left(1+(-|g|\,\beta_0\,\ell_{\mathtt{p}}^2)\,|\ddot{x}|^2\right) g_{\mu\nu}$$

Why just quantum–induced corrections

The present attempt to reconcile principles of QM and GR:

- introduces gravitational impacts to the fundamental theory of quantum mechanics,
- introduces additional curvatures to impose quantization of curved four–dimensional spacetime in GR.

and are strongly based on the classical nature of the vector measure, namely the precise GR measure of the line element

$$d\tilde{s}^2 = g_{AB}\, dx^A\, dx^B = \tilde{g}_{\mu\nu}\, d\zeta^\mu\, d\zeta^\nu$$

For a full quantum-mechanical measure, we are certainly urgent to integrate probability distribution to the metric tensor and to 1–form dx^μ. Alternative we need to suggest noncommutative relations.

Quantized Riemann curvature tensor:

The coefficients of the Riemann curvature tensor can be constructed from the Levi–Civita connections, regardless metric compatibility or torsion-free property.

$$\tilde{R}^\gamma_{\beta\mu\nu} = \tilde{\Gamma}^\gamma_{\beta\nu,\mu} - \tilde{\Gamma}^\gamma_{\beta\mu,\nu} + \tilde{\Gamma}^\gamma_{\sigma\mu}\tilde{\Gamma}^\sigma_{\beta\nu} - \tilde{\Gamma}^\gamma_{\sigma\nu}\tilde{\Gamma}^\sigma_{\beta\mu}$$

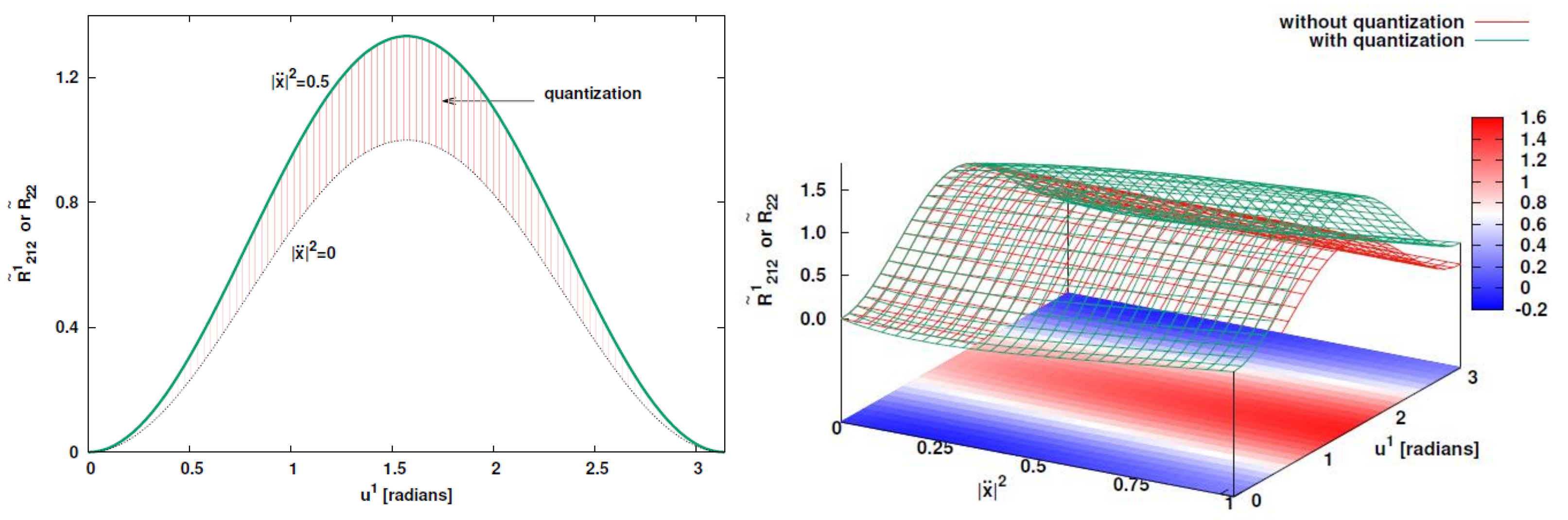
where, $\tilde{\Gamma}^\gamma_{\sigma\mu} = \dfrac{1+2\mathscr{T}|\ddot{x}|^2}{1+\beta|\ddot{x}|^2}\dfrac{g^{\gamma\alpha}}{2}\left(g_{\mu\alpha,\sigma} + g_{\sigma\alpha,\mu} - g_{\sigma\mu,\alpha}\right) = \dfrac{1+2\beta|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}\Gamma^\gamma_{\sigma\mu}$

And finally, we get

$$\tilde{R}^\gamma_{\beta\mu\nu} = \frac{1+2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}R^\gamma_{\beta\mu\nu} + \frac{(1+2\mathscr{T}|\ddot{x}|^2)\,\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left(\Gamma^\gamma_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\gamma_{\sigma\nu}\Gamma^\sigma_{\beta\mu}\right) + \frac{2\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left[\Gamma^\lambda_{\lambda\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\lambda_{\lambda\nu}\Gamma^\gamma_{\beta\mu}\right]$$

As an example, we assume 2-sphere

$$\tilde{R}^1_{212} = \frac{1+2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}\left[\sin(u^1)\right]^2 + \frac{(2\mathscr{T}|\ddot{x}|^2-1)\,\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left[\cos(u^1)\right]^2$$



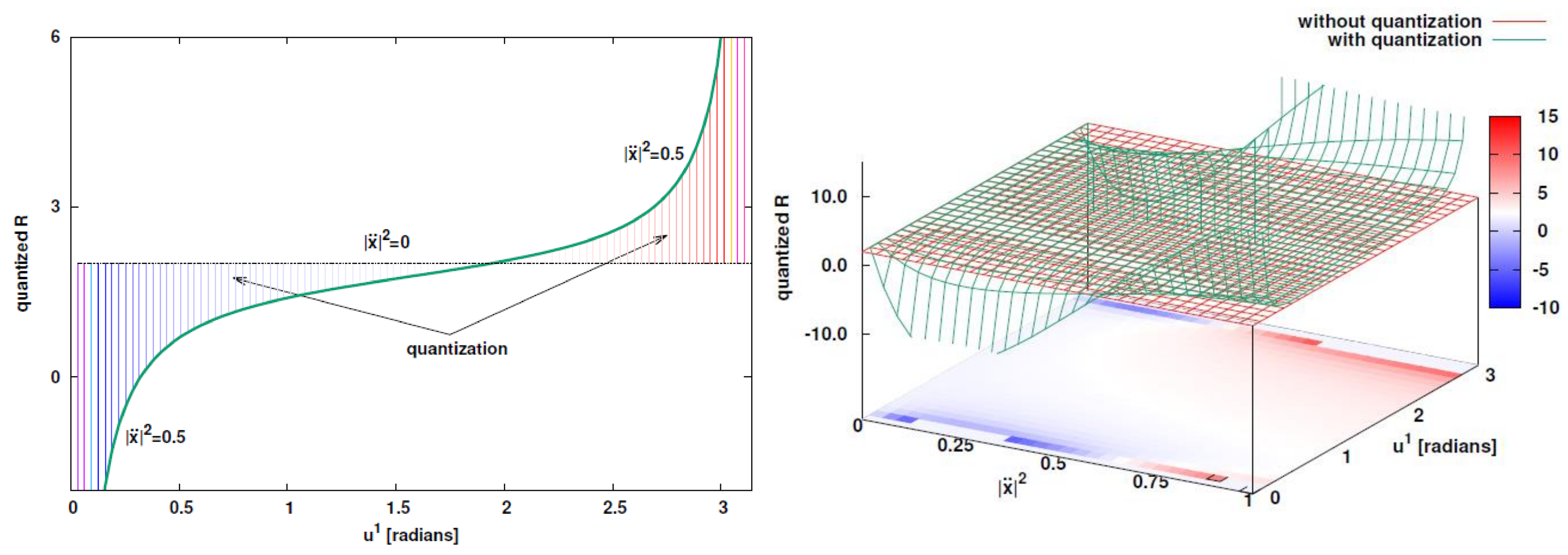
Quantized Ricci curvature tensor:

$$\tilde{R}_{\beta\nu} = \frac{1+2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}R_{\beta\nu} + \frac{(1+2\mathscr{T}|\ddot{x}|^2)\,\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left(\Gamma^\mu_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\mu_{\sigma\nu}\Gamma^\sigma_{\beta\mu}\right) + \frac{2\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left[\Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{\beta\nu} - \Gamma^\lambda_{\lambda\nu}\Gamma^\mu_{\beta\mu}\right]$$

Quantized Ricci scalar:

$$\tilde{R} = \frac{1+2\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}R + \frac{(1+2\mathscr{T}|\ddot{x}|^2)\,\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^3}g^{\beta\nu}\left(\Gamma^\mu_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\mu_{\sigma\nu}\Gamma^\sigma_{\beta\mu}\right) + \frac{2\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^3}g^{\beta\nu}\left[\Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{\beta\nu} - \Gamma^\lambda_{\lambda\nu}\Gamma^\mu_{\beta\mu}\right]$$

$$\tilde{R} = 2\frac{1+2\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2} + \frac{2\,(1+2\mathscr{T}|\ddot{x}|^2)\,\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left\{\left[\cot(u^1)\right]^2 - \cot(u^1)\right\} - \frac{4\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^3}\left[\cot(u^1)\right]^2$$



Quantized Einstein curvature tensor:

$$\tilde{G}_{\beta\nu} = \frac{1+2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}G_{\beta\nu} + \frac{(1+2\mathscr{T}|\ddot{x}|^2)\,\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left[\left(\Gamma^\mu_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\mu_{\sigma\nu}\Gamma^\sigma_{\beta\mu}\right) - \frac{1}{2}g_{\beta\nu}g^{\alpha\rho}\left(\Gamma^\mu_{\sigma\mu}\Gamma^\sigma_{\alpha\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\alpha\mu}\right)\right] + \frac{2\mathscr{T}|\ddot{x}|^2}{(1+\mathscr{T}|\ddot{x}|^2)^2}\left[\left(\Gamma^\mu_{\lambda\mu}\Gamma^\mu_{\beta\nu} - \Gamma^\lambda_{\lambda\nu}\Gamma^\mu_{\beta\mu}\right) - \frac{1}{2}g_{\beta\nu}g^{\alpha\rho}\left(\Gamma^\mu_{\lambda\mu}\Gamma^\mu_{\alpha\rho} - \Gamma^\lambda_{\lambda\rho}\Gamma^\mu_{\alpha\mu}\right)\right].$$

$$\nabla^\beta\tilde{G}_{\beta\nu} = \nabla^\beta\left(\frac{1+2\mathscr{T}|\ddot{x}|^2}{1+\mathscr{T}|\ddot{x}|^2}G_{\beta\nu}\right) = \left[\frac{\mathscr{T}\ddot{x}^i\ddot{x}^j}{(1+\mathscr{T}|\ddot{x}|^2)^2}G_{\beta\nu}\right]\eta_{ij,\beta} = 0$$

Conclusions:

The quantum-induced corrections to the metric tensor could be determined by Hessian in the extended coordinates $(x^\mu, (\sqrt{-|g|}\sqrt{\beta_0}\ell_p)\dot{x}^\mu)$. By equating the line element measures on differential manifold M_4 and on the tangent bundle TM_4, even with its infinite precision and deterministic outcomes, the quantum–induced corrections to the metric tensor $\tilde{g}_{\mu\nu}$ are determined, in which gravitational and quantum-mechanical imprints are encoded. With this single extension, we have been able to extend the Levi-Civita connections on pseudo–Riemannian manifold and thereby suggesting quantum-induced corrections to the Riemann and Ricci curvature tensors and Ricci scalar. Accordingly, we have constructed a quantized version of the Einstein tensor.