

Heavy quarkonium production in hadronic collisions

Kazuhiro Watanabe



Rencontres QGP France, July 07, 2021

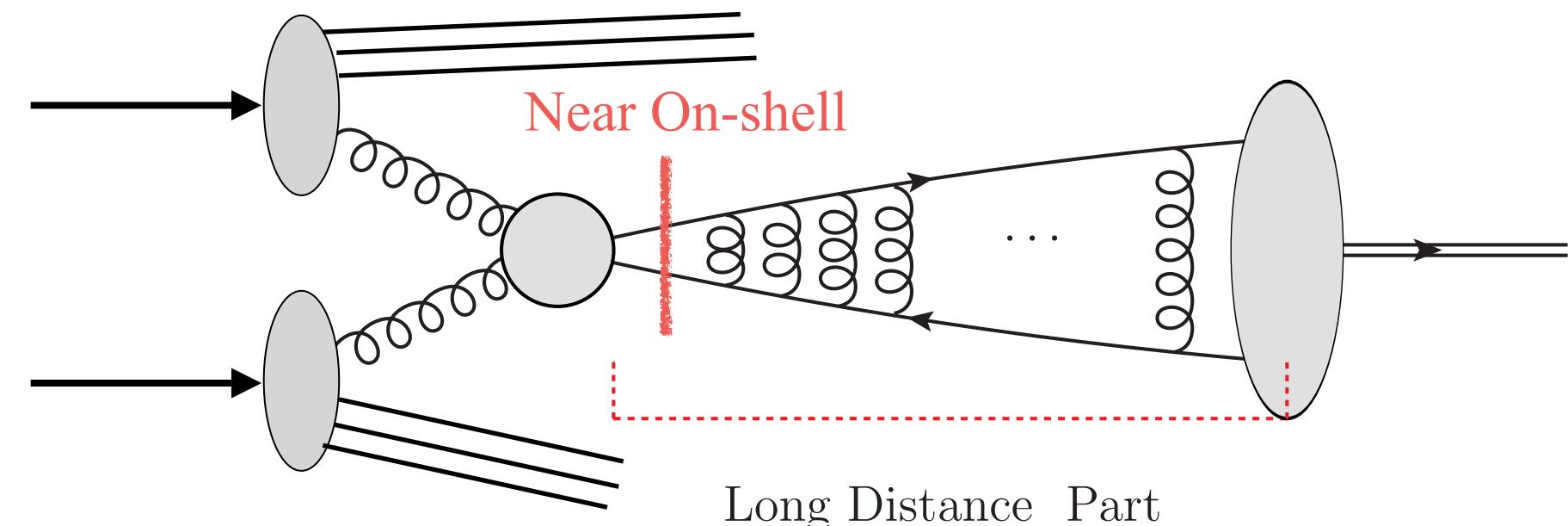
Quarkonia as probes

A+A collisions: Large systems

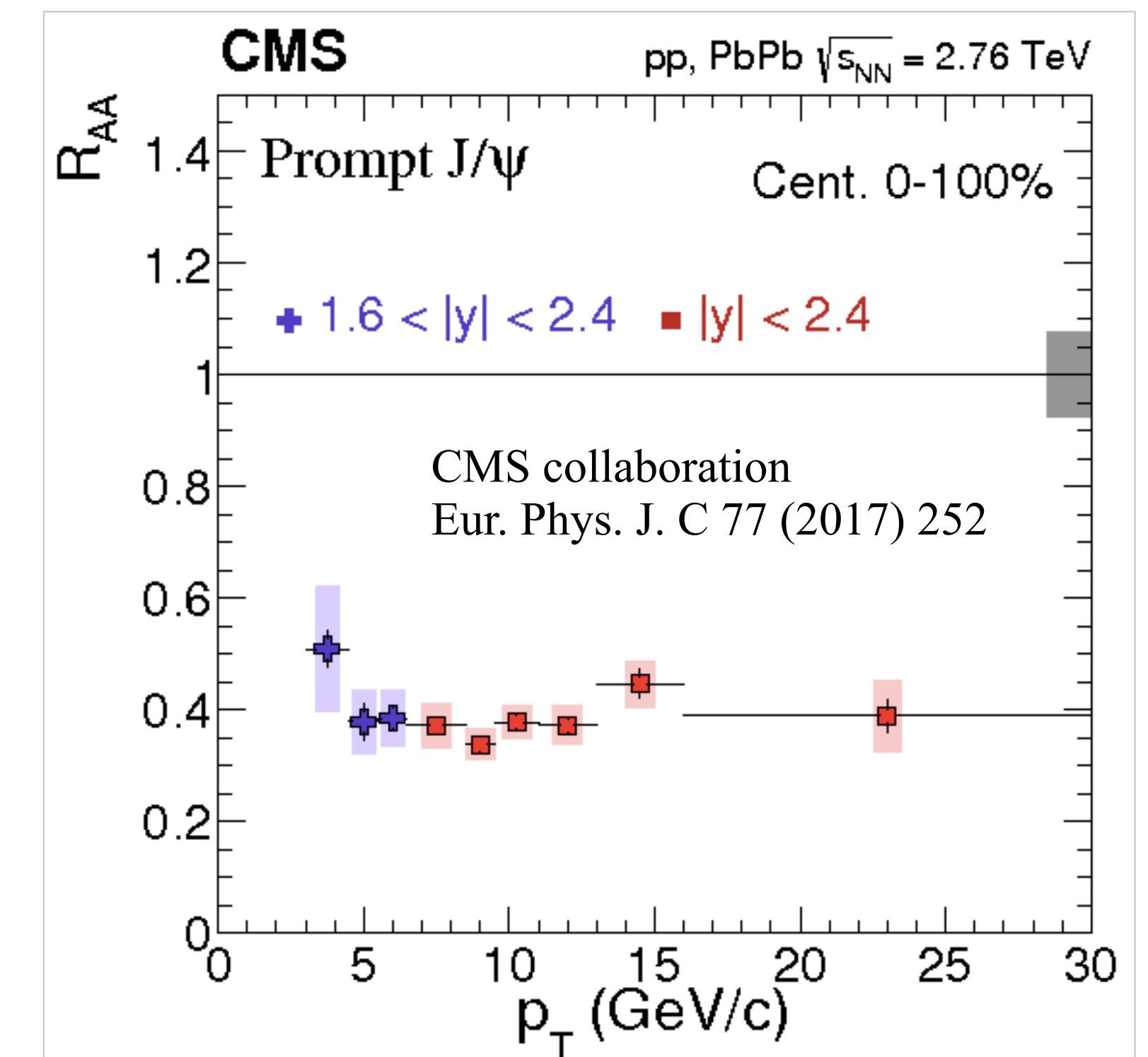
- J/ψ , Υ suppression due to the presence of QGP, strong flow effects.
- Space-time evolution of quarkonium formation.

p+A collisions: Small systems

- Cold nuclear matter (CNM) effects (nPDFs, energy-loss, saturation,...)
- Soft partons shower from projectile and target.
- Comoving partons/hadrons in final state, and possible QGP droplets.
- Quarkonium production in pp collisions + anything else; Can study using theoretical frameworks in pp collisions.



Need to look into quarkonium production xsection
by varying $\sqrt{s_{NN}}$, y , p_\perp , m , system size, N_{ch}



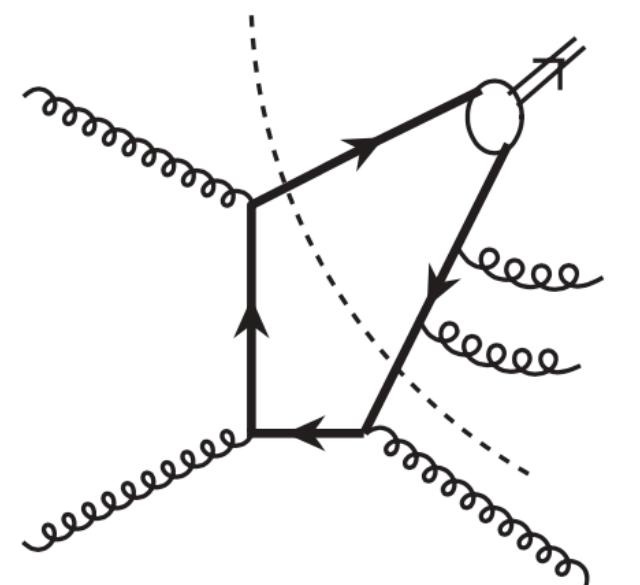
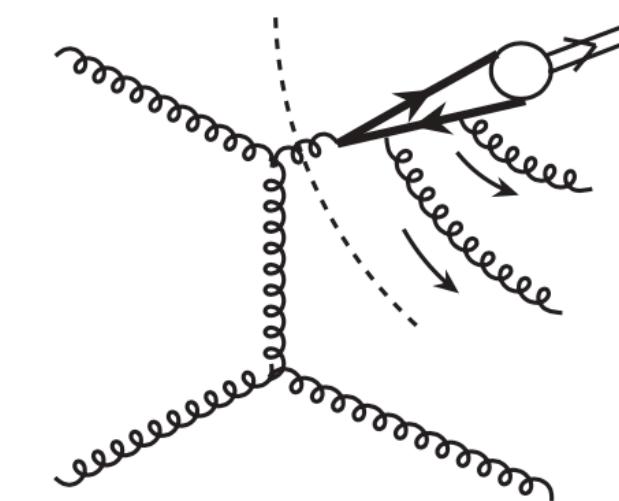
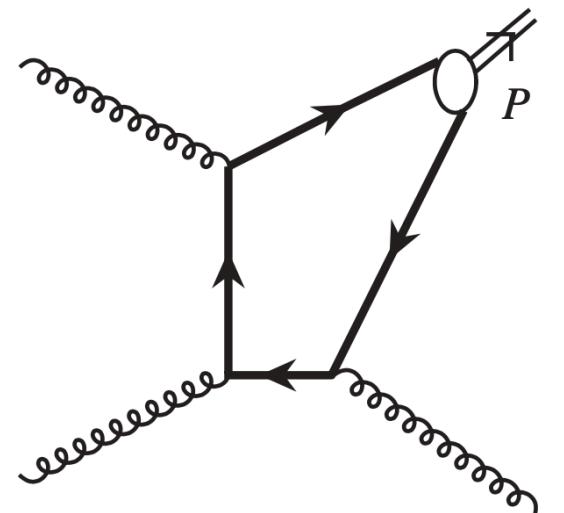
Part I: Quarkonium production at high p_T

Heavy quarkonium production of high p_T

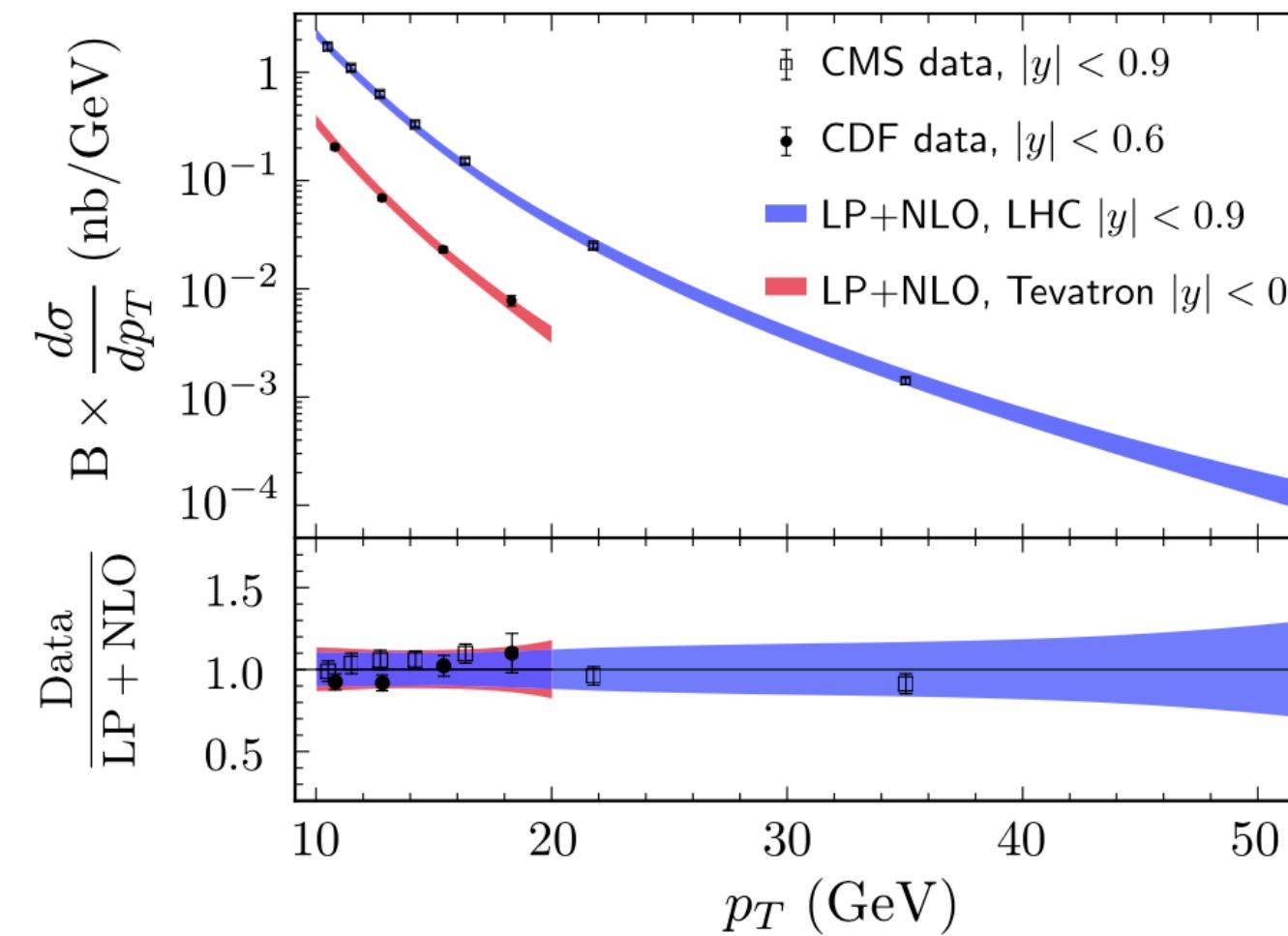
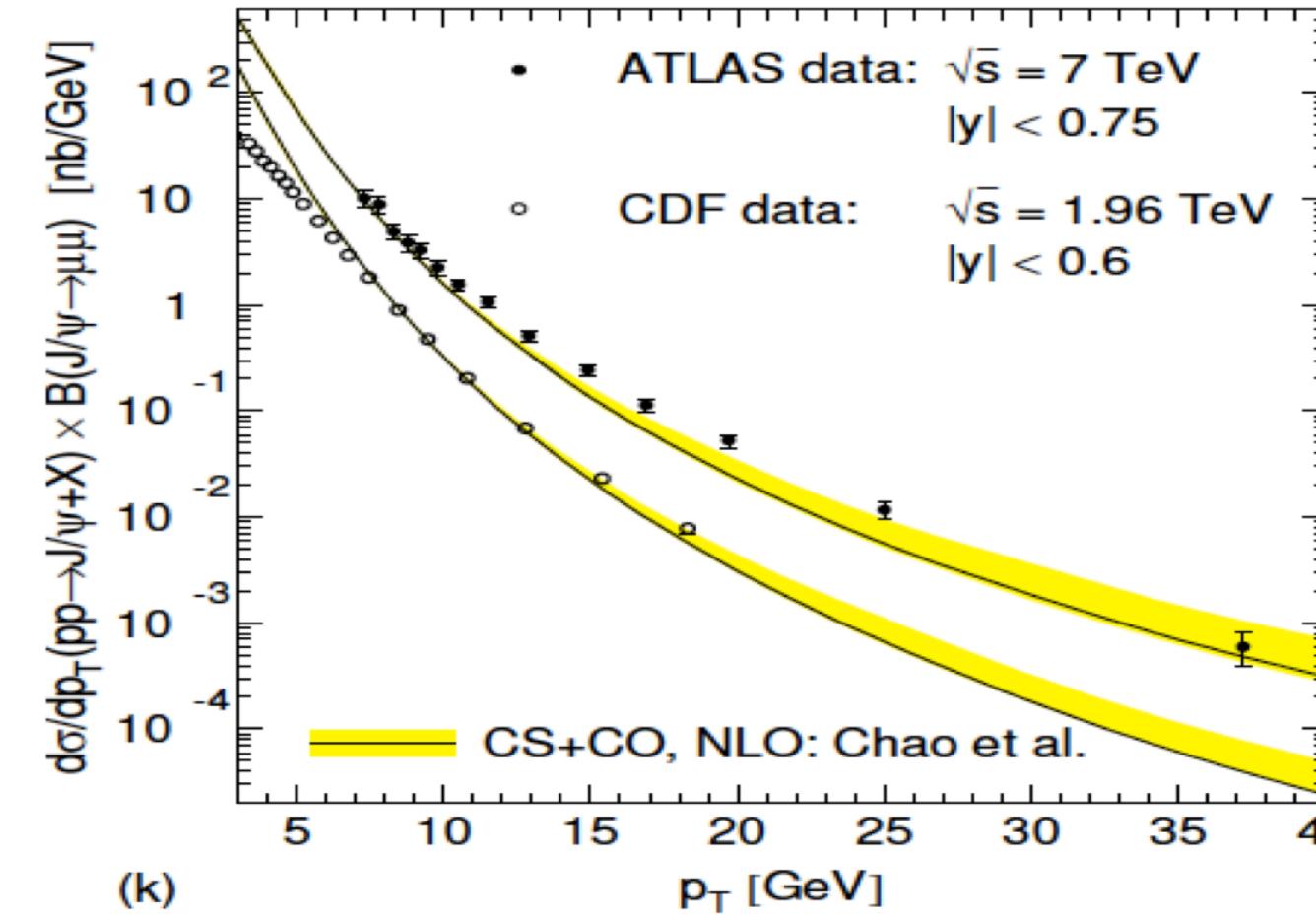
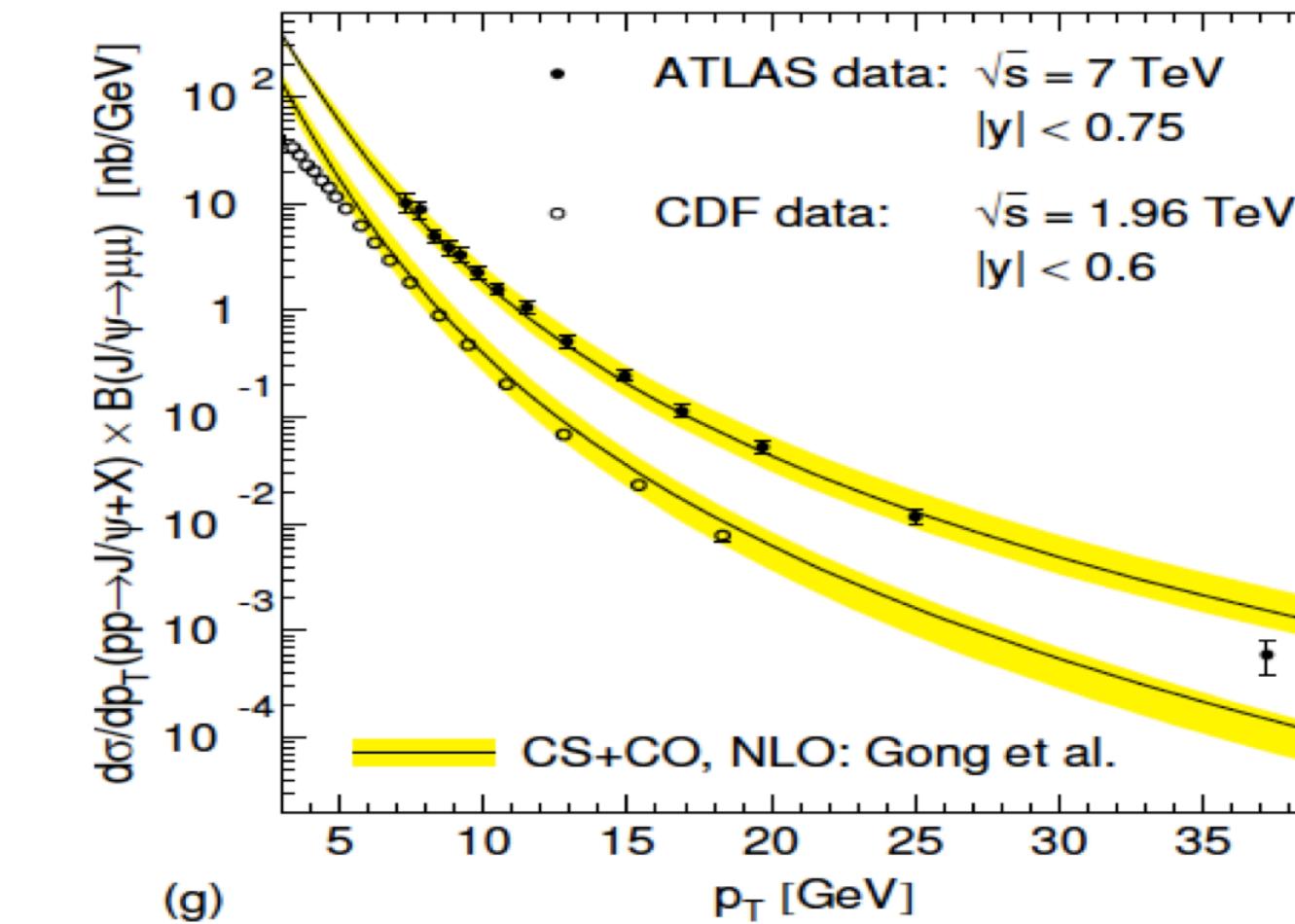
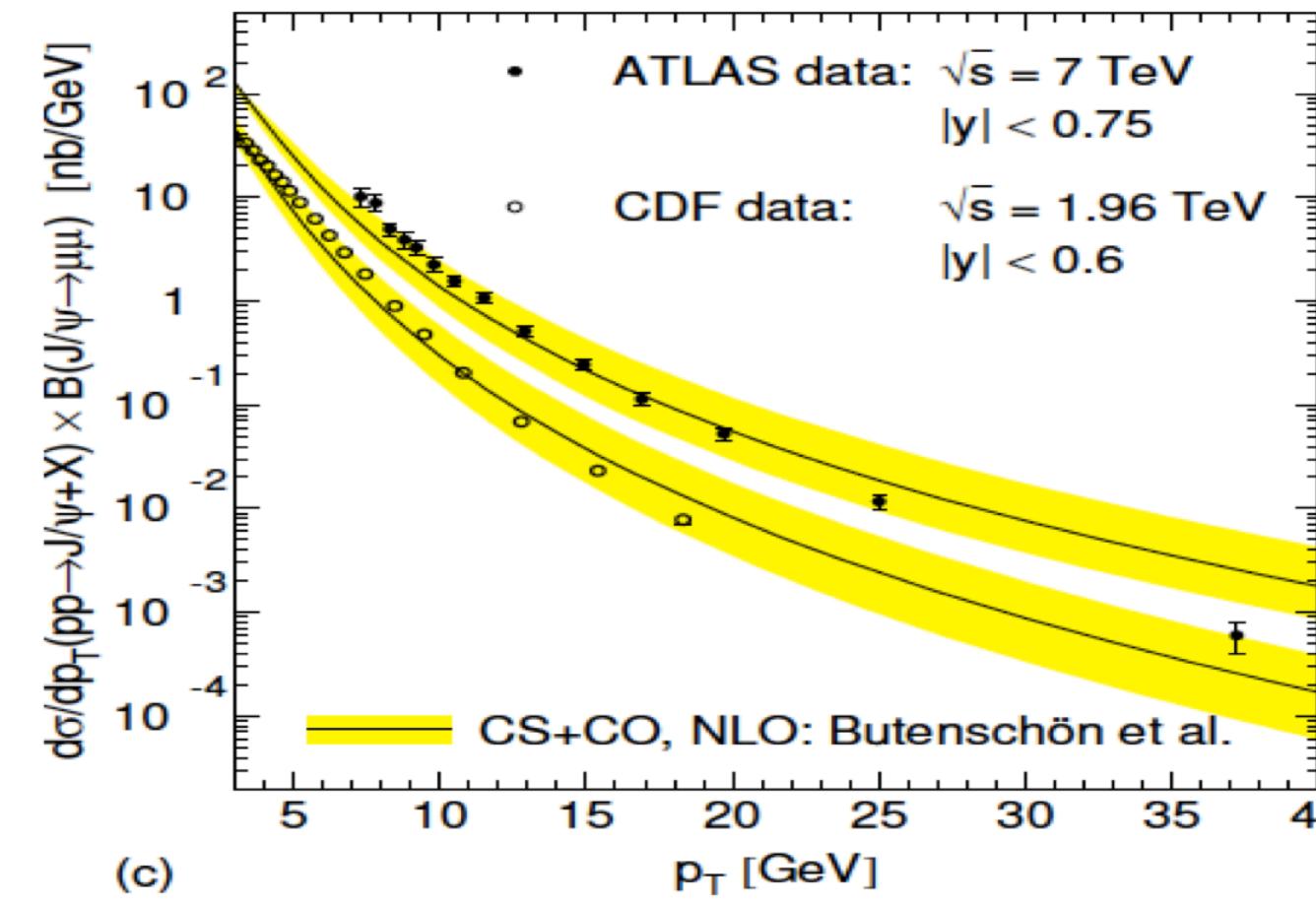
- High p_T quarkonium production ($p_T^2 \gg (2m)^2 \gg \Lambda_{\text{QCD}}^2$) allows us to separate the short distance part for a quark pair production and the long distance part for the bound state formation.
- Modern approaches for tackling the bound state formation;
 1. Color Evaporation Model: useful for phenomenology but the long distance part is blinded.
 2. Color Singlet Model (direct production at the early stage): $d\sigma(Q\bar{Q}[{}^3S_1^{[1]}]) \propto \alpha_s^3 m^4 / p_T^8$ at LO.
 3. Color Octet Model or NRQCD factorization approach: $d\sigma = \sum_{\kappa} d\hat{\sigma}_{\kappa} \langle \mathcal{O}_{\kappa} \rangle$. Long-Distance

Matrix Elements (LDMEs), $\langle \mathcal{O}_{\kappa} \rangle$, are organized by the power of the quark velocity
 $v^2 \sim q_T^2/m^2 < 1$.

- At high p_T higher order corrections must be essential:
 $d\sigma(Q\bar{Q}[{}^3S_1^{[1]}]) \propto \alpha_s^3 m^4 / p_T^8 \times \alpha_s p_T^2 / m^2 = \alpha_s^4 m^2 / p_T^6$
- More importantly, the gluon jet fragmentation gives $d\sigma \propto \alpha_s^5 / p_T^4$ as well as
 $d\sigma \propto \alpha_s^2 / p_{\perp}^4 \times \alpha_s^3 \ln(p_T^2/m^2)$. The later is enhanced even if $\alpha_s \ll 1$; we may not obtain reliable predictions by considering only diagrams in the naive α_s expansion as well as v expansion.



NRQCD vs. Data



Butenschoen, Kniehl, PRD84, 051501 (2011).
 Chao, Ma, Shao, Wang, Zhang, PRL108, 242004 (2012).
 Gong, Wan, Wang, Zhang, PRL110, 042002 (2013).
 Bodwin, Chung, Kim, Lee, PRL113, 022001 (2014).

	$\langle \mathcal{O}(^3S_1^{[1]}) \rangle$ GeV ³	$\langle \mathcal{O}(^1S_0^{[8]}) \rangle$ 10 ⁻² GeV ³	$\langle \mathcal{O}(^3S_1^{[8]}) \rangle$ 10 ⁻² GeV ³	$\langle \mathcal{O}(^3P_0^{[8]}) \rangle$ 10 ⁻² GeV ⁵
Set I (Butenschoen <i>et al.</i>)	1.32	3.04	0.16	-0.91
Set II (Chao <i>et al.</i>)	1.16	8.9	0.30	1.26
Set III (Gong <i>et al.</i>)	1.16	9.7	-0.46	-2.14
Set IV (Bodwin <i>et al.</i>)	-	9.9	1.1	1.1

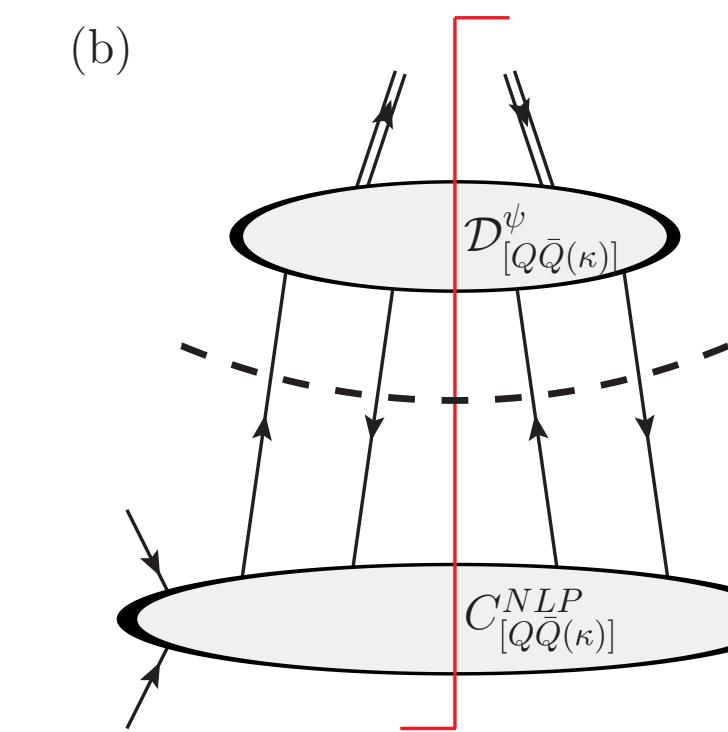
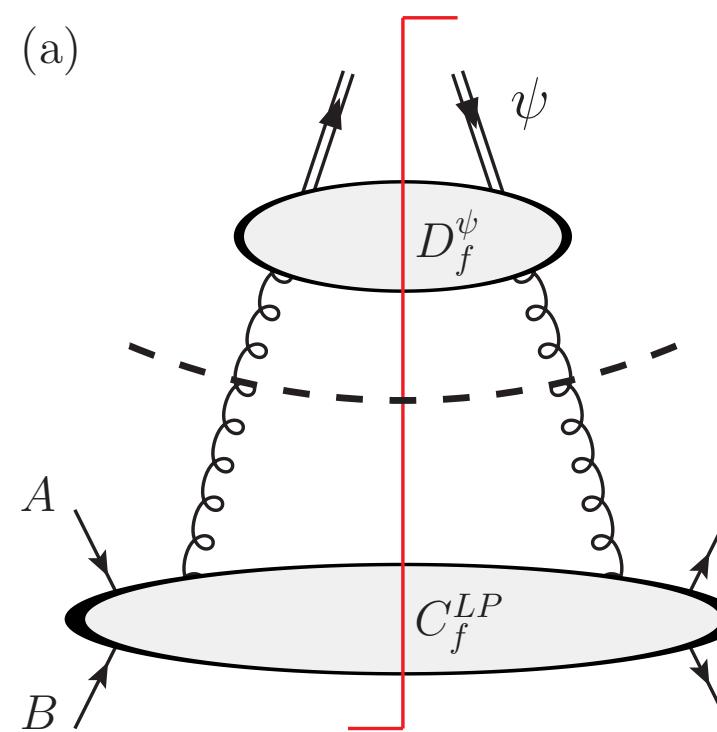
LDMEs should be universality, however:

- Numbers are not the same.
- Not even the sign.

More work is needed!

QCD factorization approach

Leading power (LP)



Nayak, Qiu, Sterman, PRD72 (2005) 114012
 Kang, Qiu, Sterman, PRL108 (2012) 102002
 Kang, Ma, Qiu, Sterman, PRD90 (2014) 3, 034006,
 PRD91 (2015) 1, 014030

Subleading power (NLP):
 critical at moderate $p_T \sim \mathcal{O}(2m)$

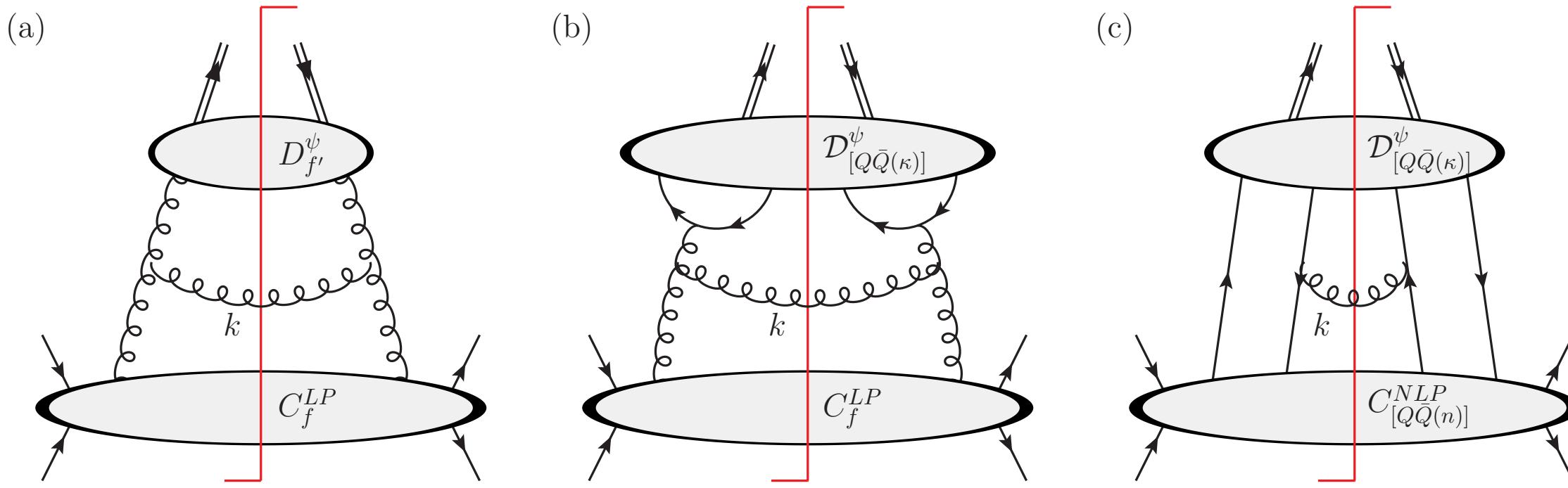
$$d\sigma_{A+B \rightarrow [f, Q\bar{Q}] \rightarrow \psi + X}^{\text{QCD-Res}}(\mu) = \sum_{f=q,\bar{q},g} C_{A+B \rightarrow [f] + X}^{\text{LP}}(\mu) \otimes D_{[f] \rightarrow \psi}(\mu) + \frac{1}{p_\perp^2} \left[\sum_n C_{A+B \rightarrow [Q\bar{Q}(n)] + X}^{\text{NLP}}(\mu) \otimes \mathcal{D}_{[Q\bar{Q}(n)] \rightarrow \psi}(\mu) \right]$$

- In hadronic collisions, short distance coefficients (SDCs) C^{LP} and C^{NLP} are available up to NLO and LO in α_s expansion, respectively. C^{NLP} at NLO has been calculated recently in e^+e^- collisions. Lee, Sterman, JHEP 09 (2020) 046
- Single (twist-2) and double parton (twist-4) fragmentation functions D_f and $\mathcal{D}_{Q\bar{Q}}$ are employed.
- Matching condition:

$$d\sigma_{A+B \rightarrow \psi + X}(m \neq 0) = d\sigma_{A+B \rightarrow \psi + X}^{\text{QCD-Evol}}(m=0) - d\sigma_{A+B \rightarrow \psi + X}^{\text{QCD-(n)}}(m=0) + d\sigma_{A+B \rightarrow \psi + X}^{\text{NRQCD-(n)}}(m \neq 0)$$

$$\Rightarrow \begin{cases} d\sigma_{A+B \rightarrow \psi + X}^{\text{QCD-Evol}} & \text{when } p_\perp \gg m; d\sigma^{\text{NRQCD-(n)}} \approx d\sigma^{\text{QCD-(n)}} \\ d\sigma_{A+B \rightarrow \psi + X}^{\text{NRQCD-(n)}} & \text{when } p_\perp \rightarrow m; d\sigma^{\text{QCD-Evol}} \approx d\sigma^{\text{QCD-(n)}} \end{cases}$$

Nonlinear QCD evolution



- Twist-2 evolution equation: DGLAP + nonlinear term

Kang, Ma, Qiu, Sterman, PRD 90 (2014) 3, 034006

$$\frac{dD_{[f] \rightarrow \psi}}{d \ln \mu^2} = \gamma_{[f] \rightarrow [f']} \otimes D_{[f'] \rightarrow \psi} + \frac{1}{\mu^2} \gamma_{[f] \rightarrow [Q\bar{Q}(\kappa)]} \otimes \mathcal{D}_{[Q\bar{Q}(\kappa)] \rightarrow \psi}$$

- Twist-4 “DGLAP like” evolution equation:

$$\frac{d\mathcal{D}_{[Q\bar{Q}(n)] \rightarrow \psi}}{d \ln \mu^2} = \Gamma_{[Q\bar{Q}(n)] \rightarrow [Q\bar{Q}(\kappa)]} \otimes \mathcal{D}_{[Q\bar{Q}(\kappa)] \rightarrow \psi}$$

Ma, Qiu, Zhang, PRD89 (2015) 094029, 094030

- Input fragmentation functions:

Long Distance Matrix Elements (LDMEs)

$$D_{f \rightarrow \psi}(z; m, \mu_0) = \sum_{[Q\bar{Q}(n)]} \pi \alpha_s \left\{ \hat{d}_{f \rightarrow [Q\bar{Q}(n)]}^{(1)}(z; m, \mu_0, \mu_\Lambda) + \frac{\alpha_s}{\pi} \hat{d}_{f \rightarrow [Q\bar{Q}(n)]}^{(2)}(z; m, \mu_0, \mu_\Lambda) + \mathcal{O}(\alpha_s^2) \right\} \frac{\langle \mathcal{O}_{[Q\bar{Q}(n)]}^\psi(\mu_\Lambda) \rangle}{m^{2L+3}}$$

$$\mathcal{D}_{[Q\bar{Q}(\kappa)] \rightarrow \psi}(z, u, v; m, \mu_0) = \sum_{[Q\bar{Q}(n)]} \left\{ \hat{d}_{[Q\bar{Q}(\kappa)] \rightarrow [Q\bar{Q}(n)]}^{(0)}(z, u, v; m, \mu_0, \mu_\Lambda) + \frac{\alpha_s}{\pi} \hat{d}_{[Q\bar{Q}(\kappa)] \rightarrow [Q\bar{Q}(n)]}^{(1)}(z, u, v; m, \mu_0, \mu_\Lambda) + \mathcal{O}(\alpha_s^2) \right\} \frac{\langle \mathcal{O}_{[Q\bar{Q}(n)]}^\psi(\mu_\Lambda) \rangle}{m^{2L+1}}$$

$\mu_0 = \mathcal{O}(2m)$: pQCD factorization scale, $\mu_\Lambda = \mathcal{O}(m)$: NRQCD factorization scale

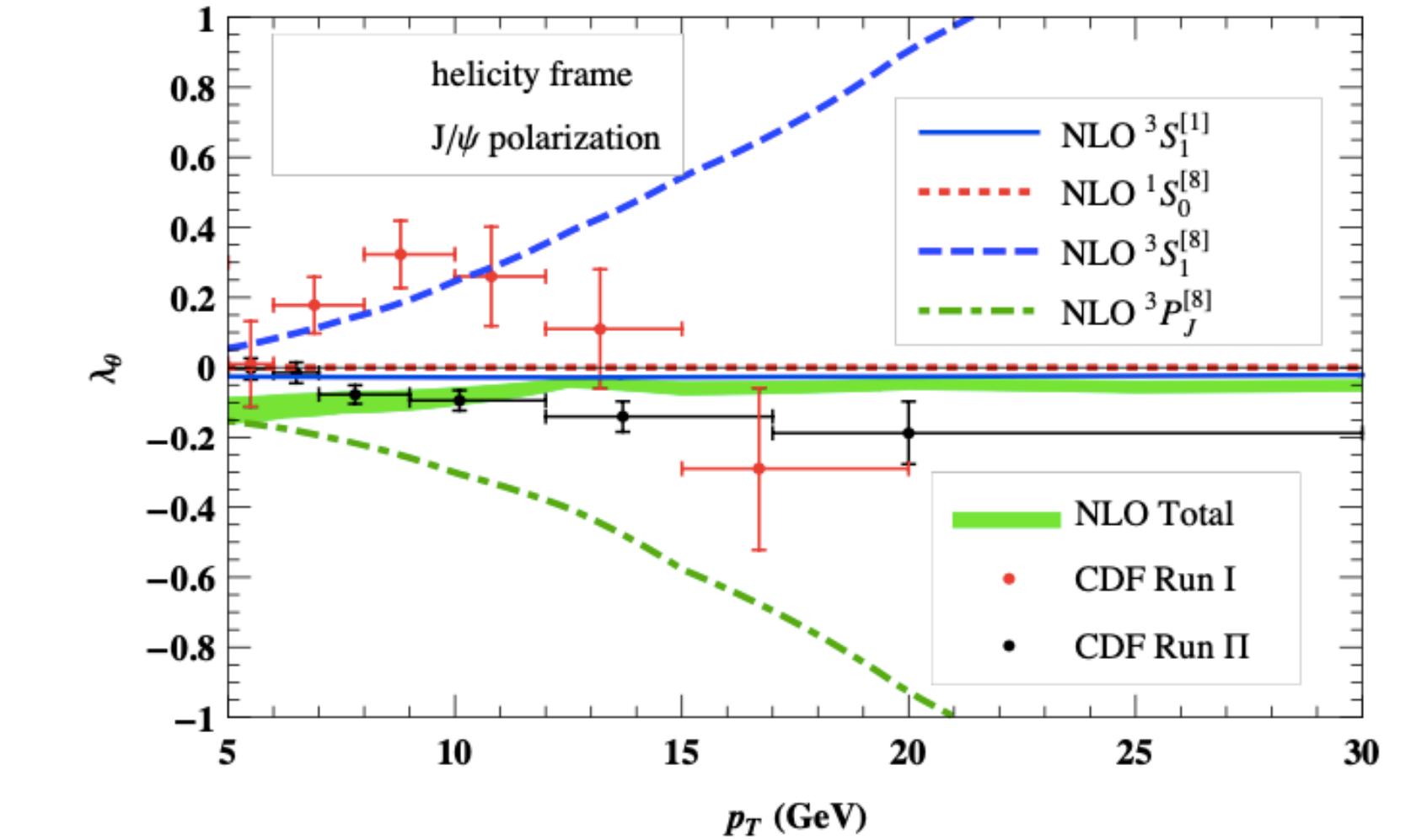
Preliminaries

i) Fixed order calculations in the NRQCD factorization at NLO

- Polarizations and p_T spectra of quarkonia can be reproduced in the NRQCD at NLO, however, it depends on LDMEs.
- Transverse components of SDCs in the NRQCD for producing $Q\bar{Q}$ in P -state can be negative: cancellations between different intermediate states happen.

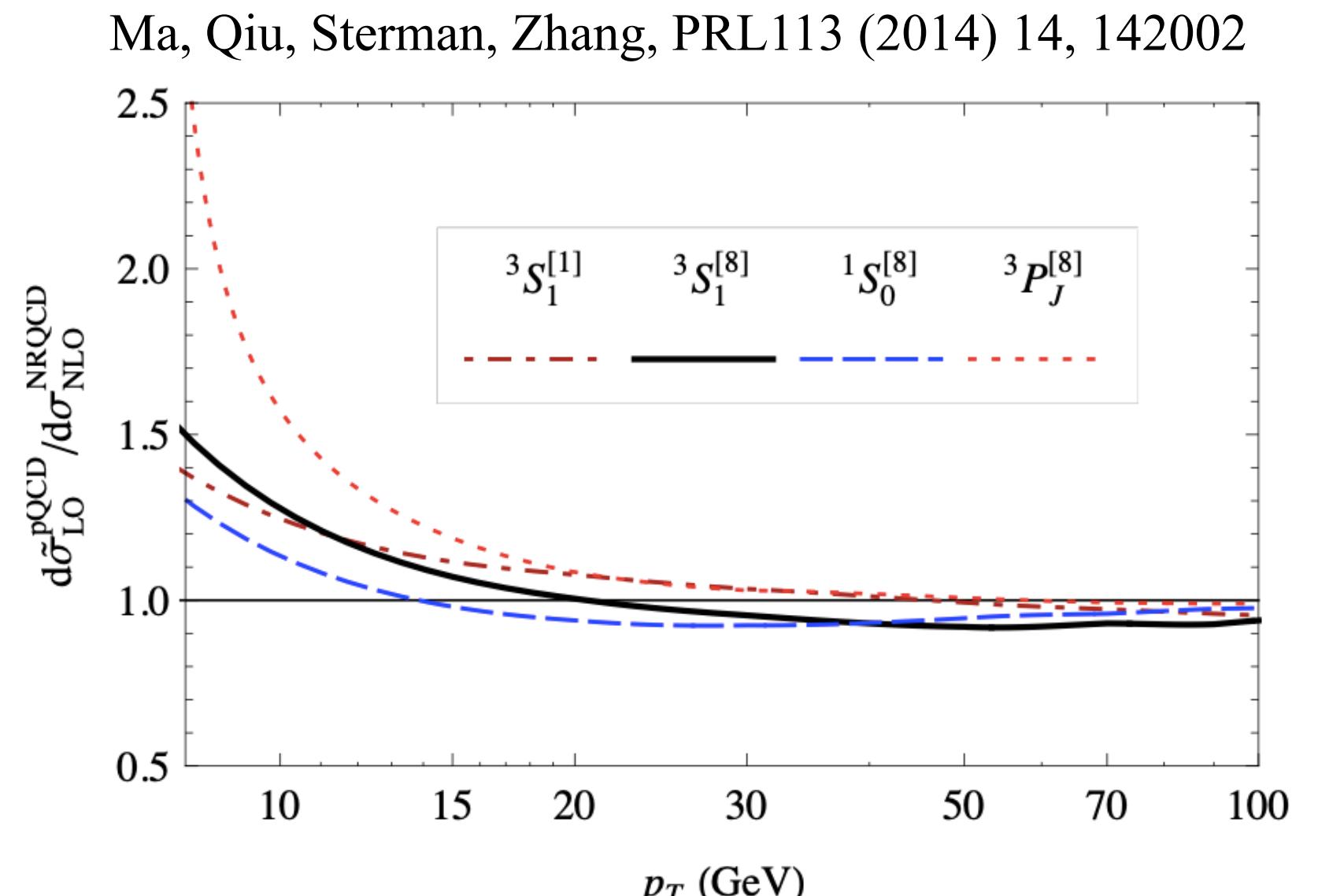
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Chao, Ma, Shao, Wang, Zhang, PRL108 (2012) 242004



ii) QCD factorization at LO with input FFs

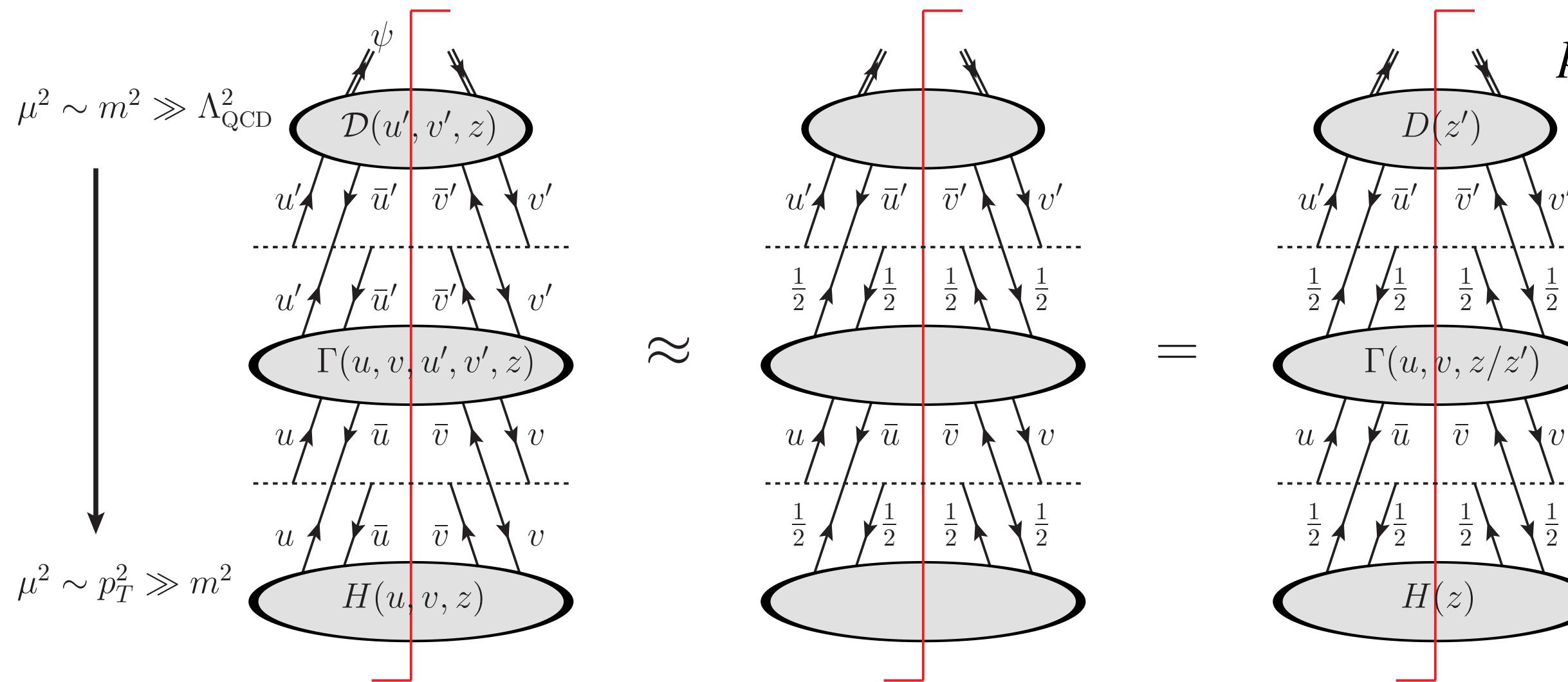
- LP (NLP) SDCs at LO (α_s^2 (α_s^3)) combined with LP (NLP) input FFs at NLO (α_s^2 (α_s)) have the same powers of α_s as the NLO NRQCD ($\alpha_s^4 v^4$).
- LO QCD factorization results with input FFs reproduce complicated NRQCD results at NLO channel-by-channel.
- Need to include the QCD evolution effects.



Ma, Qiu, Sterman, Zhang, PRL113 (2014) 14, 142002

Evolution equations in a simplified situation

Lee, Qiu, Sterman, **KW**, in preparation.



$$p_Q = u' p_c, p_{\bar{Q}} = (1 - u') p_c, z p_c^+ = p^+$$

- The cross section and FFs involve multi-integral variables; numerical calculations get complicated.
- Since the produced heavy quark pair is dominated by its on-shell state, we may expand the SDCs and evolution kernels on lower virtuality sides at each evolution step around $u = v = 1/2$.

$$\frac{d\sigma_{\text{NLP}}^\psi}{dy d^2 p_T} = \int dz du dv H_{[Q\bar{Q}]}(p_Q, p_{\bar{Q}}, \mu) \mathcal{D}_{[Q\bar{Q}] \rightarrow \psi}(u, v, z, \mu) \approx \int dz H_{[Q\bar{Q}]}(\hat{p}_Q^+ = \frac{1}{2} p_c^+, \hat{p}_{\bar{Q}}^+ = \frac{1}{2} p_c^+, \mu) \underbrace{\int du dv \mathcal{D}_{[Q\bar{Q}] \rightarrow \psi}(u, v, z, \mu)}_{\equiv D_{[Q\bar{Q}] \rightarrow \psi}(z, \mu)}$$

$$\frac{\partial}{\partial \ln \mu^2} D_{[Q\bar{Q}(\kappa)] \rightarrow \psi}(z, \mu) \approx \sum_n \int_z^1 \frac{dz'}{z'} \int_0^1 du \int_0^1 dv \Gamma_{[Q\bar{Q}(n)] \rightarrow [Q\bar{Q}(\kappa)]} \left(u, v, u' = \frac{1}{2}, v' = \frac{1}{2}, \frac{z}{z'} \right) D_{[Q\bar{Q}(\kappa)] \rightarrow \psi}(z', \mu)$$

$$\frac{\partial}{\partial \ln \mu^2} D_{f \rightarrow \psi}(z, \mu) \approx \frac{\alpha_s}{2\pi} \sum_{f'} \int_z^1 \frac{dz'}{z'} P_{f \rightarrow f'}(z/z') D_{f' \rightarrow \psi}(z') + \frac{\alpha_s^2(\mu)}{\mu^2} \sum_{[Q\bar{Q}(\kappa)]} \int_z^1 \frac{dz'}{z'} P_{f \rightarrow [Q\bar{Q}(\kappa)]} \left(u' = \frac{1}{2}, v' = \frac{1}{2}, \frac{z}{z'} \right) D_{[Q\bar{Q}(\kappa)] \rightarrow \psi}(z', \mu)$$

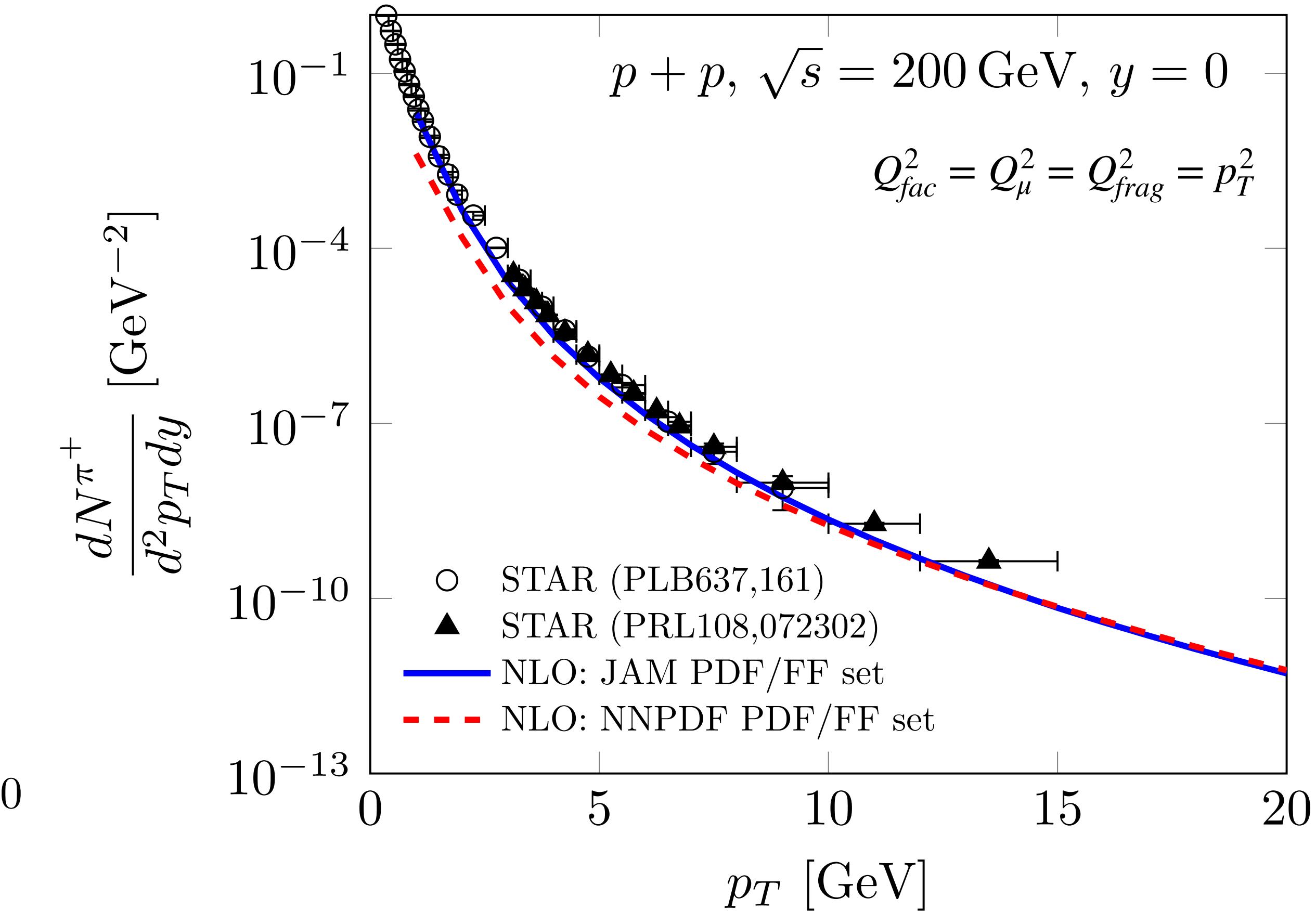
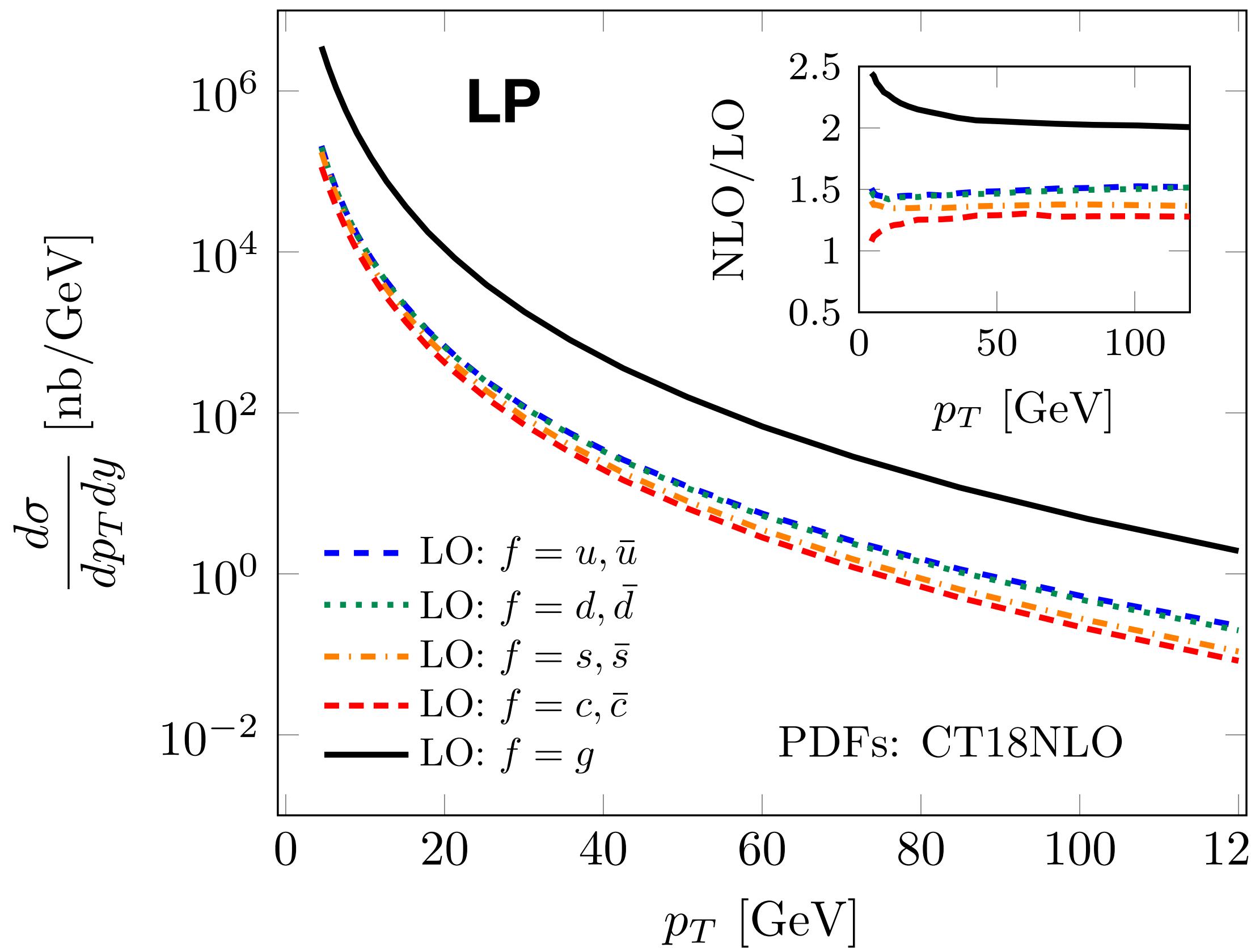
We solve these evolution equations in moment space numerically. Inverse Mellin transformation gets solutions in moment space back to those in position space.

Single parton production cross section

Preliminary results

Aversa, Chiappetta, Greco, Guillet, NPB327 (1989) 105

$p + p \rightarrow f + X, \sqrt{s} = 7 \text{ TeV}, y = 0$



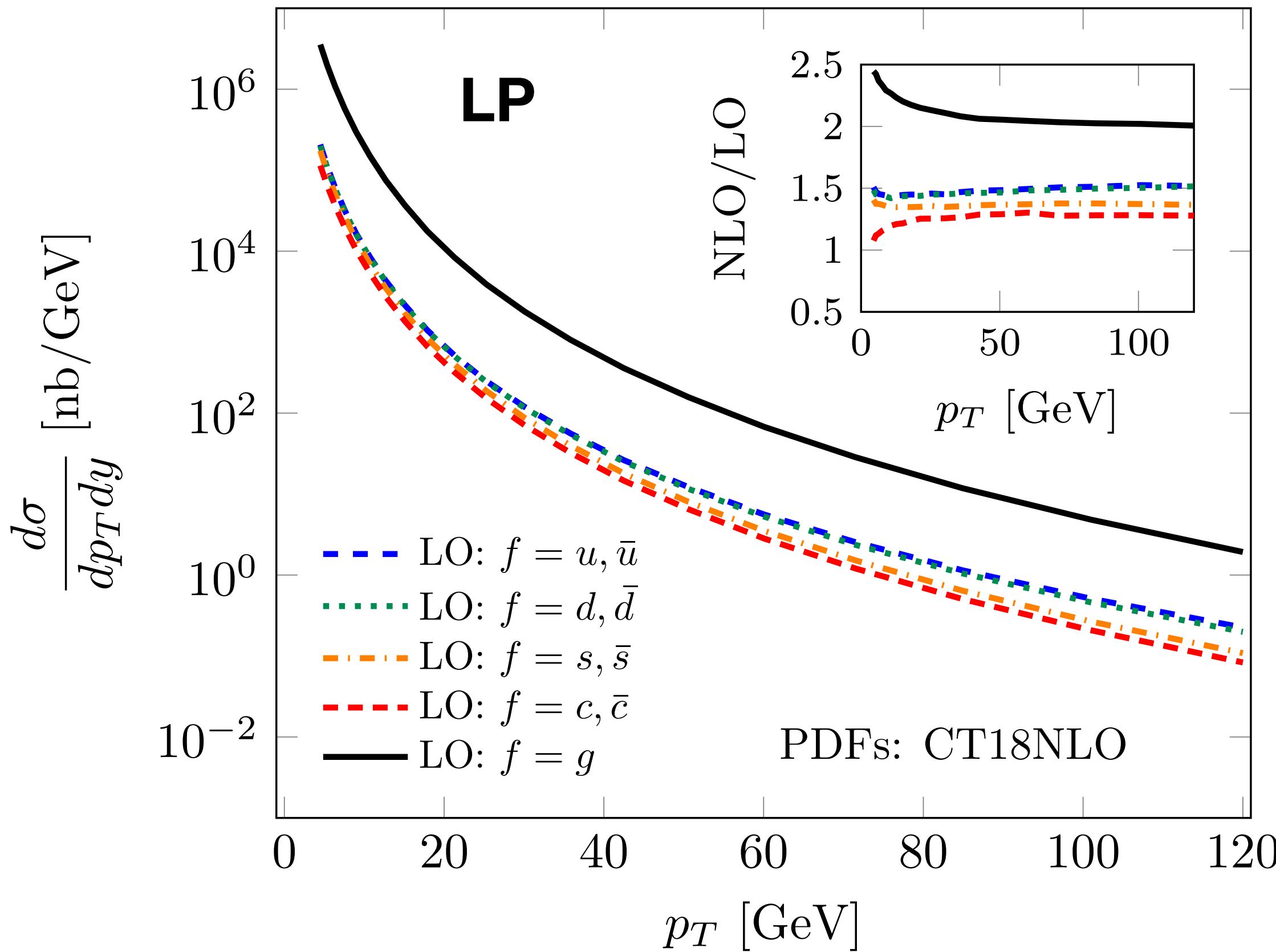
- For the single parton production, gluons are more produced than light and heavy quarks. $K = 1.5 \sim 2$.
- We can reproduce π^+ production yield in pp collisions at RHIC using the LP single parton cross section.

Power corrections at twist-4

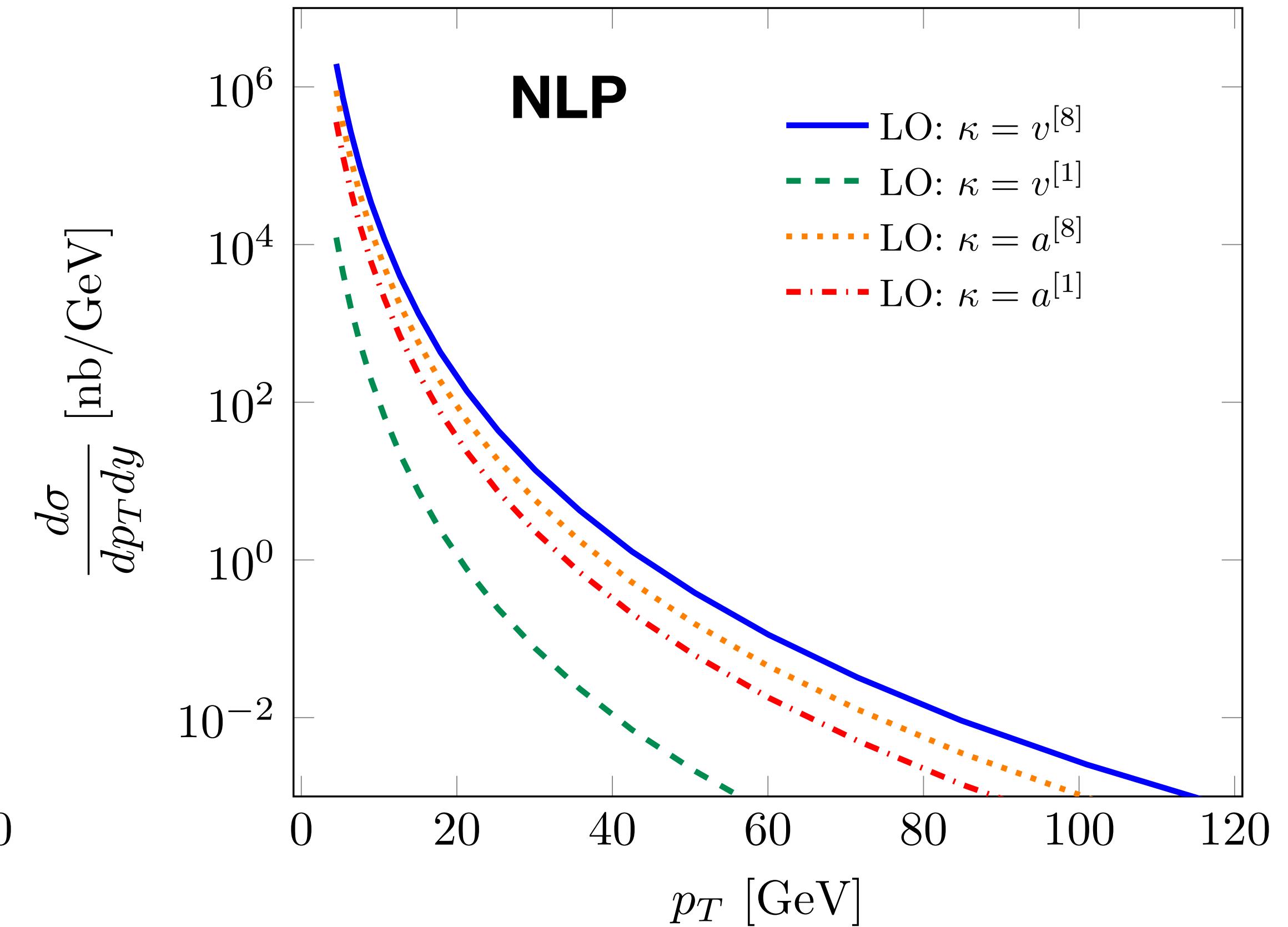
Preliminary results

Lee, Qiu, Sterman, KW, in preparation.

$p + p \rightarrow f + X, \sqrt{s} = 7 \text{ TeV}, y = 0$



$p + p \rightarrow c\bar{c}(\kappa) + X, \sqrt{s} = 7 \text{ TeV}, y = 0$

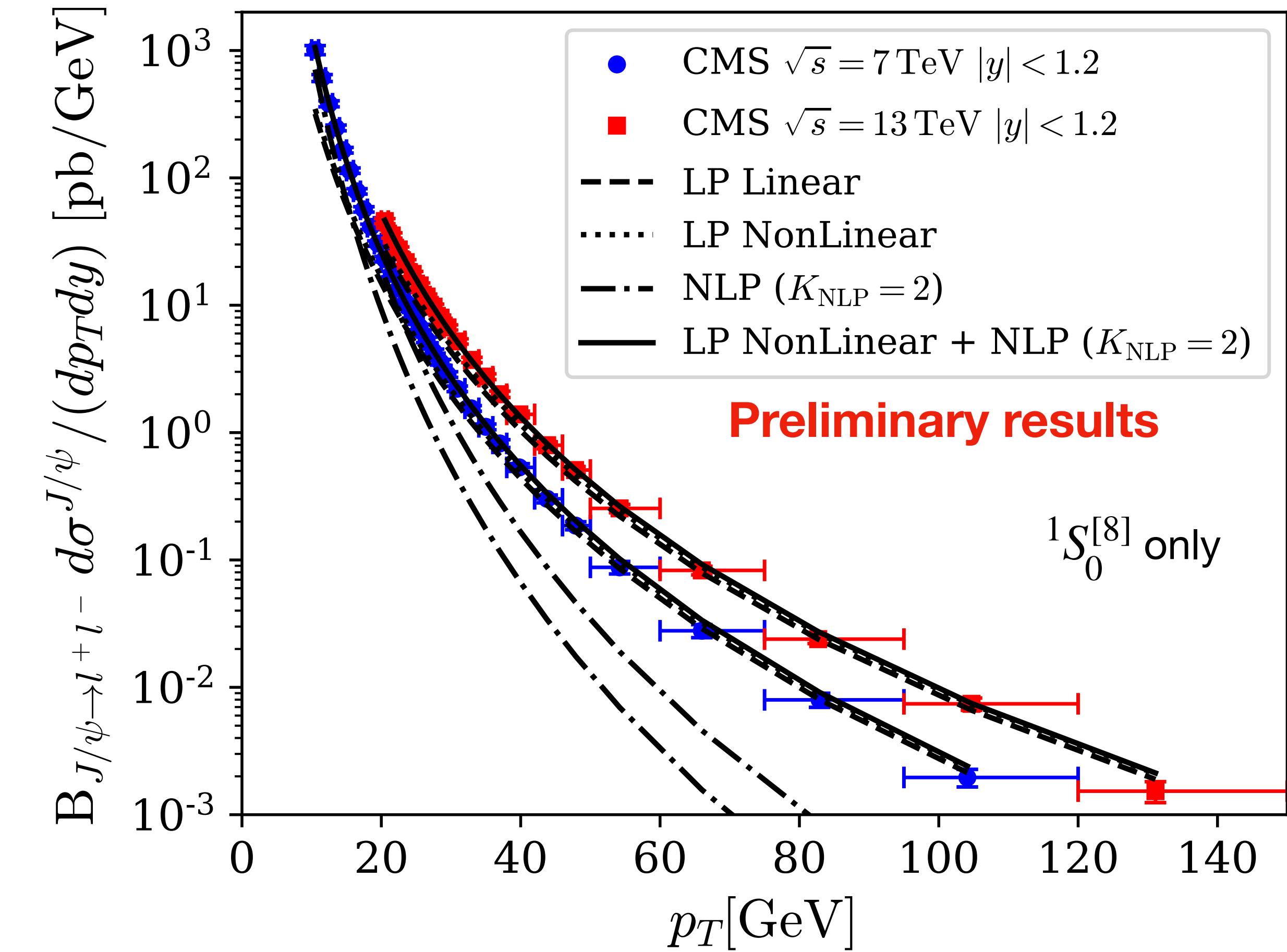


- Power corrections are suppressed by $1/p_T^2$ but sizable at moderate p_T ($p_T \sim \mathcal{O}(2m)$).
- For the double parton production, $v^{[8]}, a^{[8]}, a^{[1]}$ are important. The SDCs for $t^{[8]}$ and $t^{[1]}$ vanish.

p_T spectra of hadronic J/ψ production

Lee, Qiu, Sterman, KW, in preparation.

- $^1S_0^{[8]}$ is a dominant channel, yielding unpolarized J/ψ . Other channels do not contribute.
- NRQCD-LDMEs are overall normalization factors in the QCD factorization approach. $\langle \mathcal{O}_{^1S_0^{[8]}} \rangle$ fitted by high p_T data is similar to the one extracted using fixed order NRQCD: $\langle \mathcal{O}_{^1S_0^{[8]}}^{J/\psi} \rangle \sim 0.1 \text{ GeV}^3$.
- All SDCs are positive in the QCD factorization, while transverse components of SDCs in the NRQCD can be negative.
- $\text{LP} \gg \text{NLP}$ at high p_T : $p_\perp \gg 20 \text{ GeV}$
- $\text{NLP} \gtrsim \text{LP}$ at moderate p_T : $p_\perp \lesssim 10 \text{ GeV} = \mathcal{O}(2m)$
- High (low) p_T quarkonium is sensitive to large (small) $z = p_T/k_T$. Can improve the shapes of the FFs.



Part II: Quarkonium production at low p_T

Heavy quarkonium production of moderate and low p_T

- At moderate $p_T = \mathcal{O}(2m)$, heavy quark pair direct production via gluon fusion dominates; finite mass corrections in short distance coefficients are important → Fixed order NRQCD calculations!

At low p_T , $(2m)^2 \gg p_T^2 \gg \Lambda_{\text{QCD}}^2$, there is more phase space for gluons shower.

Soft approximation ($p_\perp = p_T$):

$$\frac{d\hat{\sigma}_{gg}^\kappa}{d^2p_\perp dy} = \hat{\sigma}_{0,gg}^\kappa \frac{C_A \alpha_s}{2\pi^2} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B f_A(\xi_A) f_B(\xi_B) \frac{1}{p_\perp^2} \left[z_B \hat{P}_{gg}^R(z_B) \delta(1 - z_A) + z_A \hat{P}_{gg}^R(z_A) \delta(1 - z_B) + 2 \ln \frac{M^2}{p_\perp^2} \delta(1 - z_A) \delta(1 - z_B) \right]$$

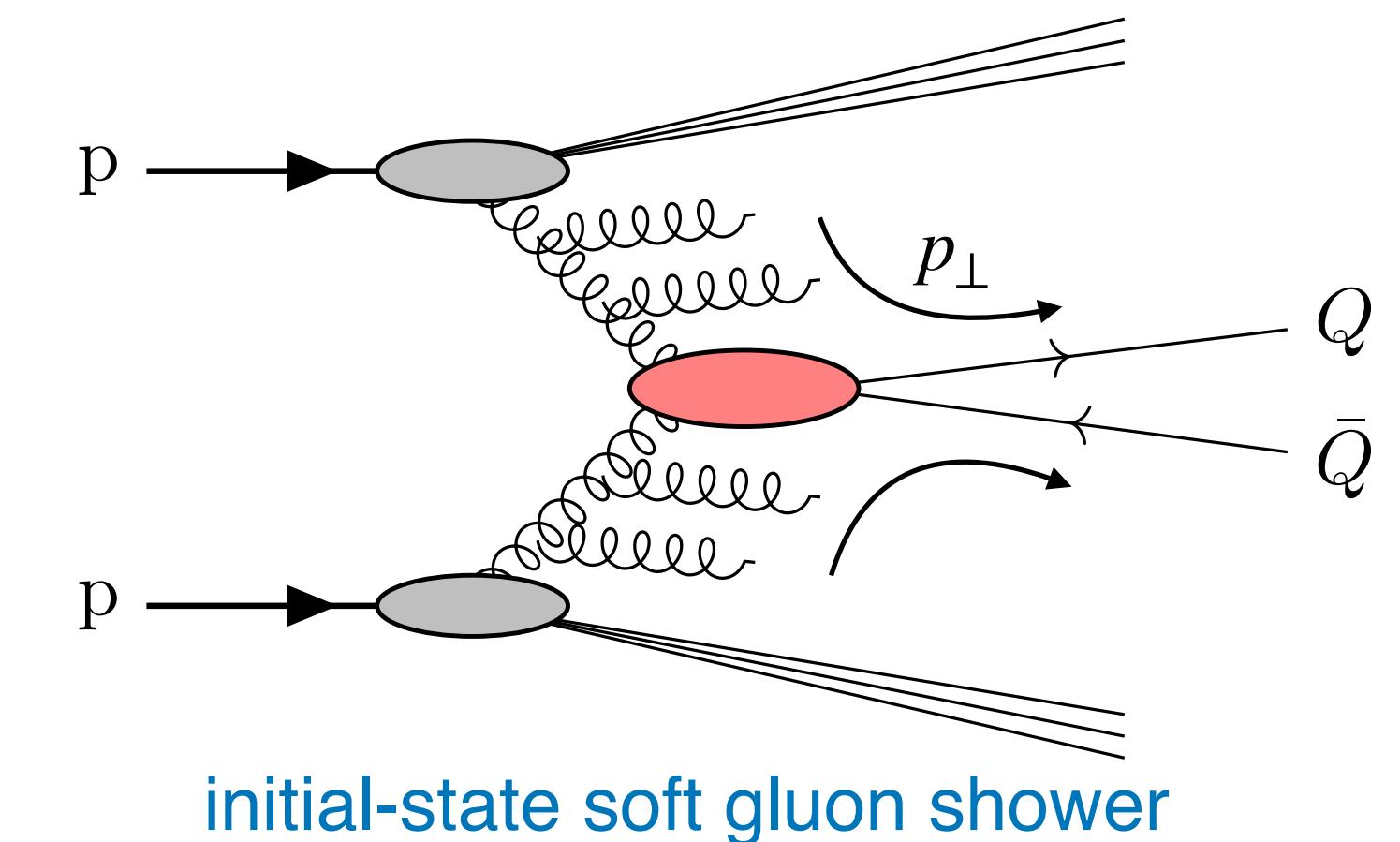
Collinear → DGLAP eq.	Collinear → DGLAP eq.	Soft and Collinear → Collins-Soper-Sterman eq.
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- Collins-Soper-Sterman (CSS) formalism:

$$\frac{d\sigma_{A+B \rightarrow Q\bar{Q}+X}}{d^2 p_\perp^2 dy} \Big|_{\text{resum}} = \int \frac{db_\perp}{2\pi} J_0(P_\perp b_\perp) \left(\sum_q b_\perp W_{q\bar{q}} d\hat{\sigma}_{q\bar{q} \rightarrow Q\bar{Q}} + b_\perp W_{gg} d\hat{\sigma}_{gg \rightarrow Q\bar{Q}} \right)$$

$$W_{ij}(b_\perp, M, x_A, x_B) = W_{ij}^{perp}(b_{max}, M, x_A, x_B) F_{ij}^{NP}(b_\perp, M, x_A, x_B; b_{max})$$

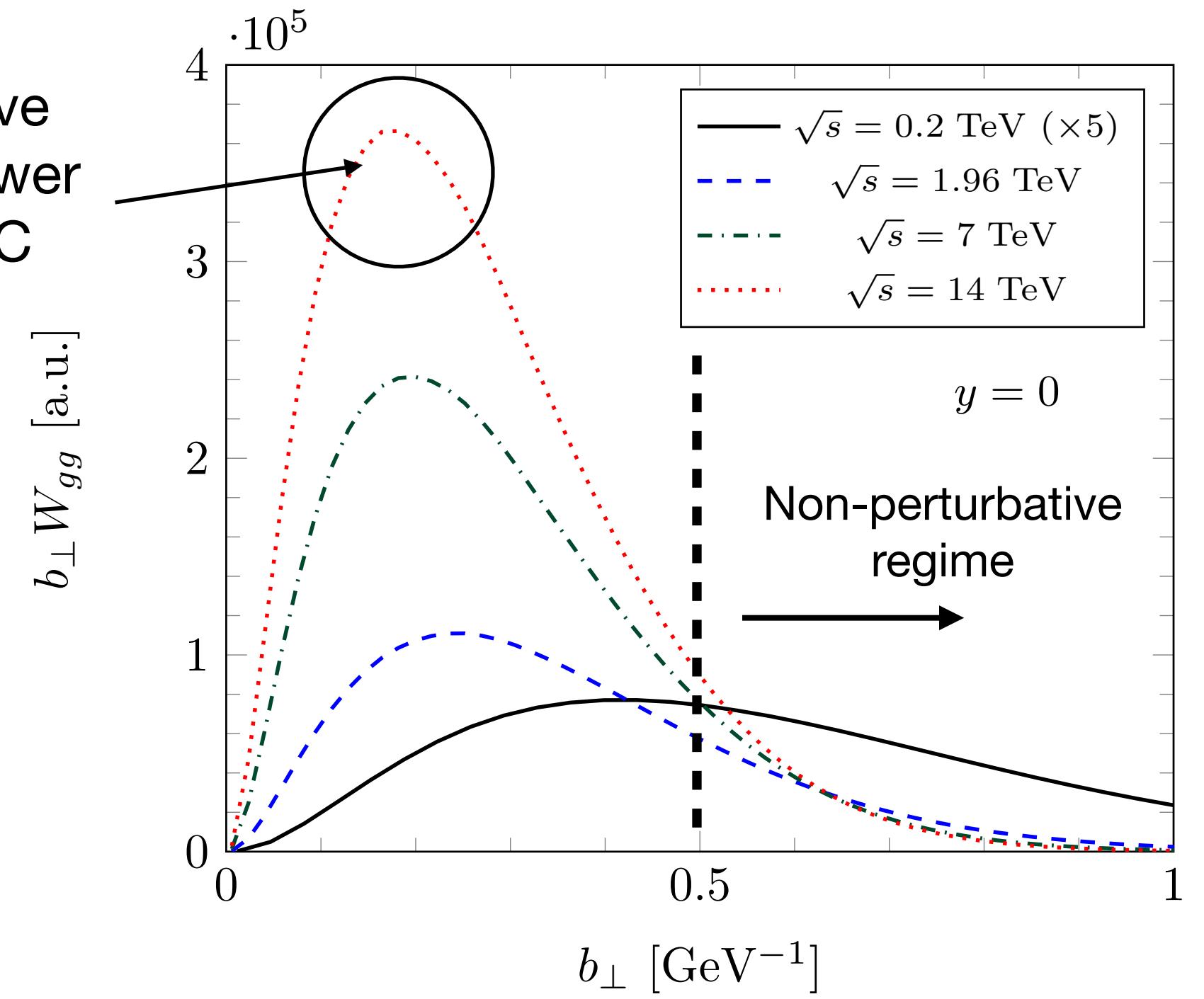
$$W_{ij}^{perp}(b_\perp, M, x_A, x_B) = e^{-S_{ij}(b_\perp, M)} f_{i/A} \left(x_A, \mu, \frac{c_0}{b_\perp} \right) f_{j/B} \left(x_B, \mu, \frac{c_0}{b_\perp} \right)$$



TMD factorization approach

Qiu, KW, PoS QCDEV2017 (2017) 024

Perturbative
gluons shower
at the LHC



Berger, Qiu, Wang, PRD71 (2005) 034007

Sun, Yuan, Yuan, PRD88 (2013) 054008

KW, Xiao, PRD92 (2015) 11, 111502

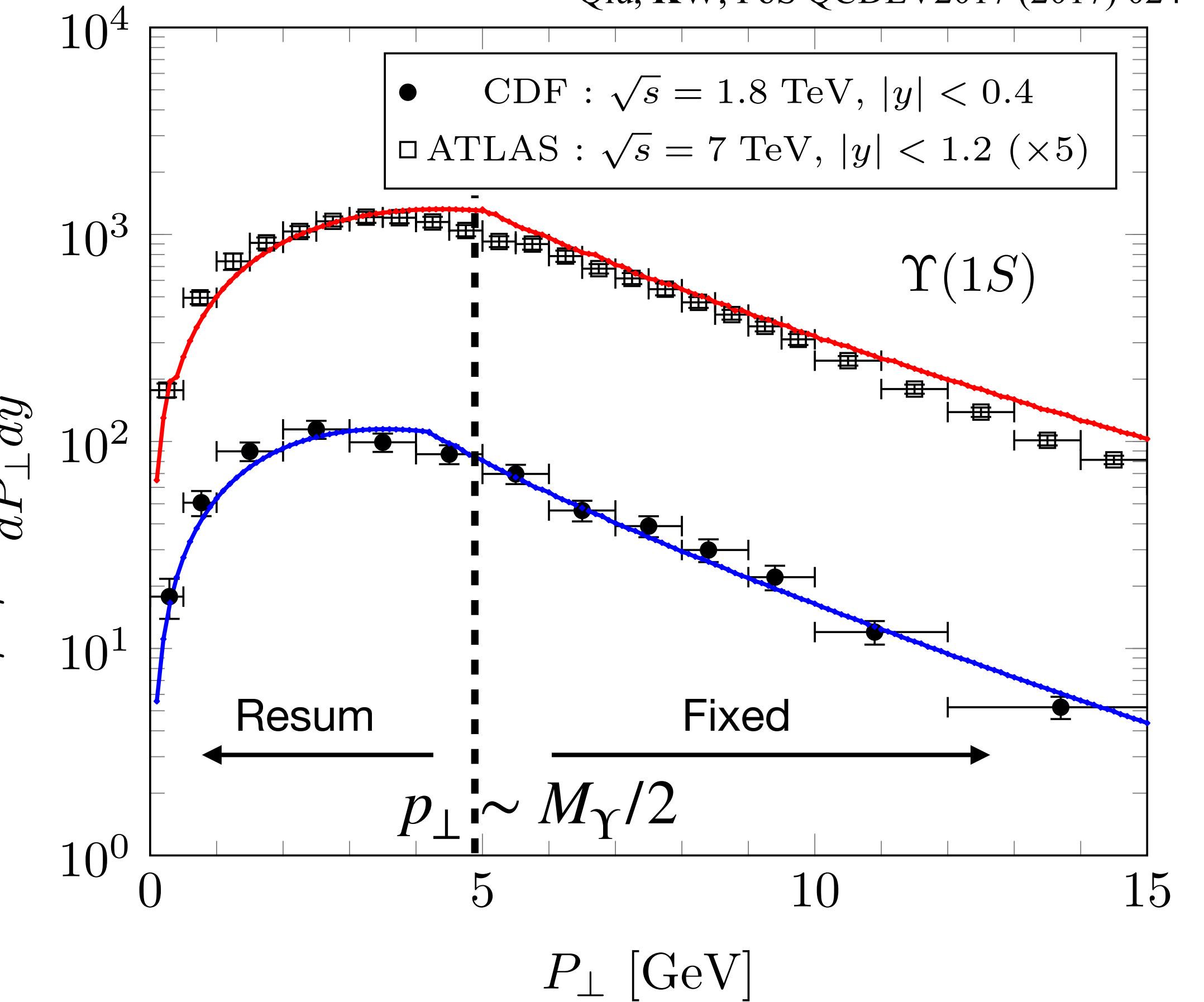
...

- Matching condition:

$$\frac{d\sigma_{A+B \rightarrow Q\bar{Q}+X}}{d^2 p_\perp dy} = \frac{d\sigma_{A+B \rightarrow Q\bar{Q}+X}}{d^2 p_\perp dy} \Big|_{\text{resum}}$$

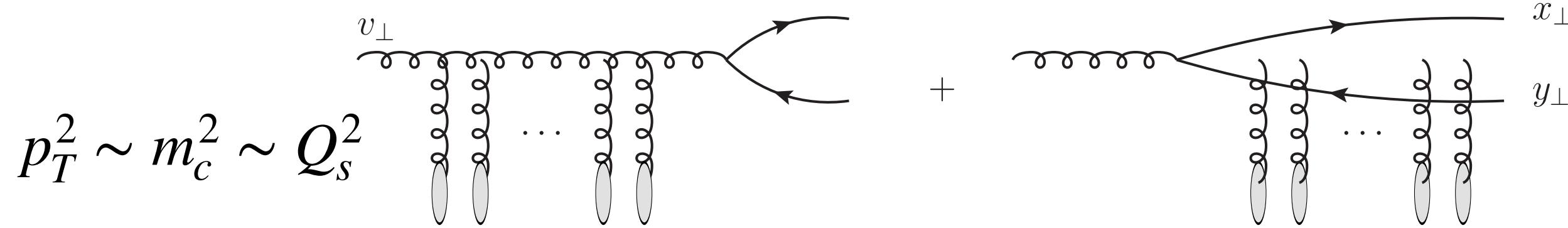
$$+ \left[\frac{d\sigma_{A+B \rightarrow Q\bar{Q}+X}}{d^2 p_\perp dy} \Big|_{\text{pert}} - \frac{d\sigma_{A+B \rightarrow Q\bar{Q}+X}}{d^2 p_\perp dy} \Big|_{\text{asym}} \right] \text{Y-term}$$

- J/ψ production is sensitive to Non-perturbative Sudakov form factor.



Forward quarkonium production

- High \sqrt{s} , large y , low p_T lead to $x_2 \ll 1 \rightarrow \alpha_s \ln 1/x_2 = \mathcal{O}(1)$



- Phase rotation: interactions with semi-classical fields.

$$U(x_\perp) \equiv \mathcal{P}_+ \exp \left[ig \int dx^+ t^a A_a^-(x^+, x_\perp) \right] = 1 + ig \int dx^+ t^a A_a^-(x^+, x_\perp) + \dots$$

- Resummation of coherent multiple scattering.

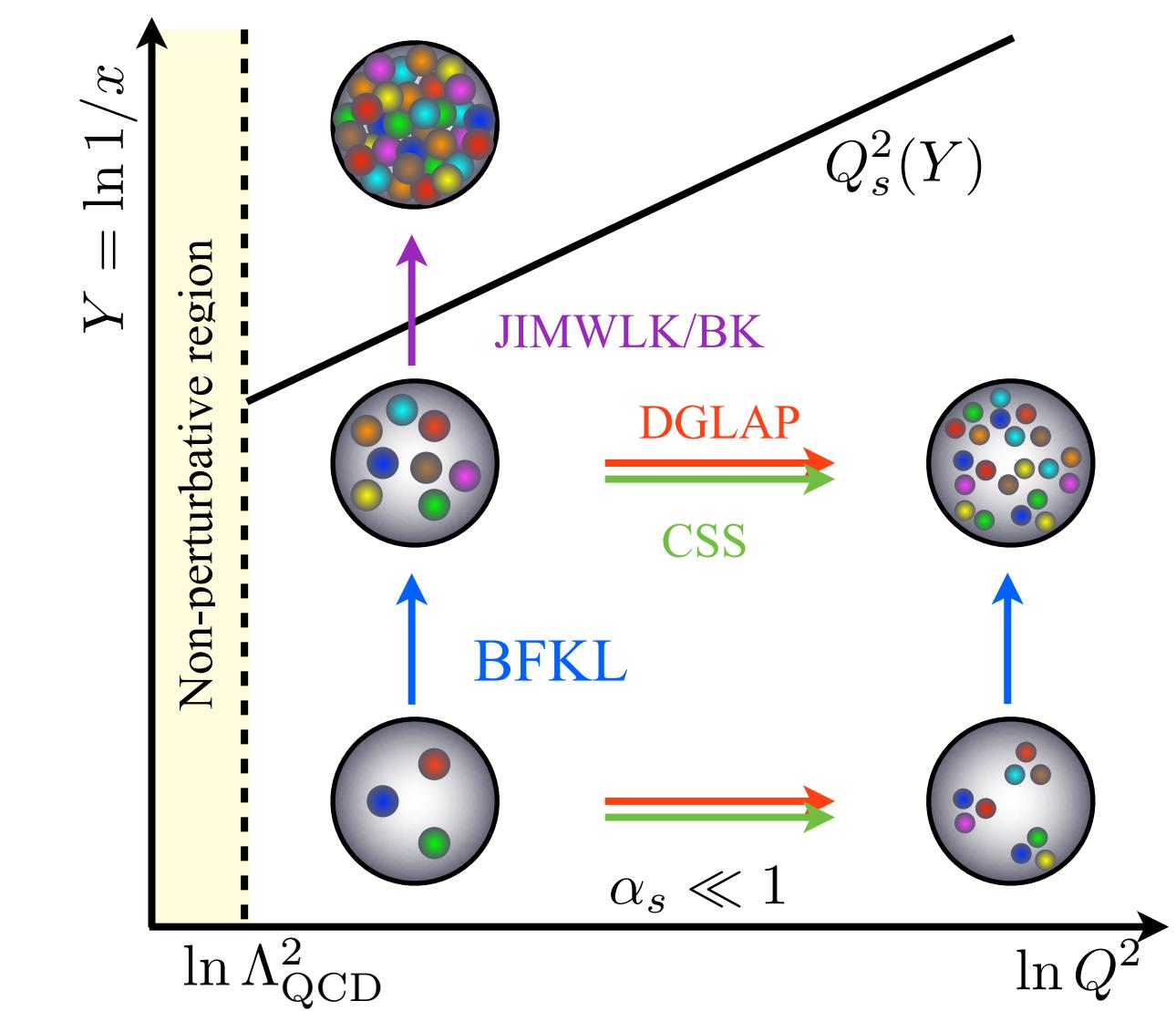
- TMD + genuine & kinematic twists = Color-Glass-Condensate (CGC)

Altinoluk, Boussarie, Kotko, JHEP 05 (2019) 156, Altinoluk, Boussarie, JHEP 10 (2019) 208
Fujii, Marquet, KW, JHEP12(2020)181, ...

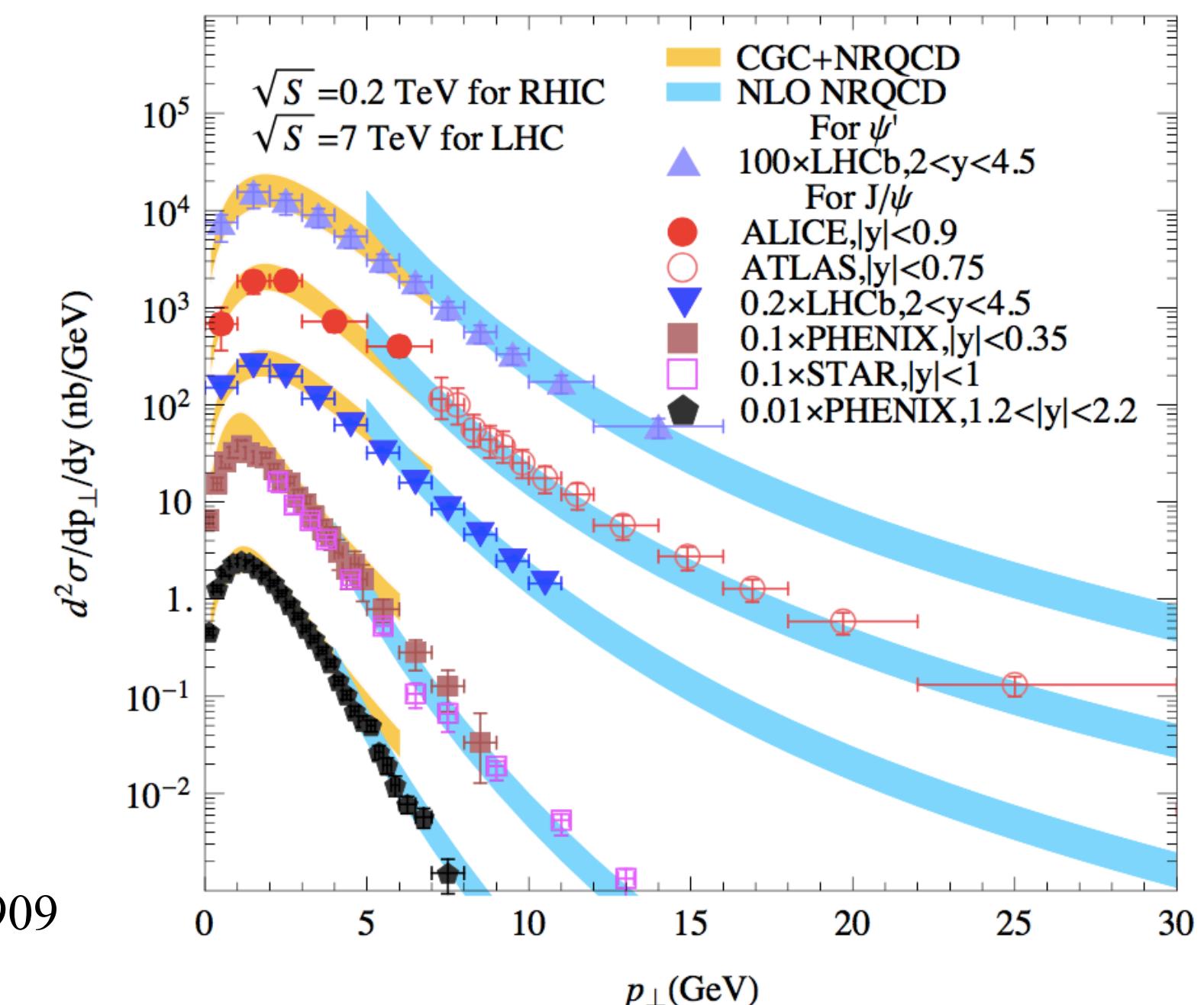
- The scale evolution of semi-hard saturation scale Q_s is controlled by the BK-JIMWLK evolution eq.
- The CGC framework gives a good parametrization of the gluon TMD distribution function at small-x.

KW, Xiao, PRD92 (2015) 11, 111502

Ma, Venugopalan, KW, Zhang, PRC97 (2018) 1, 014909



Ma, Venugopalan, PRL113 (2014) 19, 192301



Multiplicity dependence

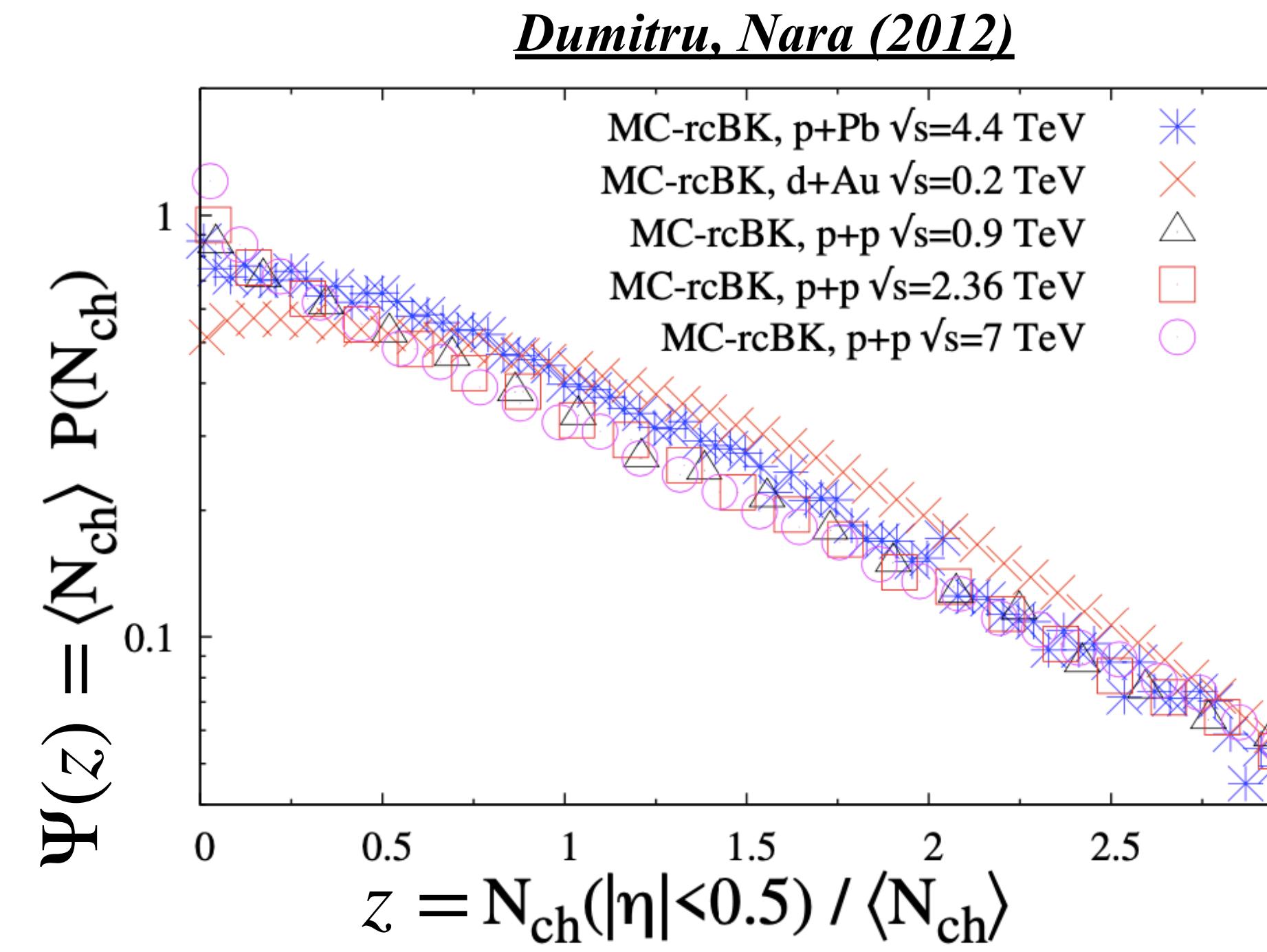
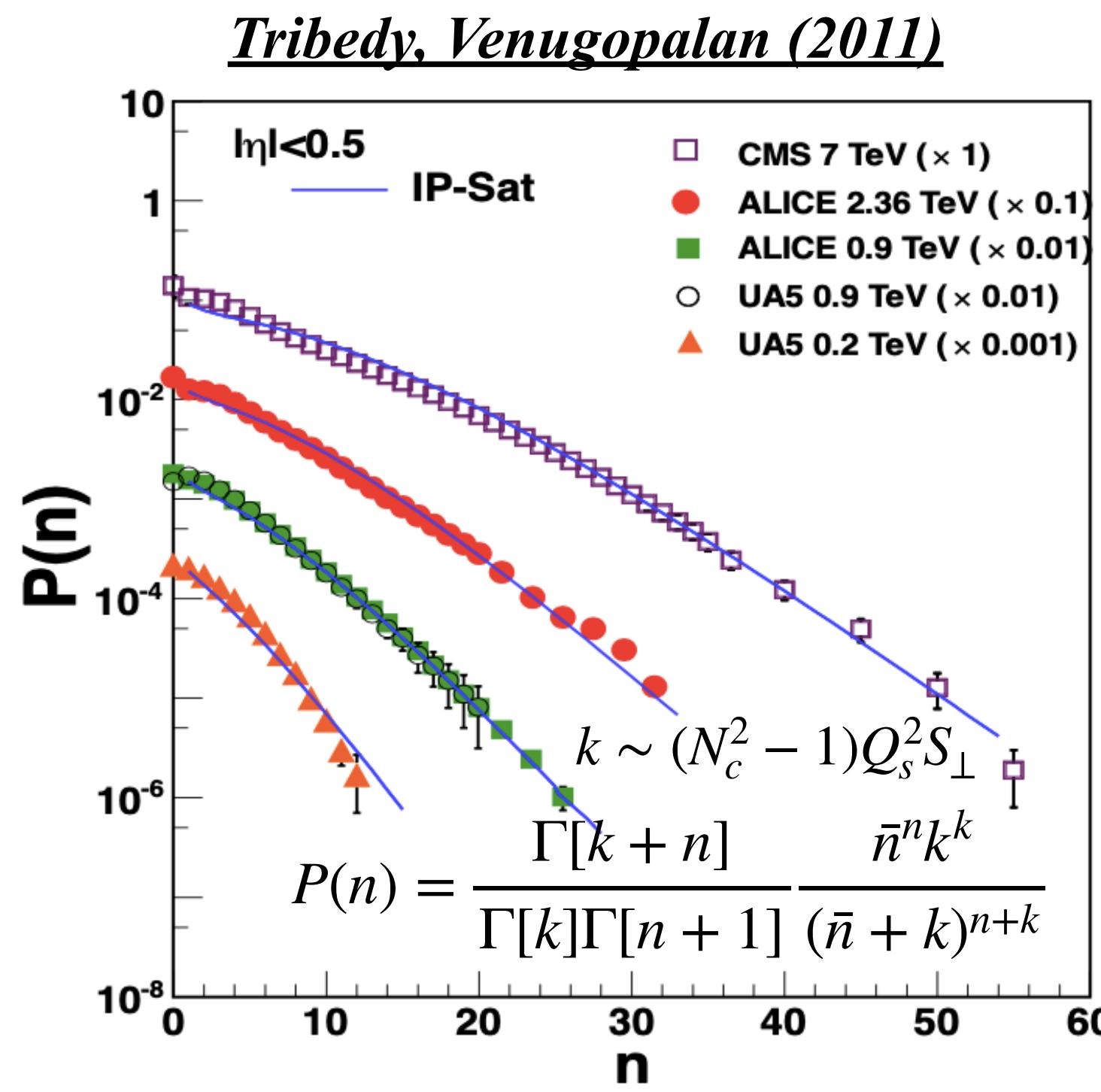
Color Glass Condensate EFT (weak coupling theory) is a robust tool to study bulk properties in hadronic and heavy-ion collisions: $\alpha_s(Q_s) < 1$ when $Q_s \gg \Lambda_{\text{QCD}}$. Qualitative estimate in a dilute-dense system A+B (p+A) or dense system (p+p or A+A) at forward rapidity:

$$\frac{dN_{ch}}{d\eta} \propto S_\perp Q_{min}^2 \sim N_{\text{part}}$$

$$Q_{min} = \min . \{ Q_{s,A}, Q_{s,B} \} \quad \text{and} \quad Q_{max} = \max . \{ Q_{s,A}, Q_{s,B} \}$$

Dumitru, McLerran (2001)

Kharzeev, Nardi (2000)



- KNO scaling was found in pp/ppbar collisions at $\sqrt{s_{NN}} \lesssim 200 \text{ GeV}$ UA5 (1986)
- If KNO scaling breaks down, soft multiple particle interactions could play an essential role.
- In the CGC framework, we can see KNO scaling in pp collisions even at the LHC up to $n/\bar{n} \leq 3$, but not in pA collisions: KNO scaling is weakly violated.
- The CGC gives semihard multiple interactions.

Particle production from Glasma flux tubes gives negative binomial distribution.

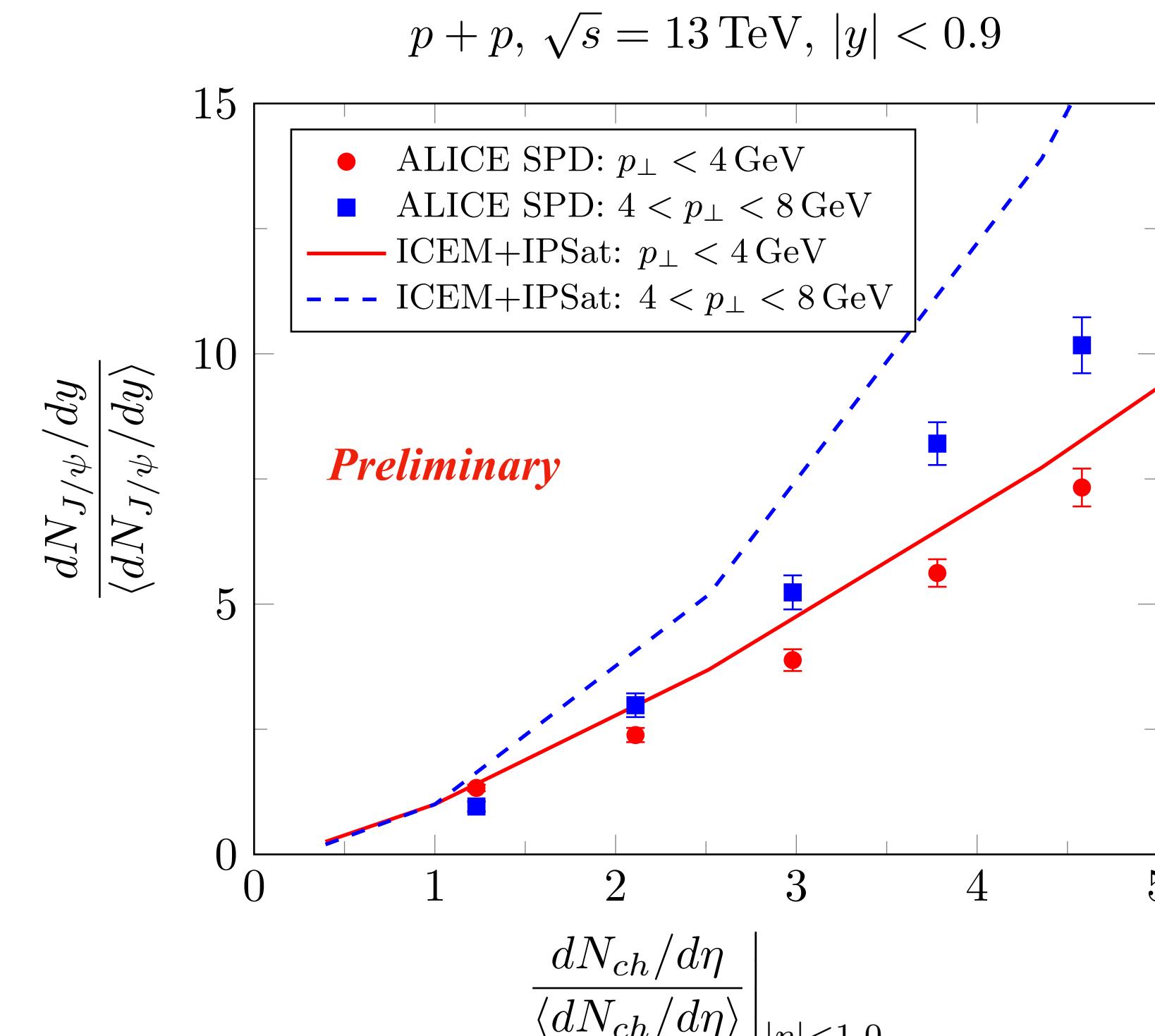
Gelis, Lappi, McLerran (2009)

Quarkonium production with bulk particles

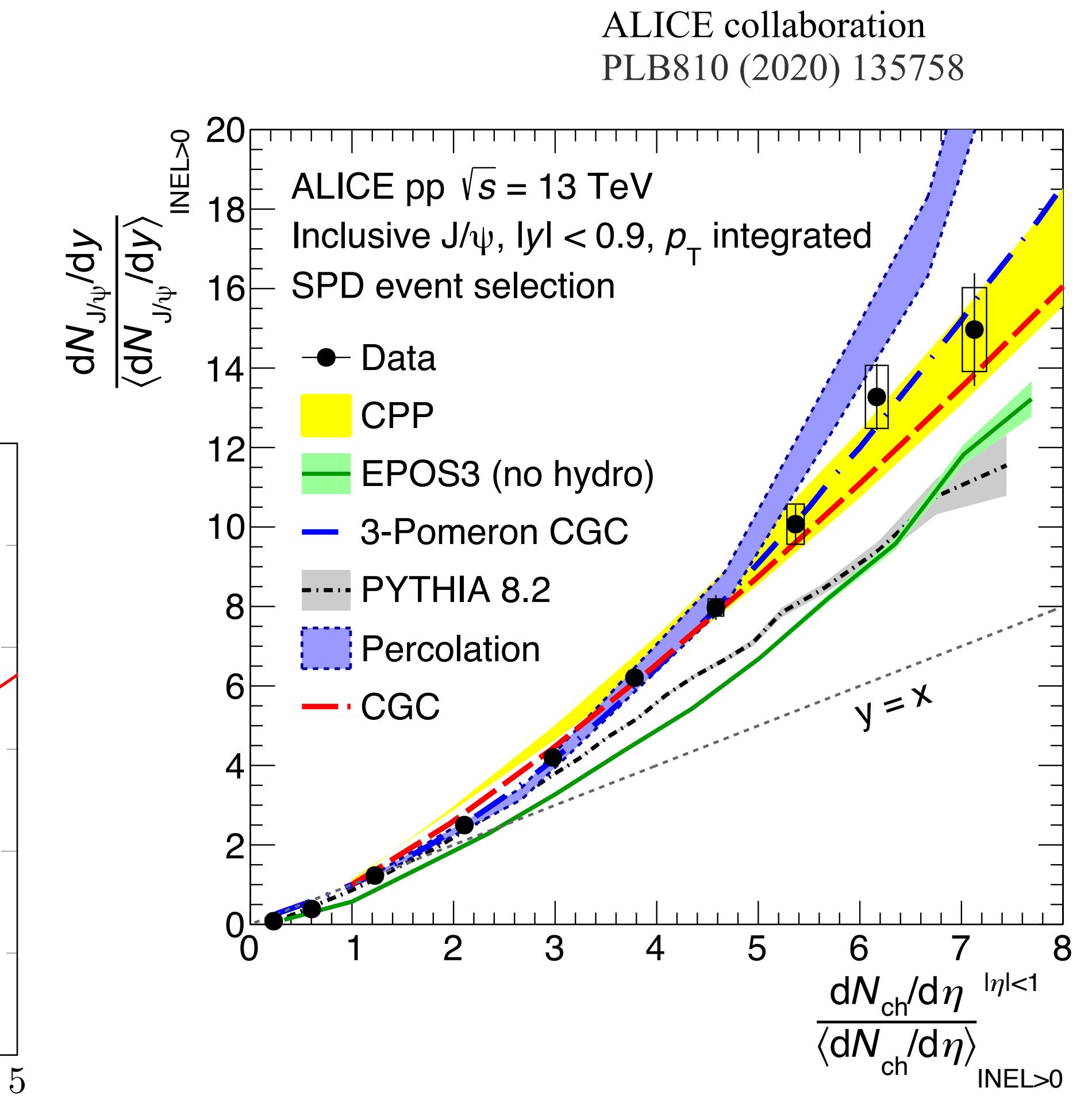
- High multiplicity events $N_{ch} \gg \langle N_{ch} \rangle$ reflect rare parton configurations inside hadrons: anisotropic, color domains.
- Phenomenological implementation in the CGC approach:
 $Q_{s,p} = c Q_0^2, c \geq 1.$

Issues:

1. LDMEs dependence: each of intermediate states for the quark pair shows different p_T spectrum.
 2. Rapidity dependence: J/ψ and charged hadron are produced in the same rapidity bins or with rapidity separation.
 3. NLO corrections in the CGC framework: important at high p_T
- ...



Stebel, Venugopalan, **KW**, in preparation



Theoretical approaches:

- Ferreiro, Pajares, PRC 86 (2012) 034903
Siddikov, Levin, Schmidt, Eur.Phys.J.C 80 (2020) 6, 560
Ma, Tribedy, Venugopalan, **KW**, PRD98 (2018) 7, 074025
...

Discussion: Toward p+A collisions.

1. High and moderate p_T quarkonium:

- High p_T quarkonium is sensitive to large- z and moderate- x : a valuable probe to constrain gluon nPDF at moderate- x .

$$x_1 = x_2 = x \sim \frac{2P_T/z}{\sqrt{s}} \sim 0.05 - 0.1 @ P_T = 300 \text{ GeV}, \sqrt{s} = 13 \text{ TeV} \text{ at mid rap}$$

- Nuclear dependence in nonperturbative Sudakov factor: unknown yet.

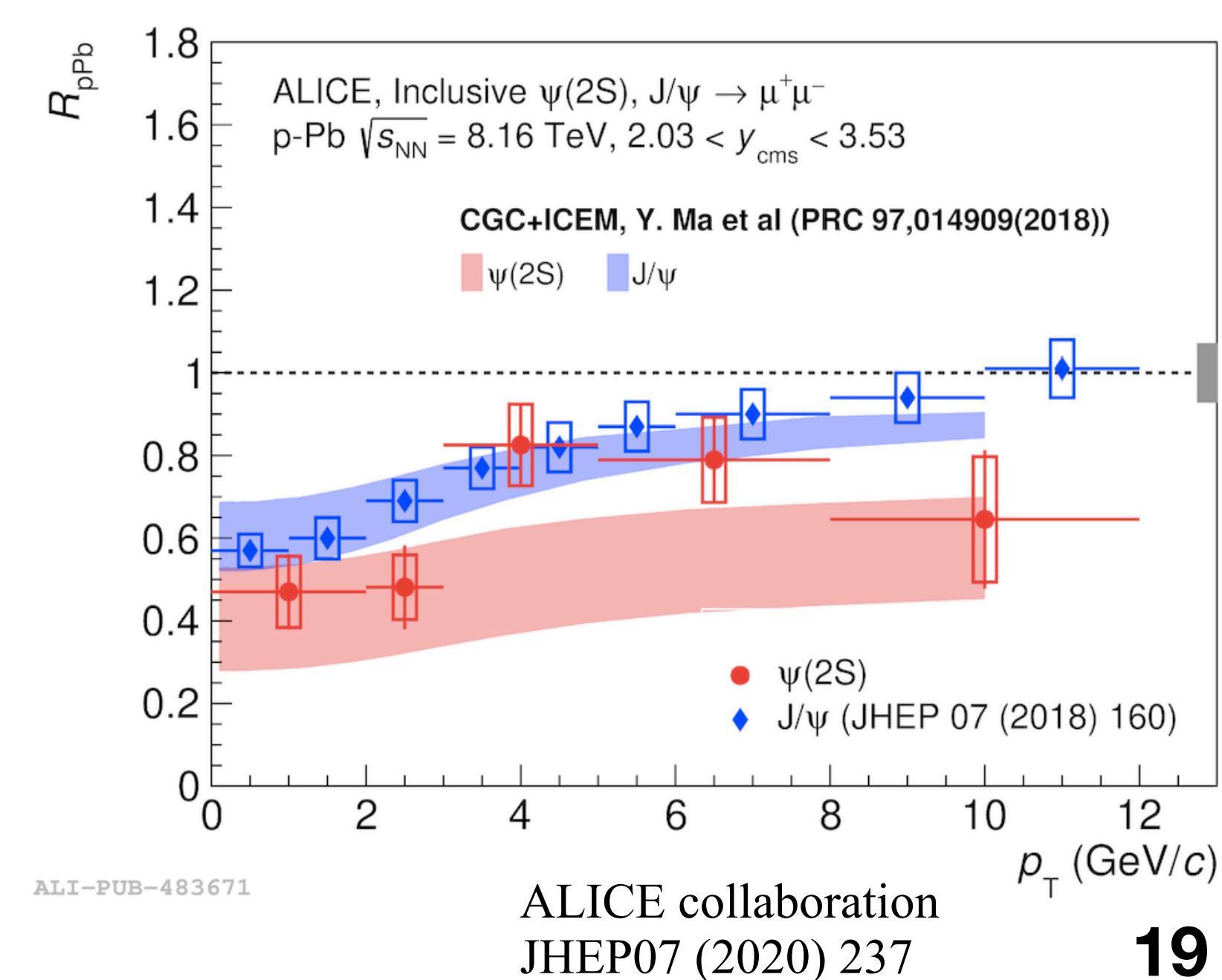
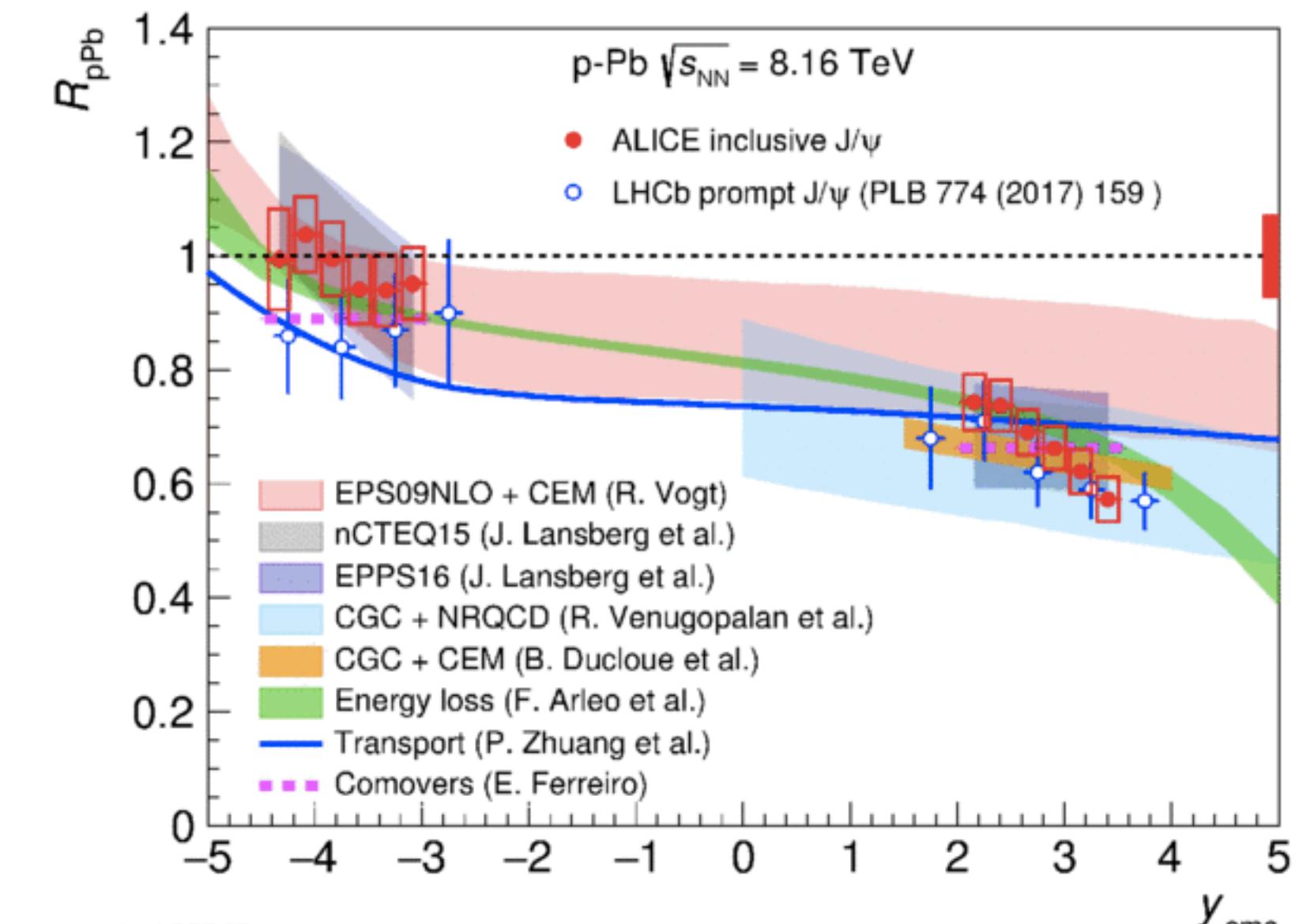
2. Low p_T quarkonium:

- Multiple-rescattering effects yield higher twist correlation functions.
- CGC: Nuclear enhanced saturation scale.
- Induced soft gluon radiations yield energy loss.

3. Additional nuclear dependent effects:

- interactions with beam remnants, comovers, or possible QGP like medium in final state.

Thank you!



ALICE collaboration
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backup

Input FFs

Perturbative SDCs of input FFs in α_s and v expansion in the NRQCD are reliable only when SDCs $\ll \mathcal{O}(1)$.

Indeed, the NRQCD factorization is not reliable as $z \rightarrow 1$ where SDCs $\hat{d}(z)$ include the following terms:

1. $\delta(1 - z)$ at LO in α_s expansion
2. $f(z)\ln(1 - z)$ with $f(z)$ being a regular function
3. $\frac{f(z)}{[1 - z]_+}$, $f(z)\left[\frac{\ln(1 - z)}{1 - z}\right]_+$ due to the perturbative cancelation of IR divergences

- For example, the transition probability $P(g \rightarrow \psi)$ should vanish at $z = 1$, but not $\delta(1 - z)$. A color octet gluon needs some phase to radiate soft particles to help neutralize the color of the quark pair, so we mimic $\delta(1 - z)$ to obtain $P(z \rightarrow 1) = 0$. This is the same for $Q\bar{Q} \rightarrow \psi$ channels, which have similar terms. We shall use an area under curve (first moment) to give normalization:

$$D^{[\delta]}(z) = C(\alpha_s)\delta(1 - z) \rightarrow C(\alpha_s) \frac{z^\alpha(1 - z)^\beta}{B[1 + \alpha, 1 + \beta]} \quad (\alpha \gg 1, 1 > \beta > 0)$$

- Even though we have $\ln(1 - z)$ -type unphysical divergence, we can use its first comment as we do for δ -function terms.

$$f(z)\ln(1 - z) \rightarrow \frac{z^\alpha(1 - z)^\beta}{B[1 + \alpha, 1 + \beta]} \int_0^1 dz f(z)\ln(1 - z)$$

- Plus-functions are defined only under integration over z , so we add their first moments to input FFs as we do for the $\delta(1 - z)$ term. If their first moments are negative, we simply take their absolute values.
- We apply the above treatments to both single and double parton input FFs for any channels.

The QCD evolution of FFs at twist-2 and twist-4

A choice for input parameters:

$$\alpha_{\text{LP}} = \alpha_{\text{NLP}} = 30$$

$$\beta_{\text{LP}} = \beta_{\text{NLP}} = 0.5$$

$$\mu_0 = 4m = 6 \text{ GeV}, \mu_\Lambda = m = 1.5 \text{ GeV}$$

