Scalar potentials beyond tree-level

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Outline

Gauge theories and spontaneous symmetry breaking

- Gauge theories at the center of our current understanding of particle physics

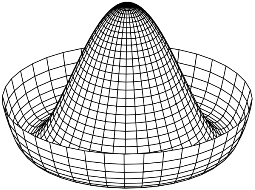
  Standard Model: $SU(3)_C \times SU(2)_L \times U(1)_Y$

- Gauge bosons can only acquire mass trough spontaneous symmetry breaking

  Standard Model: $SU(3)_C \times SU(2)_L \times U(1)_Y \longrightarrow SU(3)_C \times U(1)_Q$

- Quantities not transforming under $SO(1,3)$ (= scalars !) must acquire a symmetry-breaking vacuum expectation value

  Standard Model: $\langle H^\dagger H \rangle = \frac{1}{2} v^2 \approx \frac{1}{2}\,(246\,\mathrm{GeV})^2$

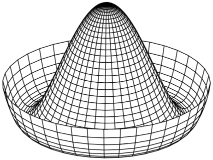


Higgs mechanism in the Standard Model

The standard point of view:

$$V(H) = -\mu^2 H^\dagger H + \lambda \left(H^\dagger H\right)^2$$

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \rightarrow \quad v^2 = \frac{\mu^2}{\lambda} = 246\,\mathrm{GeV}, \quad m_h = \sqrt{2\lambda v^2} = 125\,\mathrm{GeV}$$



- Gauge and fermion sectors experimentally well probed, but not the scalar sector

  $\rightarrow$ Theoretical & phenomenological exploration of scalar potentials and symmetry breaking is essential, in the SM and beyond.

Tree-level potentials

- In a general gauge theory with a (reducible in general) scalar multiplet ϕ, the tree-level potential takes the form

$$V(\phi) = \frac{1}{2}\mu_{ab}\phi^a\phi^b + \frac{1}{3!}\tau_{abc}\phi^a\phi^b\phi^c + \frac{1}{4!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi^d$$

- Assuming that the potential is bounded from below, the global minimum is found by solving the stationary point equations

$$\frac{\partial V}{\partial \phi^a} \equiv \partial_a V = \mu_{ab}\phi^b + \frac{1}{2}\tau_{abc}\phi^b\phi^c + \frac{1}{3!}\lambda_{abcd}\phi^b\phi^c\phi^d = 0$$

- At the vacuum $\langle\phi\rangle$, "physical" quantities may be computed, starting with the scalar spectrum

$$M_{ab} = \partial_a\partial_b V(\langle\phi\rangle) = \mu_{ab} + \tau_{abc}\langle\phi^c\rangle + \frac{1}{2}\lambda_{abcd}\langle\phi^c\rangle\langle\phi^d\rangle$$

Tree-level potentials

- And the trilinear and quartic couplings

$$\partial_a \partial_b \partial_c V(\langle\phi\rangle) = \tau_{abc} + \lambda_{abcd}\langle\phi^d\rangle$$
$$\partial_a \partial_b \partial_c \partial_d V(\langle\phi\rangle) = \lambda_{abcd}$$

- The vacuum structure of the scalar potential is tightly connected to the gauge sector through the Goldstone mechanism

$$M_{ab}(T^\lambda)_{ab}\,\langle\phi^b\rangle = 0 \quad \text{for all broken } T^\lambda$$

- The "would-be" Golstone modes $T^\lambda\phi$ become longitudinal modes of the gauge bosons V_μ^λ

Stability of tree-level potentials

- For a minimum to be global (*i.e.* corresponding to a true vacuum), the potential must be bounded from below (= stable).
- Stability is asserted by looking at the scale-invariant sector of the potential:

$$V_{\rm SI} = \lambda_{abcd}\phi^a\phi^b\phi^c\phi^d > 0, \quad \forall\phi$$

- The derivation of stability conditions is in general highly non trivial. Some simple examples:
  - One-field potential:

$$V = \lambda\phi^4 \quad \rightarrow \quad \lambda > 0$$

  - Two-field biquadratic potential:

$$V = \lambda_1\phi_1^4 + \lambda_{12}\phi_1^2\phi_2^2 + \lambda_2\phi_2^4 \quad \rightarrow \quad \lambda_1 > 0 \wedge \lambda_2 > 0 \wedge \lambda_{12} + 2\sqrt{\lambda_1\lambda_2} > 0$$

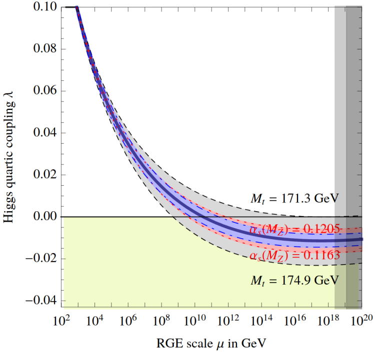
  - Two-field general potential:

$$V = \lambda_1\phi_1^4 + \sigma_1\phi_1^3\phi_2 + \lambda_{12}\phi_1^2\phi_2^2 + \sigma_2\phi_1\phi_2^3 + \lambda_2\phi_2^4 \quad \rightarrow \quad \text{Quite involved conditions!}$$

[K. Kannike, 1603.02680]

Stability of tree-level potentials

Stability across energy scales: example of the Standard Model



[G. Degrassi et al., 1205.6497]

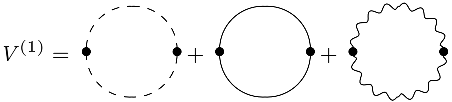
- Asserting stability beyond a unique RG-scale necessitates to go beyond tree-level
- In particular, the naive criterion of tree-level stability at all scales is erroneous
- One can resort to RG-improvement methods (see later slides)

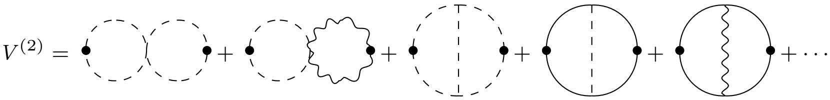
The effective potential: introduction

- V^{eff} is the quantum-corrected version of the scalar potential.
- In perturbation theory,

$$V^{\text{eff}} = V^{(0)} + \hbar V^{(1)} + \hbar^2 V^{(2)} + \cdots$$

- It is defined as the zero-momentum limit of the 1PI effective action.
- Its derivation can be performed in the path integral formalism, or by direct diagrammatic evaluation

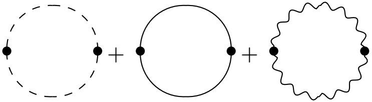




The effective potential: analytic expression

- General expression of $V^{(n)}$ is known up to 3-loop in the $\overline{\mathrm{MS}}$ scheme + landau gauge (Stephen Martin's works, *e.g.* [hep-ph/0111209], [1709.02397])
- One-loop contributions in the $\overline{\mathrm{MS}}$ scheme + landau gauge:

$$V^{(1)} = \frac{1}{16\pi^2}\frac{1}{4}\left\{\mathrm{Tr}\left[M_s^4\left(\log\frac{M_s^2}{\mu^2}-\frac{3}{2}\right)\right] - 2\mathrm{Tr}\left[M_f^4\left(\log\frac{M_f^2}{\mu^2}-\frac{3}{2}\right)\right] + 3\mathrm{Tr}\left[M_g^4\left(\log\frac{M_g^2}{\mu^2}-\frac{5}{6}\right)\right]\right\}$$



Field-dependent mass matrices:

$$\left(M_s^2\right)_{ab} = \partial_a\partial_b V^{(0)}, \quad \left(M_g^2\right)_{AB} = g_A g_B \left\{T_A, T_B\right\}_{ab} \phi^a\phi^b, \quad M_f^2 \equiv M_f^\dagger M_f$$

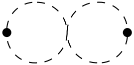
The effective potential: analytic expression

- Beyond 1-loop, the perturbative structure looks like:

$$V^{(n)} \sim \left(\frac{1}{16\pi^2}\right)^n \left[L_0(m_i) + L_1(m_i)\log\frac{m_i^2}{\mu^2} + \cdots + L_n(m_i)\left(\log\frac{m_i^2}{\mu^2}\right)^n\right]$$

- For instance, at 2-loop:

$$V^{(2)} \supset$$



$$= \frac{1}{(16\pi^2)^2}\frac{1}{8}\lambda_{iijj} m_i^2 \left(\log\frac{m_i^2}{\mu^2} - 1\right) m_j^2 \left(\log\frac{m_j^2}{\mu^2} - 1\right)$$

Important remark (see next slides):

- The leading-log contribution in $V^{(n)}$ may be computed from the leading-log contribution in $V^{(n-1)}$, using the 1-loop RGEs.
- The next-to-leading-log contribution in $V^{(n)}$ may be computed from
  - the next-to-leading-log contribution in $V^{(n-1)}$ using the 1-loop RGEs
  - + the next-to-leading-log contribution in $V^{(n-2)}$ using the 2-loop RGEs
- And so on ...

The effective potential: analytic properties

$V^{\text{eff}}(\mu; \xi; \phi)$ and/or its truncations

- are gauge-fixing dependent

$$\text{Nielsen identity:} \quad \left[\xi \frac{\partial}{\partial \xi} + C(\phi, \xi) \frac{\partial}{\partial \phi}\right] V^{\text{eff}} = 0$$

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- are RG-scale dependent

$$\text{Callan-Symanzik equation:} \quad \frac{dV^{\text{eff}}}{d\log\mu} = \left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda_I)\frac{\partial}{\partial\lambda_I} - \phi_i\gamma_{ij}\frac{\partial}{\partial\phi^j}\right] V^{\text{eff}} = 0$$

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- can develop a spurious imaginary part

$$V^{(1)} \supset \sum_i m_i^4 \log\frac{m_i^2}{\mu^2}$$

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- can exhibit IR divergences in the Goldstone sector

$$m_G \to 0 \quad \Rightarrow \quad \partial^2 V^{(1)} \sim \partial V^{(2)} \sim V^{(3)} \sim \infty$$

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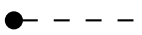
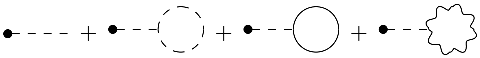
... and yet we wish to compute physical quantities from it!

Loop-corrections to tree-level minima – With or without V^{eff} ?

Let's assume that the tree-level potential has a global minimum.

$$\frac{\partial V^{(0)}}{\partial \phi}(v^{(0)}) =$$

$$= 0$$

One-loop quantities (*e.g.* self-energies, vertex corrections) can be evaluated at the tree-level vacuum $v^{(0)}$, but one needs to include tadpole diagrams, since


$$\neq 0$$

On the other hand, directly minimising $V^{\text{eff}} = V^{(0)} + V^{(1)}$ automatically yields an overall vanishing tadpole contribution at the vacuum $v^{(1)}$.

In addition:

- Depth of the potential with 1-loop corrections (= vacuum energy) ?
- What if the tree-level potential doesn't have a minimum ?
- What about the RG-scale dependence / the behaviour of the perturbative expansion?

Loop-corrections to tree-level minima – With ~~or without~~ V^{eff}

1) Prescription for the RG-scale μ

- At tree-level model has a characteristic energy scale. It is common to fix μ at or around this scale.
- Such a prescription is only valid if the whole particle spectrum is distributed around the same characteristic scale.

Loop-corrections to tree-level minima – With ~~or without~~ V^{eff}

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- At tree-level model has a characteristic energy scale. It is common to fix μ at or around this scale.
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2) Minimisation of $V^{\text{eff}} = V^{(0)} + V^{(1)}$

Although an analytic form for $V^{(1)}$ and its derivatives can be derived (see next slides), the stationary point equations cannot be solved analytically.

- Solution 1: full-fledged numerical minimisation
- Solution 2: the $\hbar$-expansion: $v = v^0 + \hbar\, v^1 + \mathcal{O}(\hbar^2)$

$$V(\phi) = V^{(0)}\left(v^0\right) + \hbar\left[V^{(1)}\left(v^0\right) + v^1\,\frac{\partial V^{(0)}}{\partial\phi}\left(v^0\right)\right] + \mathcal{O}(\hbar^2)$$

$$\frac{\partial V(\phi)}{\partial\phi}\left(v^0\right) = \frac{\partial V^{(0)}}{\partial\phi}\left(v^0\right) + \hbar\left[\frac{\partial V^{(1)}}{\partial\phi}\left(v^0\right) + v^1\,\frac{\partial^2 V^{(0)}}{\partial\phi^2}\left(v^0\right)\right] + \mathcal{O}(\hbar^2)$$

Loop-corrections to tree-level minima – With ~~or without~~ V^{eff}

Including the 1-loop corrections can **qualitatively** change the features of scalar sectors.

- Example 1: [P.M. Ferreira et al., 1910.08662]

  $\rightarrow$ Coexistence of charge-conserving and charge-breaking minima in a THDM

- Example 2: [L. Gráf et al., 1611.01021]

  $\rightarrow$ Tachyons in the scalar spectrum of $SO(10)$ models

The effective potential in classically scale-invariant models

A very interesting application for the effective potential: $V^{(0)}$ has no dimensionful couplings.

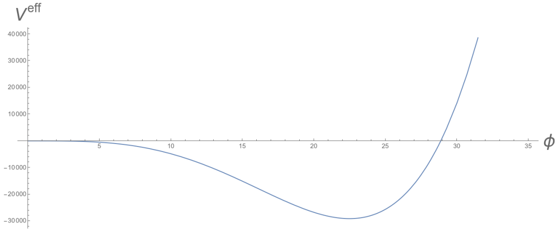
$$V^{(0)} = \frac{1}{4!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi^d$$

Hence it cannot develop a non-trivial minimum. But the 1-loop corrections may generate a minimum ! (Coleman-Weinberg mechanism)

Example in ϕ^4 theory:

$$V^{(0)} = \lambda\phi^4$$

$$V^{(1)} = \frac{1}{16\pi^2}\left(12\lambda^2\phi^2\right)\left[\log\frac{12\lambda\phi^2}{\mu^2} - \frac{3}{2}\right]$$



$\lambda = 0.5,\, \mu = 100\,\mathrm{GeV}$

Minimisation of classically scale-invariant models

Method 1 (*à la* Coleman-Weinberg):

- Identify a characteristic energy scale of the theory (in SM-like models, a natural choice is $v \sim 246\,\mathrm{GeV}$)
- Take it as a prescription to fix the RG-scale μ. If the particle spectrum is distributed around μ, the

$$\log \frac{m_i^2}{\mu^2}$$

  should stay under control.
- Minimise $V^{(0)} + V^{(1)}$
- Pros:
  - Valid for any relative size of the 1-loop contributions compared to the tree-level ones.
  - Allows in principle to identify the global minimum
- Cons:
  - Strongly dependent on the prescription for μ
  - In general not tractable analytically

Minimisation of classically scale-invariant models

Method 2 (*à la* Gildener-Weinberg):

- Identify a RG-scale μ_{GW} at which $V^{(0)}$ develops **flat directions**.
- At this scale, along the flat direction one has:

$$V^{(0)} = 0 \quad \text{and} \quad \frac{\partial V^{(0)}}{\partial \phi^i} = 0$$

- The flat direction corresponds to a "scalon", associated with classical scale invariance.
- Scale invariance is broken when quantum corrections are included, the "scalon" acquires a mass.
- Up to small $\mathcal{O}(\hbar)$ corrections, the minimum lies along the flat direction. Therefore, the ratios of all the vevs are fixed.
- It only remains to determine the value of the radial component $\varphi^2 = \phi_i \phi^i$

Minimisation of classically scale-invariant models

The 1-loop corrections can always be put in the form

$$V^{(1)} = \mathbb{A} + \mathbb{B} \log \frac{\varphi^2}{\mu^2}$$

where $\mathbb{A}$ and $\mathbb{B}$ are homogeneous functions of ϕ^i. For instance:

$$\mathbb{A} \supset \frac{1}{16\pi^2} \mathrm{Tr} \left[M_s^4 \left(\log \frac{M_s^2}{\varphi^2} - \frac{3}{2} \right) \right], \qquad \mathbb{B} \supset \frac{1}{16\pi^2} \mathrm{Tr} \left[M_s^4 \right]$$

$$M_s^2 = \frac{1}{2} \lambda_{abcd} \phi^c \phi^d$$

Hence one has (using Euler's homogeneous function theorem)

$$\frac{\partial V^{(1)}}{\partial \phi} = \phi^i \partial_i V^{(1)} = 4\mathbb{A} + 4\mathbb{B} \left(\log \frac{\varphi^2}{\mu^2} + \frac{1}{2} \right)$$

And since

$$\partial_i \left(V^{(0)} + V^{(1)} \right) = \partial_i V^{(1)} = 0$$

we have

$$\log \frac{\varphi^2}{\mu^2} = -\frac{\mathbb{A}}{\mathbb{B}} - \frac{1}{2}$$

Minimisation of classically scale-invariant models

- Some further analytical results can be derived, in particular regarding the mass spectrum (sum rule)
- Pros:
  - The occurrence of a minimum is triggered by the RG-flow (flat directions)
  - Analytically tractable, some nice (and very general) results
  - Gives a mechanism for a naturally light scalar mode
  - The RG-scale is essentially fixed
- Cons:
  - One has to assume that the one-loop corrections are somewhat negligible compared to the tree-level contributions.
  - In the opposite case, the formalism still applies but the minimum might only be local

Renormalisation group improvement

Given a truncation of V^{eff}, the dependence on μ comes with complications in practice:

- For field values (or more generally masses) strongly deviating from μ, perturbation theory breaks down:

$$\log \frac{\phi^2}{\mu^2} \gg 1 \qquad \text{or} \qquad \log \frac{m_i^2}{\mu^2} \gg 1$$

- The issue is even more severe when multiple mass scales are present in the theory.

Using the renormalisation group properties of the effective potential, it is possible to alleviate these issues.

The procedure of "renormalisation group improvement" allows to:

- Study scalar potentials over wide energy ranges / larger ranges of field values
- Cure / alleviate perturbation breakdown
- More generally, reduce the dependence of the results on the arbitrary scale μ

Effective potentials and the renormalisation group

Callan-Symanzik equation:

$$\frac{dV^{\text{eff}}}{d\log\mu} = \left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda_I)\frac{\partial}{\partial\lambda_I} - \phi_i\gamma_{ij}\frac{\partial}{\partial\phi^j}\right] V^{\text{eff}} = 0$$

$$\beta(\lambda_I) = \frac{d\lambda_I}{dt}, \quad \frac{d\phi^i}{dt} = -\gamma_{ij}\phi^j, \quad t = \log\frac{\mu}{\mu_0}$$

Introducing a shortcut notation

$$\frac{\widetilde{d}}{dt} \equiv \beta(\lambda_I)\frac{\partial}{\partial\lambda_I} - \phi_i\gamma_{ij}\frac{\partial}{\partial\phi^j}$$

One has

$$\frac{dV^{\text{eff}}}{d\log\mu} = \left[\frac{\partial}{\partial t} + \frac{\widetilde{d}}{dt}\right] V^{\text{eff}} = 0$$

Effective potentials and the renormalisation group

The Callan-Symanzik equation can be written order by order in perturbation theory. Let us first define:

$$\beta(\lambda_I) = \beta^{(1)}(\lambda_I) + \beta^{(2)}(\lambda_I) + \cdots$$
$$\gamma_{ij} = \gamma_{ij}^{(1)}(\lambda_I) + \gamma_{ij}^{(2)}(\lambda_I) + \cdots$$

and

$$\left.\frac{\widetilde{d}}{dt}\right|^{(n)} \equiv \beta^{(n)}(\lambda_I)\frac{\partial}{\partial\lambda_I} - \phi_i\gamma_{ij}^{(n)}\frac{\partial}{\partial\phi^j}$$

Then we have

$$\frac{\partial V^{(0)}}{\partial t} = 0$$
$$\frac{\partial V^{(1)}}{\partial t} + \left.\frac{\widetilde{d}V^{(0)}}{dt}\right|^{(1)} = 0$$
$$\frac{\partial V^{(2)}}{\partial t} + \left.\frac{\widetilde{d}V^{(1)}}{dt}\right|^{(1)} + \left.\frac{\widetilde{d}V^{(0)}}{dt}\right|^{(2)} = 0$$
$$\frac{\partial V^{(3)}}{\partial t} + \left.\frac{\widetilde{d}V^{(2)}}{dt}\right|^{(1)} + \left.\frac{\widetilde{d}V^{(1)}}{dt}\right|^{(2)} + \left.\frac{\widetilde{d}V^{(0)}}{dt}\right|^{(3)} = 0$$

Effective potentials and the renormalisation group

Going one step further, one can always decompose the n-loop contributions as:

$$V^{(0)} = \mathbb{A}_0$$

$$V^{(1)} = \mathbb{A}_1 + \mathbb{B}_1 \log \frac{\mathcal{M}^2}{\mu^2}$$

$$V^{(2)} = \mathbb{A}_2 + \mathbb{B}_2 \log \frac{\mathcal{M}^2}{\mu^2} + \mathbb{C}_2 \left(\log \frac{\mathcal{M}^2}{\mu^2}\right)^2$$

$$V^{(3)} = \mathbb{A}_3 + \mathbb{B}_3 \log \frac{\mathcal{M}^2}{\mu^2} + \mathbb{C}_3 \left(\log \frac{\mathcal{M}^2}{\mu^2}\right)^2 + \mathbb{D}_3 \left(\log \frac{\mathcal{M}^2}{\mu^2}\right)^3$$

$$\vdots$$

$$V^{(n)} = \sum_{k=1}^{n} \mathbb{X}_n^{(k)} \left(\log \frac{\mathcal{M}^2}{\mu^2}\right)^k$$

(Using the following "trick": $\log \frac{X}{\mu^2} = \log \frac{X}{\mathcal{M}^2} + \log \frac{\mathcal{M}^2}{\mu^2}$)

Effective potentials and the renormalisation group

Injecting these forms into the fixed-order CS equations yields as set of relations.

At 1-loop:

$$2\mathbb{B}_1 = \left.\frac{\widetilde{d}\mathbb{A}_0}{dt}\right|^{(1)}$$

At 2-loop:

$$2\mathbb{B}_2 = \left.\frac{\widetilde{d}\mathbb{A}_1}{dt}\right|^{(1)} + \left.\frac{\widetilde{d}\mathbb{A}_0}{dt}\right|^{(2)}, \qquad 4\mathbb{C}_2 = \left.\frac{\widetilde{d}\mathbb{B}_1}{dt}\right|^{(1)}$$

At 3-loop:

$$2\mathbb{B}_3 = \left.\frac{\widetilde{d}\mathbb{A}_2}{dt}\right|^{(1)} + \left.\frac{\widetilde{d}\mathbb{A}_1}{dt}\right|^{(2)} + \left.\frac{\widetilde{d}\mathbb{A}_0}{dt}\right|^{(3)}, \qquad 4\mathbb{C}_3 = \left.\frac{\widetilde{d}\mathbb{B}_2}{dt}\right|^{(1)} + \left.\frac{\widetilde{d}\mathbb{B}_1}{dt}\right|^{(2)}$$

$$6\mathbb{D}_3 = \left.\frac{\widetilde{d}\mathbb{C}_2}{dt}\right|^{(1)}$$

Effective potentials and the renormalisation group

Reminder:

$$V^{(n)} = \sum_{k=1}^{n} \mathbb{X}_n^{(k)} \left(\log \frac{\mathcal{M}^2}{\mu^2} \right)^k$$

Using the results from the previous slide, the 1-loop RG-evolution may be used to compute the leading-log contributions to all orders:

$$\mathbb{X}_n^{(n)} = \frac{1}{2^n n!} \left. \frac{\widetilde{d}^n \mathbb{A}_0}{dt^n} \right|^{(1)}$$

For instance, in the ϕ^4 theory, $\mathbb{X}_n^{(n)}$ is the coefficient in front of $\left(\log \frac{\phi^2}{\mu^2}\right)^n$ and one has:

$$\begin{aligned} V^{\text{eff}} &\supset \lambda\phi^4 + \frac{1}{2}\frac{d\lambda}{dt} \log \frac{\phi^2}{\mu^2} \phi^4 + \frac{1}{2^2}\frac{1}{2!}\frac{d^2\lambda}{dt^2} \left(\log \frac{\phi^2}{\mu^2} \right)^2 \phi^4 + \cdots \\ &\supset \phi^4 \sum_{k=0}^{\infty} \frac{1}{2^n n!} \frac{d^n\lambda}{dt^n} \left(\log \frac{\phi^2}{\mu^2} \right)^n \\ &\supset \phi^4 \sum_{k=0}^{\infty} \frac{1}{n!} \frac{d^n\lambda}{dt^n} \left(\log \frac{\phi}{\mu} \right)^n \end{aligned}$$

Effective potentials and the renormalisation group

$$V^{\text{eff}} \supset \phi^4 \sum_{k=0}^{\infty} \frac{1}{n!} \frac{d^n \lambda}{dt^n} \left(\log \frac{\phi}{\mu} \right)^n$$

The value of λ was evaluated at the RG-scale μ. The above series can be put in a more suggestive form:

$$V^{\text{eff}} \supset \phi^4 \sum_{k=0}^{\infty} \frac{1}{n!} \left(\log \phi - \log \mu \right)^n \frac{d^n \lambda}{d(\log \mu)^n} (\log \mu)$$

Effective potentials and the renormalisation group

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For the ϕ^4 theory, we may even derive a closed form, using $(16\pi^2)\beta(\lambda) = 72\lambda^2$:

$$V_{\text{RG-imp.}} = \frac{\lambda\phi^4}{1 - \frac{72}{16\pi^2}\log\frac{\phi}{\mu}}$$

RG-improvement in multi-scale theories

- For multi-scale theories, it is not clear how to best RG-improve the effective potential.
- In the last $\sim$ 30 years, many methods have been proposed (multiple renormalisation scales, implementation of decoupling, EFT approach...)
- Here I will present the method described in [L. Chataigner et al., 1801.05258]

Single-scale Renormalisation Group Improvement of Multi-scale Effective Potentials

Leonardo Chataignier,[a,c] **Tomislav Prokopec,**[a] **Michael G. Schmidt**[b] **and Bogumiła Świeżewska**[a,d]

RG-improvement in multi-scale theories

The results from the last slides imply that for a (possibly field-dependent) $\mathcal{M}$, resumming the leading powers of $\log \frac{\mathcal{M}^2}{\mu^2}$ yields

$$V^{\text{eff}} \supset V^{(0)}(\mathcal{M}) + \cdots$$

$\Rightarrow$ A way to RG-improve a truncation of the effective potential is to evaluate it at a field-dependent RG-scale $\mathcal{M}(\phi^i)$ with suitable properties.

At one-loop, we have:

$$V(\phi^i(\mu), \lambda_I(\mu); \mu) = V^{(0)}\left(\phi^i(\mu), \lambda_I(\mu)\right) + V^{(1)}\left(\phi^i(\mu), \lambda_I(\mu); \mu\right)$$

The authors of [1801.05258] propose to choose the field-dependent scale $\mathcal{M} = \mu_*$ such that the one-loop corrections vanish:

$$V^{(1)}\left(\phi^i(\mu_*), \lambda_I(\mu_*); \mu_*\right) = 0$$

Therefore, the RG-improved potential only takes a tree-level form, with couplings and field normalisations evaluated at the variable scale μ:

$$V_{\text{RG}-\text{imp.}}(\phi^i) = V^{(0)}\left(\phi^i(\mu_*), \lambda_I(\mu_*)\right) + \mathcal{O}(\hbar^2)$$

RG-improvement and effective potential stability

No matter the loop-order, one can show that in the limit of large field values, the field-dependent scale μ_* satisfies

$$\mu_* \sim \varphi$$

where

$$\varphi^2 = \phi_i \phi^i$$

This result has a profound implication regarding the stability of the effective potential. Namely

$$V^{\text{eff}}(\phi^i) \xrightarrow{\varphi \to \infty} V^{(0)}\left(\lambda_I(\varphi), \phi^i(\varphi)\right)$$

and therefore, for large field values, the asymptotic behaviour of V^{eff} is similar to the behaviour of $V^{(0)}$ at large RG-scales.

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$\Rightarrow$ One should not draw conclusions from asserting stability of the tree-level potential at some low scale μ_0 without asserting it at all larger scales where the theory remains valid.

$\Rightarrow$ Tree-level-only analyses can be misleading. Some regions of the parameter space that seem allowed based on stability constraints might not be, and vice versa.

Applications

Some interesting applications (+ advertisement!)

- [Kenneth Lane and collaborators]: Higgs alignment in the THDM, based on the Gildener-Weinberg mechanism
- [A. Held, J. Kwapisz, LS]: Radiative symmetry breaking in a GUT model based on $SO(10)$. Stability, perturbativity constraint + requirement of a viable symmetry breaking pattern
- [B. Herrmann, M. Maniatis, LS, I. Schienbein]: Radiative symmetry breaking in classically scale-invariant THDMs.

Outlook

The study of effective potentials is decades old, and is still today a very active topic. There is still progress to make:

- Analytic insights
  - Field derivatives of $V^{(n)}$
  - All-order resummations (genuine multi-field RG-improving resummations, "daisy" resummation, ...)
  - Gauge fixing dependence
- RG-improvement methods + extension to the effective action
- What about effective potentials at finite temperature ?
- Computer tools for effective potentials ?
- Systematic 1-loop analyses !

Thank you for your attention !

Any questions ?

Bonus: derivatives of the 1-loop potential

$$V^{(1)} = \frac{1}{16\pi^2} \mathrm{Tr}\left[M_s^4 \left(\log \frac{M_s^2}{\mu^2} - \frac{3}{2} \right) \right] = \frac{1}{16\pi^2} \sum_a m_a^4 \left(\log \frac{m_a^2}{\mu^2} - \frac{3}{2} \right)$$

Defining:

$$\widetilde{\log} x = x(\log x - 1) \qquad \text{and} \qquad \widetilde{X} = U X U^{\mathrm{T}} \, ,$$

one has

$$\partial_i V^{(1)} = \frac{2}{16\pi^2} \sum_a \left(\widetilde{\partial_i M^2} \right)^{aa} \widetilde{\log} m_a$$

$$\partial_i \partial_j V^{(1)} = \frac{2}{16\pi^2} \left[\sum_a \left(\widetilde{\partial_i \partial_j M^2} \right)^{aa} \widetilde{\log} m_a + \sum_{a,b} \left(\widetilde{\partial_i M^2} \right)^{ab} \left(\widetilde{\partial_j M^2} \right)^{ba} \frac{\widetilde{\log} m_a - \widetilde{\log} m_b}{m_a - m_b} \right]$$

Bonus: derivatives of the 1-loop potential

$$\partial_i\partial_j\partial_k V^{(1)} = \frac{2}{16\pi^2}\Bigg\{ \sum_a \left(\widetilde{\partial_i\partial_j\partial_k M^2}\right)^{aa} \widetilde{\log} m_a$$

$$+ \sum_{a,b} \left[\left(\widetilde{\partial_i M^2}\right)^{ab} \left(\widetilde{\partial_j\partial_k M^2}\right)^{ba} + \left(\widetilde{\partial_j M^2}\right)^{ab} \left(\widetilde{\partial_k\partial_i M^2}\right)^{ba} + \left(\widetilde{\partial_k M^2}\right)^{ab} \left(\widetilde{\partial_i\partial_j M^2}\right)^{ba} \right] \frac{\widetilde{\log} m_a - \widetilde{\log} m_b}{m_a - m_b}$$

$$+ \sum_{a,b,c} \left(\widetilde{\partial_i M^2}\right)^{ab} \left(\widetilde{\partial_j M^2}\right)^{bc} \left(\widetilde{\partial_k M^2}\right)^{ca} \left[\frac{\widetilde{\log} m_a}{(m_a - m_b)(m_a - m_c)} + \frac{\widetilde{\log} m_b}{(m_b - m_a)(m_b - m_c)} + \frac{\widetilde{\log} m_c}{(m_c - m_a)(m_c - m_b)} \right] \Bigg\}$$

and so on...