Inclusive semi-leptonic decays from lattice QCD

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Inclusive versus exclusive?

Theoretically,
- Exclusive, with lattice FF
- Inclusive, with OPE

Each has its own pros & cons

Can’t we unite them for better understanding?

\[ |V_{cb}| \text{ can be determined anywhere in the phase space} \]

\[ \text{recoil momentum} \]

\[ \text{invariant mass of the hadronic system} \]
**Inclusive semi-leptonic $B$ decays**

**Inclusive**: sum over all final states, can be computed using PT (or OPE); a number of NP MEs involved

A new method to compute the “sum” in LQCD


from the forward-Compton amplitude.

\[
\langle B(0)| \tilde{J}_\mu^\dagger(-q; t) \tilde{J}_\nu(q; 0)|B(0)\rangle
\]

all possible states contribute
Inclusive rate

Differential decay rate:

\[ d\Gamma \sim |V_{cb}|^2 l^{\mu\nu} W_{\mu\nu} \]

Structure function:

\[ W_{\mu\nu} = \sum_X (2\pi)^2 \delta^4(p_B - q - p_X) \frac{1}{2M_B} \langle B(p_B)|J_\mu^+(0)|X\rangle\langle X|J_\nu(0)|B(p_B)\rangle \]

\[ \rightarrow \langle B(0)|\tilde{J}_\mu^+(-q; t) \delta(\omega - \hat{H}) \tilde{J}_\nu(q; 0)|B(0)\rangle \]

“spectral function”

Decay rate:

\[ \Gamma \propto \int_0^{q_{\text{max}}^2} dq \int \frac{m_B - \sqrt{q^2}}{\sqrt{m_D^2 + q^2}} d\omega K(\omega; q^2) \langle B(0)|\tilde{J}^+(q)\delta(\omega - \hat{H})\tilde{J}(q)|B(0)\rangle \]

known kinematical factor
Sum over states = energy integral

\[ \Gamma \propto \int_0^{q_{\text{max}}} dq \int_{m_B - \sqrt{q^2}}^{m_B - \sqrt{q^2}} d\omega K(\omega; q^2) \langle B(0) | \tilde{J}^\dagger(-q) \delta(\omega - \hat{H}) \tilde{J}(q) | B(0) \rangle \]

\[ = \langle B(0) | \tilde{J}^\dagger(-q) K(\hat{H}; q^2) \tilde{J}(q) | B(0) \rangle \]

“smeared spectral function”

Lattice Compton amplitude:

\[ \langle B(0) | \tilde{J}^\dagger_{\mu}(-q; t) \tilde{J}_\nu(q; 0) | B(0) \rangle \rightarrow \langle B(0) | \tilde{J}^\dagger(-q)e^{-\hat{H}t} \tilde{J}(q) | B(0) \rangle \]

“smeared” in a different way depending on t
Approximation?

\[ K(\hat{H}) = k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-k_N\hat{H}} \]

Possible?

- No, if \( K(\omega) = \delta(\omega - E) \). Corresponds to the full spectral function = Famous ill-posed problem.
- More chance if \( K(\omega) \) is a smooth function, like

\[ \text{Backus-Gilbert (and its variants) or Chebyshev approx.} \]

Hansen, Meyer, Robaina, arXiv:1704.08993  
Hansen, Rupo, Tantalo, arXiv:1903.06476  
Chebyshev approx:


(shifted) Chebyshev polynomials

\[ T_0^*(x) = 1 \]
\[ T_1^*(x) = 2x - 1 \]
\[ T_2^*(x) = 8x^2 - 8x + 1 \]

\[ \vdots \]

\[ T_{j+1}^*(x) = 2(2x - 1)T_j^*(x) - T_{j-1}^*(x) \]

"Best" approximation can be obtained with

\[ c_j^* = \frac{2}{\pi} \int_0^\pi d\theta \sin^{-1} \left( \frac{1 + \cos \theta}{2} \right) \cos(j\theta) \]

"best" = maximal deviation is minimal

\[ K(\omega) \approx \frac{c_0}{2} + \sum_{j=1}^N c_j^* T_j^*(e^{-\omega}) \]

• Constraint \( |T_j^*(e^{-\omega})| \leq 1 \) stabilizes the expansion.

• Higher orders are suppressed when the coefficients are. It is the case for smooth function \( K(\omega) \)
Kernel to approximate

\[ K(\omega) \sim e^{2\omega t_0} (m_B - \omega)^{N} \theta(m_B - |q| - \omega) \]

- kinematical factor

To implement the upper limit of integ

Smear by “sigmoid” with a width \( \sigma \)

Need to take a limit of \( \sigma \to 0 \)

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Graphs showing the behavior of \( K(\omega) \) for different values of \( \sigma \) and \( N \) for narrow and wide cases.
Compton amplitude

$$\langle B(0) | \tilde{J}_\mu^\dagger (-q; t) \tilde{J}_\nu (q; 0) | B(0) \rangle$$

S-wave (D and D*)
- Very well approximated by a single-exp = no sign of excited state contrib.

P-wave (D**’s)
- Small : no wave function overlap of excited states when $m_b = m_c$ and zero recoil

Pilot lattice computation [JLQCD setup]
- On a lattice of $48^3 \times 96$ at $1/\alpha = 3.6$ GeV
- Strange spectator quark
- physical charm quark mass
- (unphysically) light b quark ~ 2.7 GeV
- 100 configs x 4 src

**Graph Description**
- $C(t_2 - t_1)$
- $t_2 - t_1$ range from 0 to 25
- Logarithmic scale for $C(t_2 - t_1)$
- Data points for $V_0V_0$, $A_1A_1$, $A_0A_0$, $V_1V_1$
- Zero recoil indicated
Inclusive decay rate

- Breakdown to individual channels: VV and AA; parallel and perp with respect to the recoil momentum
- Compared to exclusive contributions estimated from $B \to D(*)$ form factors (dashed line), that are separately calculated.
  - $VV_{\parallel}$ dominated by $B \to D$
  - All others by $B \to D^*$

$$
\frac{\text{differential decay rate}}{|q|}
$$

$d\Gamma / |q|$

[q^2 [GeV^2]]
Comparison with OPE

OPE at \(O(\alpha_s), O(1/m_b^3)\) with
- physical charm mass
- \(m_b\) to reproduce \(B_s\) mass

\[\alpha_s = 0.32(1)\]

Reasonable agreement observed. Further analysis to study the consistency between OPE and lattice.
Further test with moments

Gambino, SH, Machler, arXiv:2111.02833

e.g. Lepton energy moment $<E_l>$

Good agreement in general.
Outlook: OPE vs Lattice

- Help each other
- More cross-checks with moments; eventually determine the MEs necessary in OPE.
- Lattice limited to small $q^2$'s. Need extrapolation.
- Also, should extend towards...
  - Physical $b$ quark mass; light quarks
  - Crosscheck against exp’t with D decays
  - Formulation of $b \rightarrow u$
Outlook: incl. vs excl. puzzle

- Framework to compute inclusive decay rate on the lattice is now available. The energy integral can be reconstructed from Euclidean lattice correlators.

- Comparison with OPE will elucidate any inconsistency on the theory side. An initial result is encouraging.

- Use the phase space more widely to better control the systematic errors? Collaboration among exp, pheno, lattice would be crucial.