Neutral meson mixing from Lattice QCD

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CKM 2021, Melbourne, AU

24 November 2021

The project leading to this application has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 894103.
1. Introduction: neutral meson mixing

2. Challenges in $b$-physics on the lattice

3. Recent Results

4. Summary and Outlook
Neutral $B_{(s)}$ meson mixing - background

Neutral mesons oscillate:

\[ |B_{L,H}\rangle = p |B_q^0\rangle \pm q |\bar{B}_q^0\rangle \]

⇒ **splittings** in mass eigenstates:
- mass splitting $\Delta m_q \equiv m_H - m_L$
- width splitting $\Delta \Gamma_q \equiv \Gamma_L - \Gamma_H$

Time dependence:

\[ |B_q^0(t)\rangle = g_+(t) |B_q^0\rangle + \frac{q}{p} g_-(t) |\bar{B}_q^0\rangle \]

\[ |\bar{B}_q^0(t)\rangle = \frac{p}{q} g_-(t) |B_q^0\rangle + g_+(t) |\bar{B}_q^0\rangle \]

where $q = d, s$

Occurs at loop level in SM ⇒ sensitive probe of new physics!
Neutral $B_{(s)}$ Meson Mixing - experiment

$$|g_{\pm}(t)|^2 = \frac{e^{-\Gamma_q t}}{2} \left[ \cosh \left( \frac{\Delta \Gamma_q}{2} t \right) \pm \cos (\Delta m_q t) \right]$$

$\Delta m$ experimentally accessible as a frequency!

$B^0_d$: Many results

$B^0_s$: "Only" CDF and LHCb

$$\Delta m_d = 0.5065(19) \text{ps}^{-1}$$

$$\Delta m_s = 17.757(21) \text{ps}^{-1}$$

Well below per cent level!

[HFLAV]
Neutral $B_s$ Meson Mixing - theory

\[ \langle B_0^q | H_{\text{eff}} | \bar{B}_0^q \rangle \propto \langle B_0^q | H^{\Delta b=2} | \bar{B}_0^q \rangle + \sum_n \langle B_0^q | H^{\Delta b=1} | n \rangle \langle n | H^{\Delta b=1} | \bar{B}_0^q \rangle \]

Short distance
\[ \text{short distance} \propto \left| \sum_{q'=u,c,t} \frac{m_{q'}^2}{M_W^2} V_{q'b} V_{q'q}^* \right|^2 \approx \frac{m_t^4}{M_W^4} \left| V_{tb} V_{tq}^* \right|^2 \]

SD: Top enhanced: $m_t^2 V_{tb} V_{tq}^* \gg m_c^2 V_{cb} V_{cq}^* \gg m_u^2 V_{ub} V_{uq}^*$

LD: Only $m_c, m_u$ in intermediate states: no top + CKM suppressed
\[ \Rightarrow \text{Short distance dominated.} \]
Operator Product Expansion

OPE factorises this into

- **Perturbative model-dependent Wilson coefficients** $C_i(\mu)$
- **Non-perturbative model-independent matrix elements**

$$
\langle B^0_{(s)} \left| \mathcal{H}^{\Delta b=2} \right| \bar{B}^0_{(s)} \rangle = \sum_i C_i(\mu) \langle B^0_{(s)} \left| \mathcal{O}^{\Delta b=2}_i(\mu) \right| \bar{B}^0_{(s)} \rangle
$$

- 5 independent (parity even) operators $\mathcal{O}_i$.
- Only $\mathcal{O}_1$ is relevant for $\Delta m$:
  $$
  \mathcal{O}_1 = (\bar{b}_a \gamma_\mu (1 - \gamma_5) q_a) (\bar{b}_b \gamma_\mu (1 - \gamma_5) q_b) = \mathcal{O}_{VV+AA}
  $$

- Define bag parameters: $B_i = \langle \bar{B}_q^0 | \mathcal{O}_i | B_q^0 \rangle / \langle \bar{B}_q^0 | \mathcal{O}_i | B_q^0 \rangle_{\text{VSA}}$
  $$
  \Delta m_q = |V_{tb}^* V_{tq}|^2 \times f_{B_q}^2 \hat{B}_{B_q}^{(1)} \times m_{B_q} \mathcal{K}
  $$
  
  $\Rightarrow$ **Non-perturbative matrix elements calculable on the lattice**
Lattice QCD in a nutshell

Based on the **Path Integral** formulation.

\[
\langle O \rangle_M = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}, U] \, O[\psi, \bar{\psi}, U] \, e^{iS[\psi, \bar{\psi}, U]}
\]

**Minkowski:** Highly oscillatory, infinite dimensional integral.  

\[
\Rightarrow \text{Wick rotate to Euclidean (i.e. imaginary) time } (t \rightarrow i\tau).
\]

\[
\langle O \rangle_E = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}, U] \, O_E[\psi, \bar{\psi}, U] \, e^{-S_E[\psi, \bar{\psi}, U]}
\]

**Euclidean:** Exponentially decaying, infinite dimensional integral.  

\[
\Rightarrow \text{Discretise space-time and interpret as a probability distribution.}
\]

- Lattice spacing \( a \) (UV regulator)
- Box of length \( L \) (IR regulator)

**Lattice:** Exponentially decaying and finite dimensional
Recovering continuum physics

Lattice vs Continuum

We simulate:
- at finite lattice spacing $a$
- in finite volume $L^3$
- Euclidean space
- lattice regularised
- some bare input quark masses $am_l, am_s, am_c, am_b$

In general: $m_\pi \neq m_\pi^{\text{phys}}$

We want:
- $a = 0$
- $L = \infty$
- Minkowski space
- some continuum scheme
- $m_l = m_l^{\text{phys}}$
- $m_s = m_s^{\text{phys}}$
- $m_h = m_c^{\text{phys}}, m_b^{\text{phys}}$

$\Rightarrow$ Need to control all limits!
- particularly simultaneously control FV and discretisation.

$\Rightarrow$ Universality: Different discretisations must give same results.
Multiple scale problem on the lattice: back of the envelope

Control effects of IR (finite volume) and UV (discretisation) regulators:

\[ m_\pi L \gtrsim 4 \quad \quad a^{-1} \gg \text{Mass scale of interest} \]

For \( m_\pi = m_\pi^{\text{phys}} \sim 140 \text{ MeV} \) and \( \overline{m}_b(m_b) \approx 4.2 \text{ GeV} \):

\[ L \gtrsim 5.6 \text{ fm} \quad \quad a^{-1} \gg 4.2 \text{ GeV} \approx (0.05 \text{ fm})^{-1} \]

Requires \( N \equiv L/a \gg 120 \Rightarrow N^3 \times (2N) \gg 4 \times 10^8 \) lattice sites.

**VERY EXPENSIVE** to satisfy both constraints simultaneously... \( \ldots \) needs to be repeated for different values of \( a \).
How to simulate the $b$-quark?

For now choose between:

<table>
<thead>
<tr>
<th>Effective action for $b$</th>
<th>Relativistic action for $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Can tune to $m_b$</td>
<td>Theoretically cleaner and systematically improvable</td>
</tr>
<tr>
<td>comes with systematic errors which are hard to estimate/reduce</td>
<td>Need to control extrapolation in heavy quark mass</td>
</tr>
</tbody>
</table>

Different properties:
- computational cost
- chirality
- tuning errors
- systematic errors
- cut off effects
- renormalisation

BUT SOON:
Huge efforts in the community to produce very fine lattice spacings:
⇒ Direct simulation of $\approx m_b^{\text{phys}}$ will become possible!
Extracting CKM matrix elements

We write $\Delta m_q$ in terms of the Renormalisation Group Independent (RGI) bag parameter $\hat{B}_{B_q}$:

$$\Delta m_q = |V_{tb}^* V_{tq}|^2 \frac{G_F^2 m_W^2 m_{B_q}^2}{6\pi^2} S_0(x_t) \eta_B f_{B_q}^2 \hat{B}_{B_q}^{(1)}$$

- $\Delta m_d$ and $\Delta m_s$ are known experimentally to 0.4% and 0.1% accuracy.
- Combined other inputs are known to 0.4%
- Typical precision for $f_{B_q} \sqrt{\hat{B}_{B_q}^{(1)}}$ is a few percent.
- Part of statistic and systematic errors cancel in $SU(3)$ breaking ratios:

$$\xi^2 \equiv \frac{f_{B_s}^2 \hat{B}_{B_s}}{f_{B_d}^2 \hat{B}_{B_d}} = \left| \frac{V_{td}}{V_{ts}} \right|^2 \frac{\Delta m_s}{\Delta m_d} \frac{m_B}{m_{B_s}} \Rightarrow \text{access to } |V_{td}/V_{ts}|$$
Fewer results than in the light sector, but very complementary results from

- different gauge field configurations
- different valence light actions
- different valence heavy actions
- different methodologies

Includes computations with $m_{\pi}^{\text{phys}}$!
\( N_f = 2 + 1 \) flavours of staggered quarks (asqtad) in the sea

- 4 lattice spacings, pion masses from 177 – 555 MeV
- valence light & strange: asqtad
- Fermilab method for the \( b \)-quark
- mostly non-perturbative 1-loop lattice perturbation theory

- Computation of \( f_{Bq} \sqrt{\hat{B}_{Bq}} \) and \( \xi \)
- \( f_{Bq} \) taken from the PDG average to access to \( \hat{B}_{Bq} \)
- all 5 operators for \( B \) and \( B_s \)

\[
\begin{align*}
    f_{B_d} \sqrt{\hat{B}_{B_d}} &= 227.7(9.5) \text{ MeV} \\
    f_{B_s} \sqrt{\hat{B}_{B_s}} &= 274.6(8.4) \text{ MeV} \\
    \xi &= 1.206(18)
\end{align*}
\]
<table>
<thead>
<tr>
<th></th>
<th>statistics</th>
<th>inputs</th>
<th>$\kappa$ tuning</th>
<th>matching</th>
<th>chiral</th>
<th>LQ disc</th>
<th>HQ disc</th>
<th>fit</th>
<th>total</th>
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<tbody>
<tr>
<td>$\langle O_1^d \rangle$</td>
<td>4.2</td>
<td>0.4</td>
<td>2.1</td>
<td>3.2</td>
<td>2.3</td>
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<td>4.6</td>
<td>7.7</td>
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<td>4.8</td>
<td>1.2</td>
<td>9.9</td>
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<td>0.4</td>
<td>1.7</td>
<td>2.2</td>
<td>1.9</td>
<td>0.5</td>
<td>3.9</td>
<td>6.4</td>
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<td>4.9</td>
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<td>3.4</td>
<td>1.9</td>
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<td>3.6</td>
<td>6.7</td>
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<tr>
<td>$\xi$</td>
<td>0.8</td>
<td>0.4</td>
<td>0.3</td>
<td>0.5</td>
<td>0.4</td>
<td>0.1</td>
<td>0.7</td>
<td>1.4</td>
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</tr>
</tbody>
</table>

Uncertainty dominated by chiral-continuum limit fit, in particular

- statistical
- heavy quark discretisation errors
- matching
RBC/UKQCD: set-up [1812.08791]

- $N_f = 2 + 1$ flavours of chirally symmetric domain wall fermions in the sea and the valence
- 3 lattice spacings
- 2 physical pion mass ensembles with $m_\pi$ up to 430 MeV
- Domain wall heavy quarks with $m_c^{\text{phys}} \lesssim m_h^{\text{sim}} \lesssim \frac{m_b^{\text{phys}}}{2}$
- Slightly different DWF parameters between heavy and light sector
- Computation of $SU(3)$ breaking ratios $f_{B_s}/f_{B_d}$, $B_{B_s}/B_{B_d}$ and $\xi$
- Renormalisation constants cancel

\[
\begin{align*}
  f_{B_s}/f_{B_d} &= 1.1949(60)_{\text{stat}}(^{+95}_{-175})_{\text{sys}} \\
  B_{B_s}/B_{B_d} &= 0.9984(45)_{\text{stat}}(^{+80}_{-63})_{\text{sys}} \\
  \xi &= 1.1939(67)_{\text{stat}}(^{+95}_{-177})_{\text{sys}}
\end{align*}
\]
RBC/UKQCD: error budget [%] [1812.08791]

Combined fit in $m_\pi$, $m_H$ and $a^2$

Uncertainty dominated by

- chiral-continuum fit
- heavy quark extrapolation
- estimates of higher order $1/m_H$ terms

<table>
<thead>
<tr>
<th></th>
<th>$f_{D_s}/f_D$</th>
<th>$f_{B_s}/f_B$</th>
<th>$\xi$</th>
<th>$B_{B_s}/B_{B_d}$</th>
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<tr>
<td>central</td>
<td>1.1740</td>
<td>1.1949</td>
<td>1.1939</td>
<td>0.9984</td>
</tr>
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<td>stat</td>
<td>0.43%</td>
<td>0.50%</td>
<td>0.56%</td>
<td>0.45%</td>
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<tr>
<td>fit chiral-CL</td>
<td>+0.31%</td>
<td>+0.34%</td>
<td>+0.38%</td>
<td>+0.42%</td>
</tr>
<tr>
<td></td>
<td>-0.32%</td>
<td>-0.54%</td>
<td>-0.45%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>fit heavy mass</td>
<td>+0.07%</td>
<td>+0.00%</td>
<td>+0.00%</td>
<td>+0.27%</td>
</tr>
<tr>
<td></td>
<td>-0.09%</td>
<td>-0.82%</td>
<td>-0.87%</td>
<td>-0.22%</td>
</tr>
<tr>
<td>H.O. heavy</td>
<td>0.00%</td>
<td>0.47%</td>
<td>0.35%</td>
<td>0.21%</td>
</tr>
<tr>
<td>H.O. disc.</td>
<td>0.01%</td>
<td>0.01%</td>
<td>0.12%</td>
<td>0.17%</td>
</tr>
<tr>
<td>$m_u \neq m_d$</td>
<td>0.08%</td>
<td>0.07%</td>
<td>0.08%</td>
<td>0.01%</td>
</tr>
<tr>
<td>finite size</td>
<td>0.18%</td>
<td>0.18%</td>
<td>0.18%</td>
<td>0.18%</td>
</tr>
<tr>
<td>total systematic</td>
<td>+0.38%</td>
<td>+0.61%</td>
<td>+0.56%</td>
<td>+0.66%</td>
</tr>
<tr>
<td>total sys+stat</td>
<td>+0.58%</td>
<td>+0.79%</td>
<td>+0.80%</td>
<td>+0.80%</td>
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</tbody>
</table>
- $N_f = 2 + 1 + 1$ flavours of staggered quarks (HISQ) in sea light quarks using HISQ in the valence
- 3 lattice spacings, 2 physical pion mass ensembles
- improved nonrelativistic QCD action for the $b$ all 5 operators for $B_d$ and $B_s$
- blinded analysis
- Computation of the $\hat{B}_{B_q}^{(i)}$
- $\xi$ and $f_{B_q} \sqrt{\hat{B}_{B_q}}$ accessed by using decay constants taken from a different computation

\[
\begin{align*}
\hat{B}_{B_d}^{(1)} &= 1.222(61) \\
\hat{B}_{B_s}^{(1)} &= 1.232(53) \\
\hat{B}_{B_s}^{(1)}/\hat{B}_{B_d}^{(1)} &= 1.008(25)
\end{align*}
\]
Uncertainty dominated by matching terms $\alpha_s^2$ and $\alpha_s \Lambda_{QCD}/m_b$. 

<table>
<thead>
<tr>
<th>$n_f$</th>
<th>$B_{B_s}^{(1)} / B_{B_d}^{(1)}$</th>
<th>$B_{B_s}^{(2)} / B_{B_d}^{(2)}$</th>
<th>$B_{B_s}^{(3)} / B_{B_d}^{(3)}$</th>
<th>$B_{B_s}^{(4)} / B_{B_d}^{(4)}$</th>
<th>$B_{B_s}^{(5)} / B_{B_d}^{(5)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.01(2)</td>
<td>1.06(2)</td>
<td>1.08(3)</td>
<td>0.96(2)</td>
<td>0.97(2)</td>
</tr>
<tr>
<td>3</td>
<td>1.01(2)</td>
<td>1.06(2)</td>
<td>1.08(3)</td>
<td>0.96(2)</td>
<td>0.97(2)</td>
</tr>
<tr>
<td>2</td>
<td>1.01(2)</td>
<td>1.06(2)</td>
<td>1.08(3)</td>
<td>0.96(2)</td>
<td>0.97(2)</td>
</tr>
<tr>
<td>1</td>
<td>1.01(2)</td>
<td>1.06(2)</td>
<td>1.08(3)</td>
<td>0.96(2)</td>
<td>0.97(2)</td>
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<table>
<thead>
<tr>
<th>$n_f$</th>
<th>$B_{B_s}^{(1)}$</th>
<th>$B_{B_s}^{(2)}$</th>
<th>$B_{B_s}^{(3)}$</th>
<th>$B_{B_s}^{(4)}$</th>
<th>$B_{B_s}^{(5)}$</th>
<th>$B_{B_s}^{(1)} / B_{B_s}^{(1)}$</th>
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<tr>
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<td>0.84(3)</td>
<td>0.83(4)</td>
<td>0.85(5)</td>
<td>1.03(4)</td>
<td>0.94(3)</td>
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<td>0.83(4)</td>
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<table>
<thead>
<tr>
<th>Source</th>
<th>$\eta_i^q$</th>
<th>$\eta_i^{\alpha_s^2}$</th>
<th>$\alpha_s \Lambda_{QCD}/m_b$</th>
<th>$\alpha_s \Lambda_{QCD}/m_b$</th>
<th>$m_l$ extrapolation</th>
<th>Total</th>
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<tbody>
<tr>
<td>Lattice data</td>
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<td>1.5</td>
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<td>$\eta_i^q$</td>
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<td>2.3</td>
<td>2.1</td>
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<td>$\eta_i^{\alpha_s^2}$</td>
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<td>2.9</td>
<td>5.2</td>
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<td>1.9</td>
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<td>$m_l$ extrapolation</td>
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<td>0.4</td>
<td>0.7</td>
<td>0.5</td>
<td>0.4</td>
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<tr>
<td>Total</td>
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<td>7.0</td>
<td>4.6</td>
<td>4.1</td>
<td>2.5</td>
</tr>
</tbody>
</table>
CKM matrix elements

|                  | $f_{B_d}\sqrt{\hat{B}_{B_d}}$ [GeV] | $f_{B_s}\sqrt{\hat{B}_{B_s}}$ [GeV] | $\xi$  | $|V_{td}|$  | $|V_{ts}|$  | $|V_{ts}/V_{td}|$ |
|------------------|-----------------------------------|-----------------------------------|-------|------------|------------|-----------------|
| HPQCD19          | 0.2106(55)                        | 0.2561(57)                        | 1.216(16) | 0.00867(23) | 0.04189(93) | 0.2071(27)     |
| FNAL/MILC16      | 0.2277(98)                        | 0.2746(88)                        | 1.206(18) | 0.00800(35) | 0.0390(13) | 0.2052(33)     |
| RBC/UKQCD18      |                                   |                                   | 1.194(+12\_19) |         |            |                |

- Reasonable agreement between lattice results, but some spread
- Tree-only fit somewhat differs
- Uncertainty is still dominated by theory

⇒ There is some work yet to be done!
Aside: Non-perturbative renormalisation on the lattice

We want

\[ \text{Amplitude} = C_{\overline{\text{MS}}}^{\text{MS}}(\mu) \langle O \rangle_{\overline{\text{MS}}}^{\text{MS}}(\mu) \]

- Wilson coefficients (typically) computed in $\overline{\text{MS}}$ at some scale $\mu$.
- Operators $\langle O \rangle^{\text{bare}}(a)$ computed with lattice regulator $a^{-1}$.
- Renormalise $\langle O \rangle^{\text{bare}}(a)$ at scale $\mu$ in regularisation independent (RI) scheme, by computing a non-pert. renormalisation factor $Z_{\text{RI}}^{\mu, a}$. 

\[ \langle O \rangle_{\text{RI}}^{\mu} = \lim_{a^2 \to 0} Z_{\text{RI}}^{\mu, a} \langle O \rangle^{\text{bare}}(a) \]

- Match to preferred scheme (e.g. $\overline{\text{MS}}$) using P.T. at $\mu$: $R_{\overline{\text{MS}} \leftarrow \text{RI}}^{\mu}$
- If the operators mix: $C$ and $\langle O \rangle$ become vectors, $R$ and $Z$ matrices.

\[ \text{Amplitude} = C_{i}^{\overline{\text{MS}}} R_{ij}^{\overline{\text{MS}} \leftarrow \text{RI}}(\mu) \lim_{a \to 0} Z_{jk}^{\text{RI}}(\mu, a) \langle O_k \rangle^{\text{bare}}(a) \]

- Chirally symmetric fermions $\Rightarrow$ $R$ and $Z$ are block diagonal.
RBC/UKQCD18 budget dominated
- by chiral-continuum fit
- by heavy quark extrapolation
- by estimates of higher order \(1/m_H\) terms

- Supplement RBC/UKQCD dataset with very fine ensembles from JLQCD ⇒ reduce extrapolation in \(m_H\) significantly
- all domain wall fermion set-up ⇒ mixed action NPR in RI-SMOM scheme in progress ⇒ all 5 operators \(\hat{B}_{B_d}^{(i)}\) and \(\hat{B}_{B_d}^{(i)}\)
- 6 lattice spacings, 2 ensembles with physical pion mass ⇒ good control over all required limits
- new correlator fitting strategy
unfitted data, i.e. a function of \((a, m_\pi, m_K, m_H)\). But promising since

- required extrapolation to \(m_b^{\text{phys}}\) small
- benign behaviour with \(1/m_H\)
Un-renormalised ratios \( \langle B_q | O_i | \bar{B}_q \rangle / \langle B_q | O_1 | \bar{B}_q \rangle \)

- More ensembles to be fitted soon
- Required extrapolation to \( m_b^{\text{phys}} \) small
- Benign behaviour with \( 1/m_H \)
- NPR in progress
- Developing fit strategy to take limits
### Disclaimer
- Focussed on the three most recent results (two new since CKM 18)
- Many technical lattice details omitted
- Only covered lattice results of dim 6 operators. Omitted
  - ⇒ new sum rules result \([1904.00940]\)
  - ⇒ new lattice result for width difference \([HPQCD\ 1910.00970]\)

### Summary
- Comparably few but very complementary lattice results:
  - ensembles
  - light quark action
  - heavy quark action
  - renormalisation
- Physical pion mass results
- Results for full operator basis
- First results without need for effective action for the \(b\)-quark

### Status and Future
- \(|V_{td}|\) and \(|V_{ts}|\) known at \(\approx 2.5\%\) level, \(|V_{td}/V_{ts}|\) at \(\approx 1.5\%\) level
- Uncertainty theory dominated - work is ongoing