QCD corrections to $\Delta \Gamma_s$

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1 Motivation

2 B-meson mixing
   - Theory
   - Calculation
   - Phenomenology

3 Summary and Outlook
Neutral meson systems can oscillate between their flavor eigenstates: $K^0 - \bar{K}^0$, $D^0 - \bar{D}^0$ and $B^0_q - \bar{B}^0_q$ with $q = s, d$.

- Loop-induced FCNC processes
- $B$ meson properties equally well accessible to theory and experiment
- Many new exciting experimental measurements, see talks by Alessandro Gaz, Niels Tuning, Ramon Angel Ruiz Fernandez, Anna Lupato, Lukas Novotny, Alibordi Muhammad, Thibaud Humair, . . .
**B-meson mixing: Theory**

- $B_s^0 - \bar{B}_s^0$ oscillations between flavor eigenstates $|B_s^0\rangle$ and $|\bar{B}_s^0\rangle$

\[
\frac{i}{\hbar} \frac{d}{dt} \left( |B_s^0(t)\rangle \right) = \left( \hat{M} - \frac{i}{2} \hat{\Gamma} \right) \left( |B_s^0(t)\rangle \right),
\]

\[
\hat{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}, \quad \hat{\Gamma} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{22} \end{pmatrix}
\]

- Diagonalize the matrices

\[
|B_{s,L}\rangle = p |B_s^0\rangle + q |\bar{B}_s^0\rangle
\]

\[
|B_{s,H}\rangle = p |B_s^0\rangle - q |\bar{B}_s^0\rangle
\]

- Mass eigenstates: $|B_{s,L}\rangle$ (lighter) and $|B_{s,H}\rangle$ (heavier)
Physical observables depend on: $|M_{12}|, |\Gamma_{12}|, \phi_s$

- $\Delta m_s$: $B^0_s - \bar{B}^0_s$ oscillation frequency
  \[ \Delta M_s = M_H - M_L \approx 2|M_{12}| \]
  $t$ quark is dominant in SM, sensitivity to NP in the loops

- $\Delta \Gamma_s$: $B^0_s - \bar{B}^0_s$ width difference
  \[ \Delta \Gamma_s = \Gamma_L - \Gamma_H \approx 2|\Gamma_{12}| \cos \phi_s \]
  only $u$ and $c$ contribute, precision probe of SM, little room for NP

- $\phi_s$: CP-asymmetry in the mixing
  \[ a_{fs} = \text{Im} \left( \frac{\Gamma_{12}}{M_{12}} \right) = \left| \frac{\Gamma_{12}}{M_{12}} \right| \sin \phi_s \]
Our interest: $\Delta \Gamma_s$ from $B_s^0 - \bar{B}_s^0$

Experimental value (HFLAV 2020 average)

$$\Delta \Gamma_{\text{exp}} = (0.085 \pm 0.004) \text{ ps}^{-1}$$

Theory prediction (NLO + $n_f$-piece of NNLO QCD corrections) [Beneke et al., 1999; Ciuchini et al., 2002, 2003; Lenz & Nierste, 2007; Asatrian et al., 2020, 2017]

$$\Delta \Gamma_{\text{OS}} = (0.077 \pm 0.015_{\text{pert.}} \pm 0.002_{B, \bar{B}_S} \pm 0.017_{\Lambda_{QCD}/m_b}) \text{ ps}^{-1}$$

$$\Delta \Gamma_{\text{MS}} = (0.088 \pm 0.011_{\text{pert.}} \pm 0.002_{B, \bar{B}_S} \pm 0.014_{\Lambda_{QCD}/m_b}) \text{ ps}^{-1}$$

Large perturbative uncertainty from the uncalculated NNLO corrections (pert.)

Can be reduced by including relevant 2- and 3-loop QCD corrections

Theory under pressure, full NNLO corrections highly desirable
Overview of the matching calculation

- $|\Delta B| = 1$ EFT ($m_b \ll m_W, m_t$)

  $b \to s c\bar{c}$ [Chetyrkin et al., 1998]

  Representative diagrams in the $|\Delta B| = 1$ EFT needed for the NNLO accuracy

- $|\Delta B| = 2$ EFT (via HQE)

  Matched to the $|\Delta B| = 2$ EFT

\[ \Gamma_{12} \sim \frac{1}{m_b^3} \sum_i \left( \frac{\alpha_s}{4\pi} \right)^i \Gamma_3^{(i)} + \frac{1}{m_b^4} \sum_i \left( \frac{\alpha_s}{4\pi} \right)^i \Gamma_4^{(i)} + \ldots \]
**B-meson mixing: Calculation**

$|\Delta B| = 1$ side of the matching: operator basis

Effective Hamiltonian of the $|\Delta B| = 1$ theory in the CMM basis [Chetyrkin et al., 1998]

$$
\mathcal{H}_{\text{eff}}^{|\Delta B|=1} = \frac{4G_F}{\sqrt{2}} \left[ -V_{ts}^* V_{tb}^\dagger \left( \sum_{i=1}^{6} C_i Q_i + C_8 Q_8 \right) - V_{us}^* V_{ub}^\dagger \sum_{i=1}^{2} C_i (Q_i - Q_i^u) \right. \\
+ V_{us}^* V_{cb} \sum_{i=1}^{2} C_i Q_i^{cu} + V_{cs}^* V_{ub} \sum_{i=1}^{2} C_i Q_i^{uc} \left. \right] + \text{h.c.},
$$

**Current operators**

- $Q_1 = \bar{s}_L \gamma_\mu T^a c_L \bar{c}_L \gamma_\mu T^a b_L,$
- $Q_2 = \bar{s}_L \gamma_\mu c_L \bar{c}_L \gamma_\mu b_L,$
- $Q_1^u = \bar{s}_L \gamma_\mu T^a u_L \bar{u}_L \gamma_\mu T^a b_L,$
- $Q_2^u = \bar{s}_L \gamma_\mu u_L \bar{u}_L \gamma_\mu b_L,$
- $Q_1^{cu} = \bar{s}_L \gamma_\mu T^a u_L \bar{c}_L \gamma_\mu T^a b_L,$
- $Q_2^{cu} = \bar{s}_L \gamma_\mu u_L \bar{c}_L \gamma_\mu b_L,$
- $Q_1^{uc} = \bar{s}_L \gamma_\mu T^a c_L \bar{u}_L \gamma_\mu T^a b_L,$
- $Q_2^{uc} = \bar{s}_L \gamma_\mu c_L \bar{u}_L \gamma_\mu b_L,$

**Penguin operators**

- $Q_3 = \bar{s}_L \gamma_\mu b_L \sum_q \bar{q} \gamma_\mu q,$
- $Q_4 = \bar{s}_L \gamma_\mu T^a b_L \sum_q \bar{q} \gamma_\mu T^a q,$
- $Q_5 = \bar{s}_L \gamma_\mu_1 \gamma_\mu_2 \gamma_\mu_3 b_L \sum_q \bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} q,$
- $Q_6 = \bar{s}_L \gamma_\mu_1 \gamma_\mu_2 \gamma_\mu_3 T^a b_L \sum_q \bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} T^a q,$
- $Q_8 = \frac{g_s}{16\pi^2} m_b \bar{s}_L \sigma^{\mu\nu} T^a b_R G_{\mu\nu}^a,$
Effective Hamiltonian of the $|\Delta B| = 1$ theory in the CMM basis [Chetyrkin et al., 1998]

$$\mathcal{H}_{\text{eff}}^{\Delta B = 1} = \frac{4G_F}{\sqrt{2}} \left[ -V_{ts}^* V_{tb}^\dagger \left( \sum_{i=1}^{6} C_i Q_i + C_8 Q_8 \right) - V_{us}^* V_{ub}^\dagger \sum_{i=1}^{2} C_i (Q_i - Q_i^u) \right] + V_{us}^* V_{cb} \sum_{i=1}^{2} C_i Q_i^{cu} + V_{cs}^* V_{ub} \sum_{i=1}^{2} C_i Q_i^{uc} \right] + \text{h.c.},$$

- 4-fermion vertices generate Dirac structures with multiple insertions of $\gamma$ matrices

\[
(P_L)_{ij} \times (P_L)_{kl}, \quad (\gamma^\mu P_L)_{ij} \times (\gamma^\mu P_L)_{kl}, \quad (\gamma^\mu \gamma^\nu P_L)_{ij} \times (\gamma^\mu \gamma^\nu P_L)_{kl}, \quad (\gamma^\mu \gamma^\nu \gamma^\rho P_L)_{ij} \times (\gamma^\mu \gamma^\nu \gamma^\rho P_L)_{kl}, \quad (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma P_L)_{ij} \times (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma P_L)_{kl}, \quad \ldots
\]

- 4-dimensions: Products of $\gamma$ matrices reducible using Fierz and Chisholm identities

\[
\gamma^\mu \gamma^\nu \gamma^\rho = g^{\mu \nu} \gamma^\rho + g^{\nu \rho} \gamma^\mu - g^{\mu \rho} \gamma^\nu + i \epsilon^{\mu \nu \rho \sigma} \gamma_\sigma \gamma^5
\]

- $d$-dimensions: Fierz and Chisholm identities become ambiguous

- Proper treatment using evanescent operators [Dugan & Grinstein, 1991; Herrlich & Nierste, 1995]
B-meson mixing: Calculation

$|\Delta B| = 1$ side of the matching: operator basis

Effective Hamiltonian of the $|\Delta B| = 1$ theory in the CMM basis [Chetyrkin et al., 1998]

$$
\mathcal{H}_{\text{ef}}^{|\Delta B|=1} = \frac{4G_F}{\sqrt{2}} \left[ -V_{ts}^* V_{tb}^\dagger \left( \sum_{i=1}^{6} C_i Q_i + C_8 Q_8 \right) - V_{us}^* V_{ub}^\dagger \sum_{i=1}^{2} C_i (Q_i - Q_i^u) \right. \\
+ V_{us}^* V_{cb} \sum_{i=1}^{2} C_i Q_i^{cu} + V_{cs}^* V_{ub} \sum_{i=1}^{2} C_i Q_i^{uc} \left. \right] + \text{h.c.},
$$

$|\Delta B| = 1$ LO evanescent operators

$$
E_{1}^{(1)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} T^a_{cb} \bar{c}_L \gamma_1 \gamma_2 \gamma_3 T^a_{cL} b_L - 16Q_1, \\
E_{2}^{(1)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} c_L \bar{c}_L \gamma_1 \gamma_2 \gamma_3 b_L - 16Q_2, \\
E_{3}^{(1)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} b_L \sum_q \bar{q} \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 q - 20Q_5 + 64Q_3, \\
E_{4}^{(1)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} T^a_{cb} b_L \sum_q \bar{q} \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 T^a_{cL} q - 20Q_6 + 64Q_4
$$
**B-meson mixing: Calculation**

$|\Delta B| = 1$ side of the matching: operator basis

Effective Hamiltonian of the $|\Delta B| = 1$ theory in the CMM basis [Chetyrkin et al., 1998]

$$
\mathcal{H}_{\text{eff}}^{|\Delta B|=1} = \frac{4G_F}{\sqrt{2}} \left[ -V_{ts}^* V_{tb}^\dagger \left( \sum_{i=1}^{6} C_i Q_i + C_8 Q_8 \right) - V_{us}^* V_{ub}^\dagger \sum_{i=1}^{2} C_i (Q_i - Q_i^u) \right. \\
+ \left. V_{us}^* V_{cb} \sum_{i=1}^{2} C_i Q_i^{cu} + V_{cs}^* V_{ub} \sum_{i=1}^{2} C_i Q_i^{uc} \right] + \text{h.c.},
$$

$|\Delta B| = 1$ NLO evanescent operators

$$
E_1^{(2)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} T^a c_L \bar{c} \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5} T^a b_L - 20E_1^{(1)} - 256Q_1,
$$

$$
E_2^{(2)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} c_L \bar{c} \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5} b_L - 20E_2^{(1)} - 256Q_2,
$$

$$
E_3^{(2)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} \gamma^{\mu_6} \gamma^{\mu_7} b_L \sum_q \bar{q} \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5} \gamma_{\mu_6} \gamma_{\mu_7} q - 336Q_5 + 1280Q_3,
$$

$$
E_4^{(2)} = \bar{s}_L \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} T^a b_L \sum_q \bar{q} \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5} T^a q - 336Q_6 + 1280Q_4.
$$

Evanescent operators are of $\mathcal{O}(\varepsilon)$, formally vanishing in the $d \rightarrow 4$ limit

However: A pole multiplying tree-level matrix element of an ev. operator $\langle E_i^{(j)} \rangle / \varepsilon$ is $\mathcal{O}(\varepsilon^0)$
$|\Delta B| = 1$ side of the matching: representative diagrams

3-loop $O_{1,2} \times O_{1,2}$ correlators

2-loop $O_{1,2} \times O_{3-6}$ correlators

2-loop $O_{1,2} \times O_{8}$ correlators
\[ \Delta B = 2 \] side of the matching: operator basis

- **\( \Delta \Gamma_s \) described by local \( |\Delta B| = 2 \) operators** [Beneke et al., 1999; Lenz & Nierste, 2007; Asatrian et al., 2017]
- **Using Heavy Quark Expansion** [Khoze & Shifman, 1983; Shifman & Voloshin, 1985; Khoze et al., 1987; Chay et al., 1990; Bigi & Uraltsev, 1992; Bigi et al., 1992, 1993; Blok et al., 1994; Manohar & Wise, 1994] (expansion in \( \Lambda_{\text{QCD}}/m_b \)) one arrives at
  \[
  \Gamma_{12} = - (\lambda_c^q)^2 \Gamma_{12}^{cc} - 2\lambda_c^q \lambda_u^q \Gamma_{12}^{uc} - (\lambda_u^q)^2 \Gamma_{12}^{uu}, \quad \lambda_{q'}^q \equiv V_{q'q}^\ast V_{q'b}
  \]
  \[
  \Gamma_{12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[ H^{ab}(z) \langle B_s | Q | \bar{B}_s \rangle + H^{ab}_S(z) \langle B_s | \bar{Q}_S | \bar{B}_s \rangle \right] + \mathcal{O}(\Lambda_{\text{QCD}}/m_b)
  \]
- **Physical \( |\Delta B| = 2 \) operators**
  \[
  Q = \bar{s}_i \gamma^\mu (1 - \gamma^5) b_i \bar{s}_j \gamma^\mu (1 - \gamma^5) b_j \quad \bar{Q}_S = \bar{s}_i (1 - \gamma^5) b_j \bar{s}_j (1 - \gamma^5) b_i
  \]

- **Additional operators needed at intermediate stages** (e.g. basis changes, def. of evanescent operators)
  \[
  \tilde{Q} = \bar{s}_i \gamma^\mu (1 - \gamma^5) b_j \bar{s}_j \gamma^\mu (1 - \gamma^5) b_i \quad \bar{Q}_S = \bar{s}_i (1 - \gamma^5) b_i \bar{s}_j (1 - \gamma^5) b_j
  \]
- **Not shown here:** evanescent \( |\Delta B| = 2 \) operators and the \( 1/m_b \) suppressed operator \( R_0 \)
- **\( H(z) \) and \( \tilde{H}_S(z) \):** Wilson coefficients from the perturbative matching of \( |\Delta B| = 1 \) to \( |\Delta B| = 2 \), \( z \equiv m_c^2/m_b^2 \)
- **Nonperturbative ME** \( \langle B_s | Q | \bar{B}_s \rangle \) and \( \langle B_s | \bar{Q}_S | \bar{B}_s \rangle \) (also for \( B_d \) mesons) from QCD/HQET sum rules [Ovchinnikov & Pivovarov, 1988; Reinders & Yazaki, 1988; Korner et al., 2003; Mannel et al., 2011; Grozin et al., 2016; Kirk et al., 2017; King et al., 2019], lattice QCD [Bazavov et al., 2016; Dowdall et al., 2019] or combined [Di Luzio et al., 2019]
$|\Delta B| = 2$ side of the matching: representative diagrams

Wilson coefficients of the $|\Delta B| = 2$ theory determined in the matching to $|\Delta B| = 1$

$$\Gamma_{12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[ H^{ab}(z) \langle B_s | Q | B_s \rangle + \tilde{H}_S^{ab}(z) \langle B_s | \tilde{Q}_S | B_s \rangle \right] + O(\Lambda_{QCD}/m_b)$$
\[ |\Delta B| = 1 \text{ contributions needed for NNLO (always 2 insertions from } \mathcal{H}_{\text{eff}} |\Delta B| = 1) \]

\[
C_i O_i \sim \begin{cases} 
1 & \text{for } i = 1, 2 \\
\alpha_s & \text{for } i = 3, 4, 5, 6 \quad (C_{3-6} \text{ numerically small}) \\
\alpha_s & \text{for } i = 8 \quad (\text{explicit strong coupling in the definition of } O_8)
\end{cases}
\]

Important scale: \( z \equiv m_c^2/m_b^2 \)

\[ \text{LO contributions to } \Delta \Gamma_s \]
- 1-loop \( O_{1-2} \times O_{1-2} \) correlators (\( z \)-exact) [Hagelin, 1981; Franco et al., 1982; Chau, 1983; Buras et al., 1984; Khoze et al., 1987; Datta et al., 1987, 1988]

\[ \text{NLO contributions to } \Delta \Gamma_s \ (z \text{-exact}) \]
- 2-loop \( O_{1-2} \times O_{1-2} \) correlators (\( z \)-exact) [Beneke et al., 1999]
- 1-loop \( O_{1-2} \times O_{3-6} \) correlators (\( z \)-exact) [Beneke et al., 1999]
- 1-loop \( O_{1-2} \times O_8 \) correlators (\( z \)-exact) [Beneke et al., 1999]
$|\Delta B| = 1$ contributions needed for NNLO (always 2 insertions from $H_{\text{eff}}^{\Delta B=1}$)

$$C_i O_i \sim \begin{cases} 
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\alpha_s & \text{for } i = 8 \quad (\text{explicit strong coupling in the definition of } O_8)
\end{cases}$$

Important scale: $z \equiv m_c^2/m_b^2$

**NNLO contributions to $\Delta \Gamma_s$**

- 3-loop $O_{1-2} \times O_{1-2}$ correlators [Asatrian et al., 2017, 2020] ($n_f$ piece only, $O(z^3)$)
- 2-loop $O_{1-2} \times O_{3-6}$ correlators [Asatrian et al., 2017, 2020] ($n_f$ piece only, $z$-exact)
- 2-loop $O_{1-2} \times O_8$ correlators [Asatrian et al., 2017, 2020] ($n_f$ piece only, $z$-exact)
- 1-loop $O_{3-6} \times O_{3-6}$ correlators ($z$-exact) [Beneke et al., 1996]
- 1-loop $O_{3-6} \times O_8$ correlators [Asatrian et al., 2017, 2020] ($n_f$ piece only, $z$-exact)
- 1-loop $O_8 \times O_8$ correlators [Asatrian et al., 2017, 2020] ($n_f$ piece only, $z$-exact)

**This work**

- Full ($n_f + \text{non-}n_f$) results for all 2-loop correlators at $O(z)$ (including $O_8 \times O_8 \Rightarrow N^3\text{LO}$)
- Full ($n_f + \text{non-}n_f$) results for the 3-loop $O_{1-2} \times O_{1-2}$ at $O(z^0)$
- WIP: Final checks for the 3-loop result, higher order expansions in $z$, possibly $z$-exact results for selected correlators
**Calculation**

**Matching strategy**
- Matching done on-shell: $p_b^2 = m_b^2$
- The $s$-quark mass is neglected $\Rightarrow p_s = 0$
- Asymptotic expansion in $z \equiv m_c^2 / m_b^2$ (at first up to $\mathcal{O}(z)$ for 2-loop and $\mathcal{O}(z^0)$ for 3-loop)
- Only the imaginary part of the $|\Delta B| = 1$ diagrams enters the matching

**Regularization**
- Dimensional regularization used both for UV- and IR-divergences
- Cross-check: massive gluons in IR-divergent diagrams at 2-loops
- $\varepsilon_{\text{UV}} + m_g$: renormalized amplitudes manifestly finite $\Rightarrow$ the limit $d \rightarrow 4$ is safe
- $\varepsilon = \varepsilon_{\text{UV}} = \varepsilon_{\text{IR}}$: products of $1/\varepsilon_{\text{IR}}$ and evanescent ME are of $\mathcal{O}(\varepsilon^0)$
NLO matching with $\varepsilon = \varepsilon_{IR} = \varepsilon_{UV}$ (no gluon mass) [Ciuchini et al., 2002]

- Normally, only the matching coefficients of physical $|\Delta B| = 2$ operators are relevant
- Here matching coefficients of evanescent operators are also needed (at intermediate stages)
- $|\Delta B| = 2$ matching coefficients obtain $O(\varepsilon)$ pieces

\[
C = f_0^{(0)} + \varepsilon f_1^{(0)} + \frac{\alpha_s}{4\pi} f_0^{(1)}, \quad C_E = f_{E,0}^{(0)} + \varepsilon f_{E,1}^{(0)} + \frac{\alpha_s}{4\pi} f_{E,0}^{(1)}
\]

- LO matching must be carried out up to $O(\varepsilon)$: fixes $f_0^{(0)}, f_1^{(0)}, f_{E,0}^{(0)}, f_{E,1}^{(0)}$
- At NLO we only need $O(\varepsilon^0)$
- Upon inserting $f_0^{(0)}, f_1^{(0)}, f_{E,0}^{(0)}, f_{E,1}^{(0)}$ at NLO all $1/\varepsilon_{IR}$ poles must cancel.
- Finally, the difference

\[
A_{|\Delta B|=1}^{\text{ren}} - A_{|\Delta B|=2}^{\text{ren}}
\]

is manifestly finite $\Rightarrow$ determine $f_0^{(1)}$

- Only $f_0^{(0)}$ and $f_0^{(1)}$ enter the physical matching coefficient
- What about $f_{E,1}^{(0)}$? Not needed at NLO, must be determined for the NNLO calculation!
- At NNLO, the LO matching must be done up $O(\varepsilon^2)$, the NLO matching up to $O(\varepsilon)$
- The explicit cancellation of IR poles (and of $\xi$) is a highly nontrivial cross-check of the whole calculation
B-meson mixing: Calculation

All computations done using our well-tested automatic setup

- Diagram generation with QGRAF [Nogueira, 1993]
- Insertion of Feynman rules and topology identification using Q2E/EXP [Seidensticker, 1999; Harlander et al., 1998] or TAPIR [Gerlach, Herren, 2022]
- Feynman amplitude evaluation: in-house CALC setup written in FORM [Ruijl et al., 2017]
- IBP-reduction: FIRE 6 [Smirnov & Chuharev, 2020]
- All master integrals checked numerically using FIESTA [Smirnov, 2016] and pySecDec [Borowka et al., 2018]

Cross-checks of selected intermediate results using FeynArts [Hahn, 2001], FeynRules [Christensen & Duhr, 2009; Alloul et al., 2014] and FeynCalc [VS et al., 2020]

Two complementary approaches to the treatment of tensor integrals in FORM

- Explicit decomposition formulas (1 ext. momentum, max. rank 10), calculated using FeynCalc and FERMAT [Lewis]
- Projections to the occurring 4-fermion Dirac structures

\[ \{(P_L)_{ij}, (\gamma^\mu P_L)_{ij}, (\gamma^\mu \gamma^\nu P_L)_{ij}, \ldots\} \otimes \{(P_L)_{kl}, (\gamma^\mu P_L)_{kl}, (\gamma^\mu \gamma^\nu P_L)_{kl}, \ldots\} \]

- Both methods lead to the same results!
New on-shell 3-loop integrals with massive (solid) lines

Only imaginary parts are relevant and turn out to be very simple

Appearing constants

\[ \pi, \ln(2), \zeta_2, \zeta_3, \zeta_4, Cl_2(\pi/3), \sqrt{3}, \]
\[ \text{Li}_4(1/2), \ln \left( (1 + \sqrt{5})/2 \right) \]

Real parts (obtained as a byproduct) more complicated but irrelevant for \( \Delta \Gamma_s \)
Handling of master integrals facilitated using new **FeynCalc** functions added in the course of this project (see my talk at ACAT 2021 next Wednesday)

Graph representation from propagator representation: FCLoopIntegralToGraph, FCLoopGraphPlot

Derivation of the Feynman parametrization: FCFeynmanParametrize

Mappings between master integrals: FCLoopFindIntegralMappings

Official in the upcoming **FeynCalc** 10, however already publicly available and documented
New contributions to $\Gamma_{12}^{s}$ computed in the course of this project ($z = m_c^2/m_b^2$)

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Literature result</th>
<th>This work</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{1,2} \times Q_{3-6}$</td>
<td>2 loops, $z$-exact, $n_f$-part only [Asatrian et al., 2020]</td>
<td>2 loops, $O(z)$, full</td>
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<tr>
<td>$Q_{1,2} \times Q_{1,2}$</td>
<td>3 loops, $O(\sqrt{z})$, $n_f$-part only [Asatrian et al., 2017]</td>
<td>3 loops, $O(z^0)$, full</td>
</tr>
</tbody>
</table>
All 2-loop contributions to the NNLO correction already computed and cross-checked

New theory predictions for the width difference $\Delta \Gamma_s$ and the CP asymmetry $a^{s}_{fs}$ under way

\[
\frac{\Delta \Gamma_s}{\Delta M_s} = -\text{Re} \left( \frac{\Gamma_{12}^s}{M_{12}^s} \right), \quad a^{s}_{fs} = \text{Im} \left( \frac{\Gamma_{12}^s}{M_{12}^s} \right)
\]

Ingredients

\[
\Gamma_{12}^s = -(\lambda_t^s)^2 \left[ \Gamma_{12}^{s,cc} + 2\frac{\lambda_u^s}{\lambda_t^s} (\Gamma_{12}^{s,cc} - \Gamma_{12}^{s,uc}) + \left( \frac{\lambda_u^s}{\lambda_t^s} \right)^2 (\Gamma_{12}^{s,uu} + \Gamma_{12}^{s,cc} - 2\Gamma_{12}^{s,uc}) \right]
\]

\[
\Gamma_{s,12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}^2} \left[ H^{ab}(z) \langle B_s | Q | \bar{B}_s \rangle + \tilde{H}^{ab}(z) \langle B_s | \bar{Q}_S | \bar{B}_s \rangle \right] + \mathcal{O}(\Lambda_{\text{QCD}}/m_b)
\]

\[
M_{12} = (\lambda_t^s)^2 \frac{G_F^2 M_{B_s}}{12\pi^2} M_W^2 \hat{\eta}_B S_0 \left( \frac{m_t^2}{M_W^2} \right) f_{B_s}^2 B_{B_s}
\]

Cancellation of $(\lambda_t^s)^2 = (V_{ts}^* V_{tb}^\dagger)^2$, $f_{B_s}$, $M_{B_s}$ and to large extent bag parameters in the ratio $\Gamma_{12}^s / M_{12}^s$

Following [Asatrian et al., 2020] we can can calculate

\[
\Delta \Gamma_s = \left( \frac{\Delta \Gamma_s}{\Delta M_s} \right) \Delta M_s^{\exp}
\]

$|V_{cb}|$ controversy irrelevant!
Theoretical predictions for the $\overline{\text{MS}}$ and pole schemes

- $m_b^2$ in the prefactor of $\Gamma_{12}$ treated as $(m_b^{\text{OS}})^2$ in the pole scheme and $(m_b^{\text{MS}})^2$ in the $\overline{\text{MS}}$ scheme
- In both schemes we use $\overline{z} = (m_c^{\text{MS}}/m_b^{\text{MS}})^2$

**Numerical input** [Tanabashi et al., 2018; Dowdall et al., 2019; Bazavov et al., 2018; Amhis et al., 2021]

\[
M_{B_s} = 5366.88 \text{ MeV} \quad f_{B_s} = (0.2307 \pm 0.0013) \text{ GeV},
\]

\[
B_{B_s} = 0.813 \pm 0.034, \quad \tilde{B}'_{S,B_s} = 1.31 \pm 0.09,
\]

\[
\frac{\lambda^s_u}{\lambda^s_t} = -(0.00865 \pm 0.00042) + (0.01832 \pm 0.00039)i
\]

\[
\Delta M_s^\text{exp} = (17.749 \pm 0.020) \text{ ps}^{-1}
\]

**Numerical estimate on the impact of the new 2-loop $O_{1,2} \times O_{3-6}$ contribution** [Gerlach, Nierste, VS, Steinhauser, 2021]

1-loop (already known):
\[
\frac{\Delta \Gamma_{s,12 \times 36,\alpha_s^0}}{\Delta \Gamma_s} = 7.0\% \quad \text{(pole)}
\]

full 2-loops (new):
\[
\frac{\Delta \Gamma_{s,12 \times 36,\alpha_s}}{\Delta \Gamma_s} = 0.2\% \quad \text{(pole)},
\]

\[
\frac{\Delta \Gamma_{s,12 \times 36,\alpha_s^0}}{\Delta \Gamma_s^\text{\overline{MS}}} = 6.1\% \quad \text{(MS)}
\]

\[
\frac{\Delta \Gamma_{s,12 \times 36,\alpha_s}}{\Delta \Gamma_s^\text{\overline{MS}}} = 1.4\% \quad \text{(MS)}
\]
Summary

- Experimental precision of $\Delta \Gamma_s$ calls for the NNLO calculation!
- We calculated all building blocks needed to obtain the NNLO correction to $B^0_s - \bar{B}^0_s$ mixing
- All the occurring 3-loop MI from the current-current contribution calculated analytically (for $m_c = 0$)
- The result for the 2-loop current-penguin contribution already published [Gerlach, Nierste, VS, Steinhauser, 2021]

Outlook

- Results for all the remaining 2-loop contributions and the 3-loop current-current piece under way
- New theory predictions for $\Delta \Gamma_s$ and the CP asymmetry $a_{\ell s}^s$
- Higher order expansions in $z \equiv m_c^2 / m_b^2$, ideally $z$-exact results at least for the 2-loop contributions