11th International Workshop on the CKM Unitary Triangle (CKM 2021)
21-26 November 2021

Rare D decays at LHCb

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on behalf of the LHCb Collaboration

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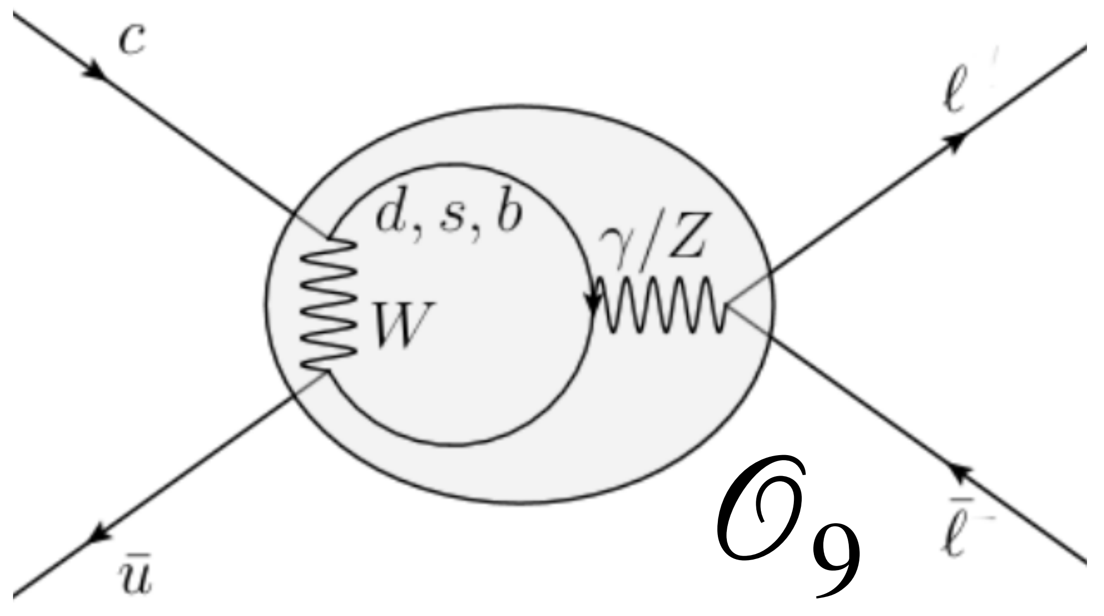
Outline

- Introduction to rare charm decays
- Rare charm decays in LHCb
- Most recent LHCb measurements
  - Search for 25 rare and forbidden decays of D^+ and D_s^+ [JHEP 06 44 (2021)]
  - Angular analysis and search for CP violation in $D^0 \to h^+h^-\mu^+\mu^-$ $(h = K, \pi)$ arXiv:2111.03327 [hep-ex]

Why study rare charm decays?



- **Unique probe:** FCNC $|\Delta c| = 1$ transitions are unique probes in up-type quark sector, complementary to studies in K and B decays.

- **Promising discovery tool:**
  - Extremely suppressed rates $\sim \mathcal{O}(10^{-10})$ (no tree level, GIM suppressed),
  - Negligible CP asymmetries (CKM suppressed),
  - Well known angular distributions (e.g. no lepton axial-vector couplings).

SM

  - Short-distance branching fractions potentially enhanced up to $\mathcal{O}(10^{-7})$,
  - CP asymmetries potentially up to few %,
  - Modified angular distributions.

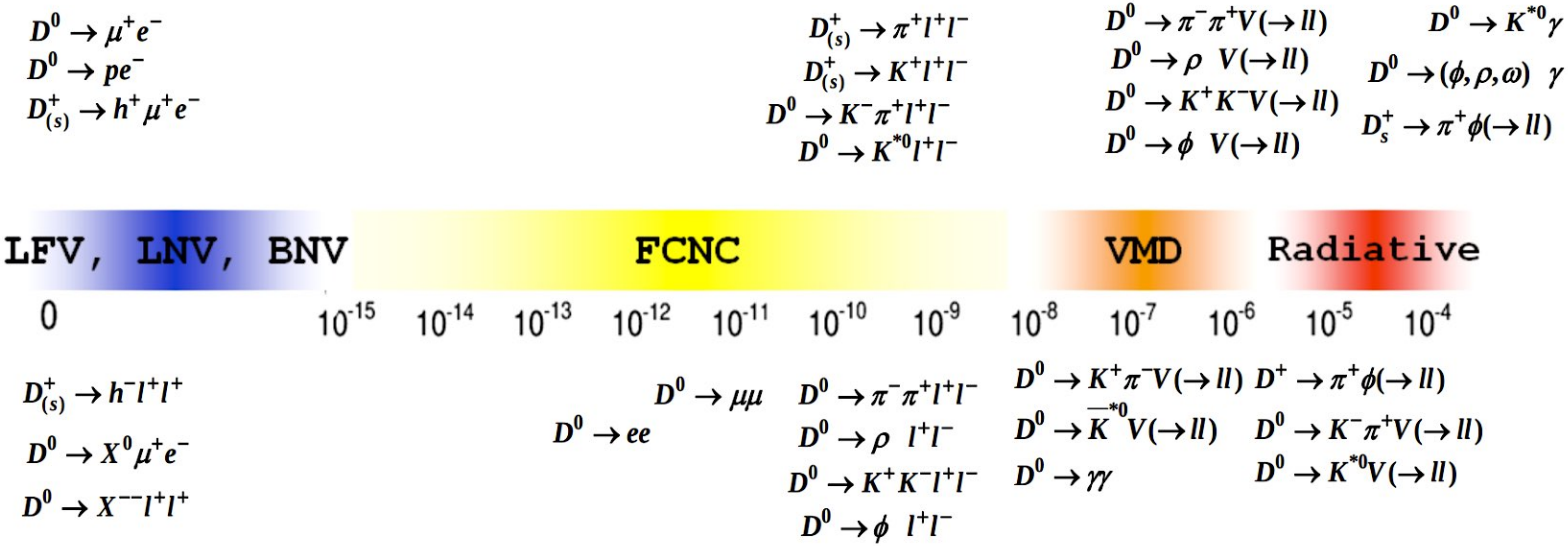
NP

[MPLA 36 (2021) 2130002]

- **Challenging**: large resonant contributions, difficult to obtain precise theoretical predictions. Experimentally: need large charm samples and control over fully hadronic backgrounds.

Spectrum of rare charm decays

Wide scale of rates, from forbidden to not-so-rare decays. Branching fractions dominated by long-distance contributions.

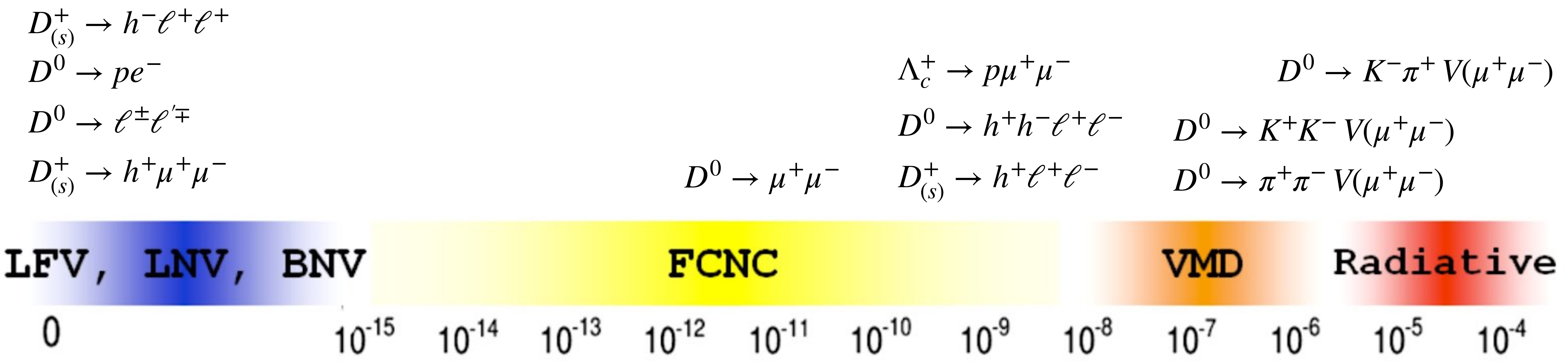


Difficult to search for NP effects in the branching fractions, more promising strategies are:

- Exploit exact symmetries with search for forbidden decays,
- Exploit approximate symmetries with clean SM null-tests in long-distance dominated decays.

Charm rare decays in LHCb

- Large sample of $c\bar{c}$ in LHCb, $N(c\bar{c}) > 10^{13}$,
- Excellent vertexing, tracking and identification. LHCb focused mostly on final states with 2 muons.



Search for NP in branching fractions

- search for $D^0 \to \mu^+\mu^-$ [PLB 725 15-24 (2013)]
- search for $D^+_{(s)} \to h^+\ell^+\ell^-$ [PLB 724 203-212 (2013)] , [JHEP 06 44 (2021)]
- search for $D^0 \to \pi^+\pi^-\mu^+\mu^-$ [PLB 728 234-243 (2014)]
- observation of $D^0 \to K^-\pi^+\mu^+\mu^-$ [PLB 757 558-567 (2016)]
- search for $\Lambda_c^+ \to p\mu^+\mu^-$ [PRD 97 091101 (2018)]
- observation of $D^0 \to h^+h^-\mu^+\mu^-$ [PRL 119 181805 (2017)]

Search for NP based on symmetries

- search for $D^0 \to \mu^+e^-$ [PLB 754 167 (2016)]
- search for $D^+_{(s)} \to h^-\ell^+\ell'^+$ [JHEP 06 44 (2021)]
- search for CPV and angular asymmetries in $D^0 \to h^+h^-\mu^+\mu^-$ [PRL 121 091801 (2018)]
- full angular analysis and search for CPV in $D^0 \to h^+h^-\mu^+\mu^-$ arXiv:2111.03327 [hep-ex]

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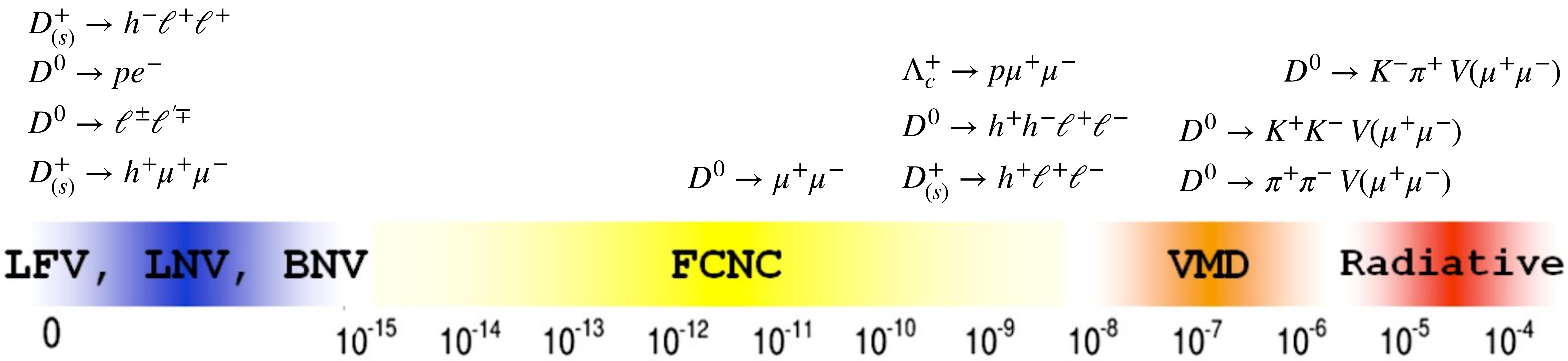

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Searches for 25 rare and forbidden decays of D^+ and D_s^+

LHCB-PAPER-2020-007 - [JHEP 06 44 (2021)]

$D^+ \to \pi^+\mu^+\mu^-$	$D^+ \to \pi^+e^+e^-$	$D^+ \to K^+e^+e^-$	$D_s^+ \to \pi^+e^+\mu^-$	$D_s^+ \to K^+\mu^+e^-$
$D^+ \to \pi^-\mu^+\mu^+$	$D^+ \to \pi^-e^+e^+$	$D_s^+ \to \pi^+\mu^+\mu^-$	$D_s^+ \to \pi^+e^+e^-$	$D_s^+ \to K^-\mu^+e^+$
$D^+ \to \pi^+\mu^+e^-$	$D^+ \to K^+\mu^+\mu^-$	$D_s^+ \to \pi^-\mu^+\mu^+$	$D_s^+ \to \pi^-e^+e^+$	$D_s^+ \to K^+e^+\mu^-$
$D^+ \to \pi^-\mu^+e^+$	$D^+ \to K^+\mu^+e^-$	$D_s^+ \to \pi^+\mu^+e^-$	$D_s^+ \to K^+\mu^+\mu^-$	$D_s^+ \to K^+e^+e^-$
$D^+ \to \pi^+e^+\mu^-$	$D^+ \to K^+e^+\mu^-$	$D_s^+ \to \pi^-\mu^+e^+$	$D_s^+ \to K^-\mu^+\mu^+$	$D_s^+ \to K^-e^+e^+$

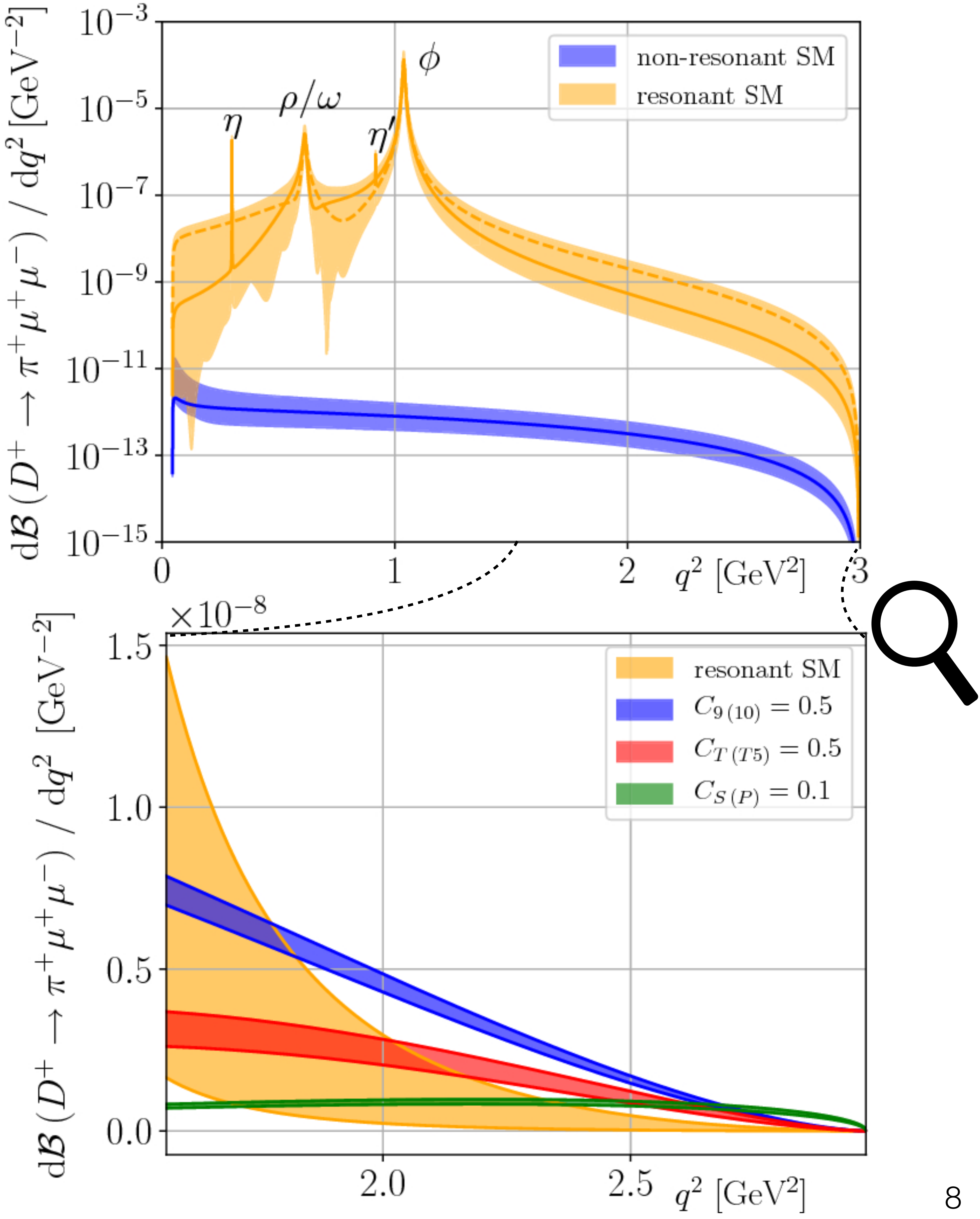
Allowed in the standard model

Effectively forbidden in the standard model

From C.Burr presentation at ICHEP 2020 in Prague

Predictions for rare and forbidden decays of D^+ and D_s^+

- Predictions for $D^+_{(s)}$ forbidden and rare semileptonic decays **[EPJC, Vol.80, 65 (2020)]**,
- Wide range of rates for SM-forbidden decays $(10^{-11} - 10^{-8})$, depending on the NP model.
- For non-forbidden decays, resonant SM highly dominates the decay rate. NP enhancements are possible far away, especially in high $m(\ell^+\ell^-)$.

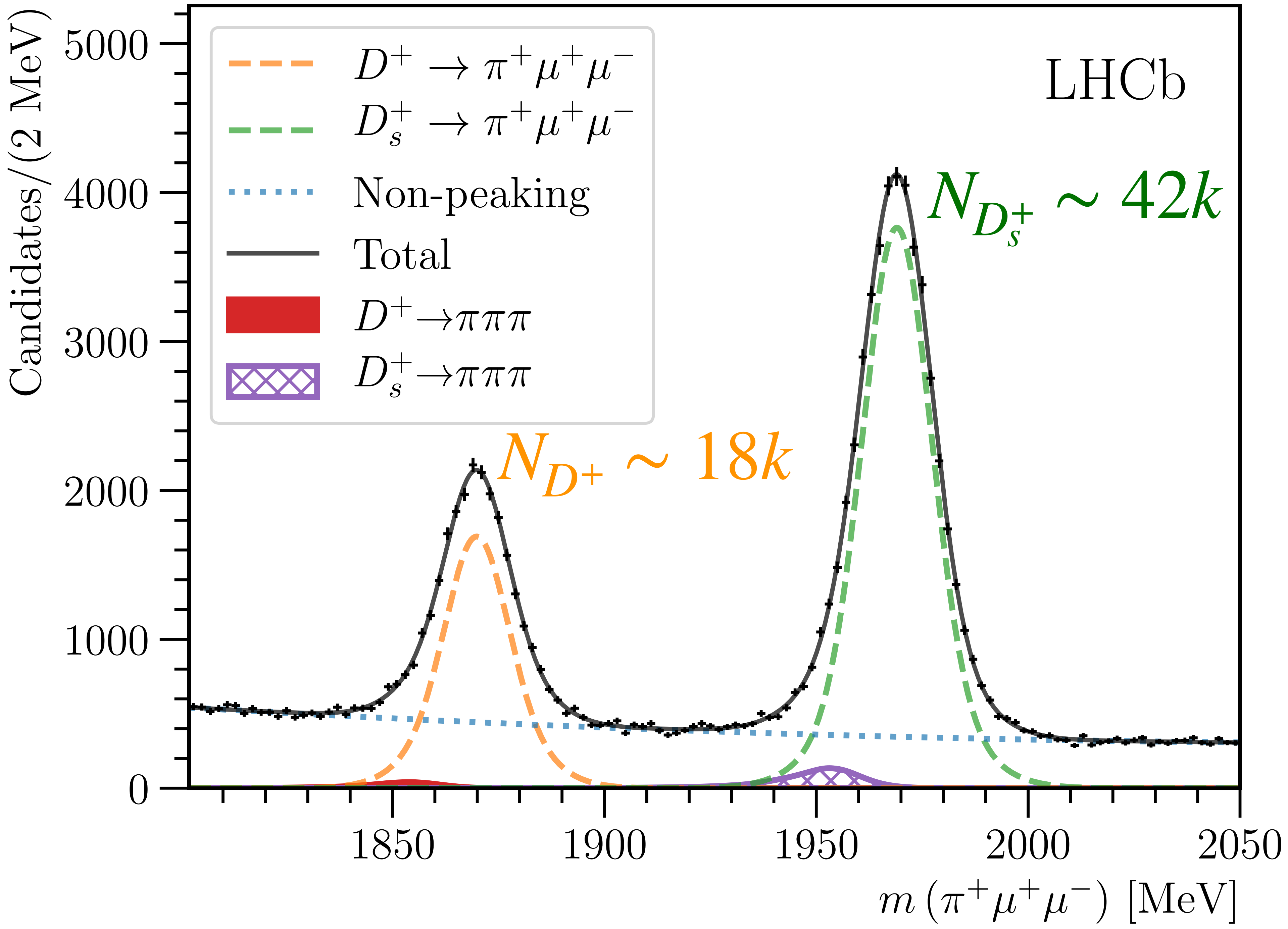


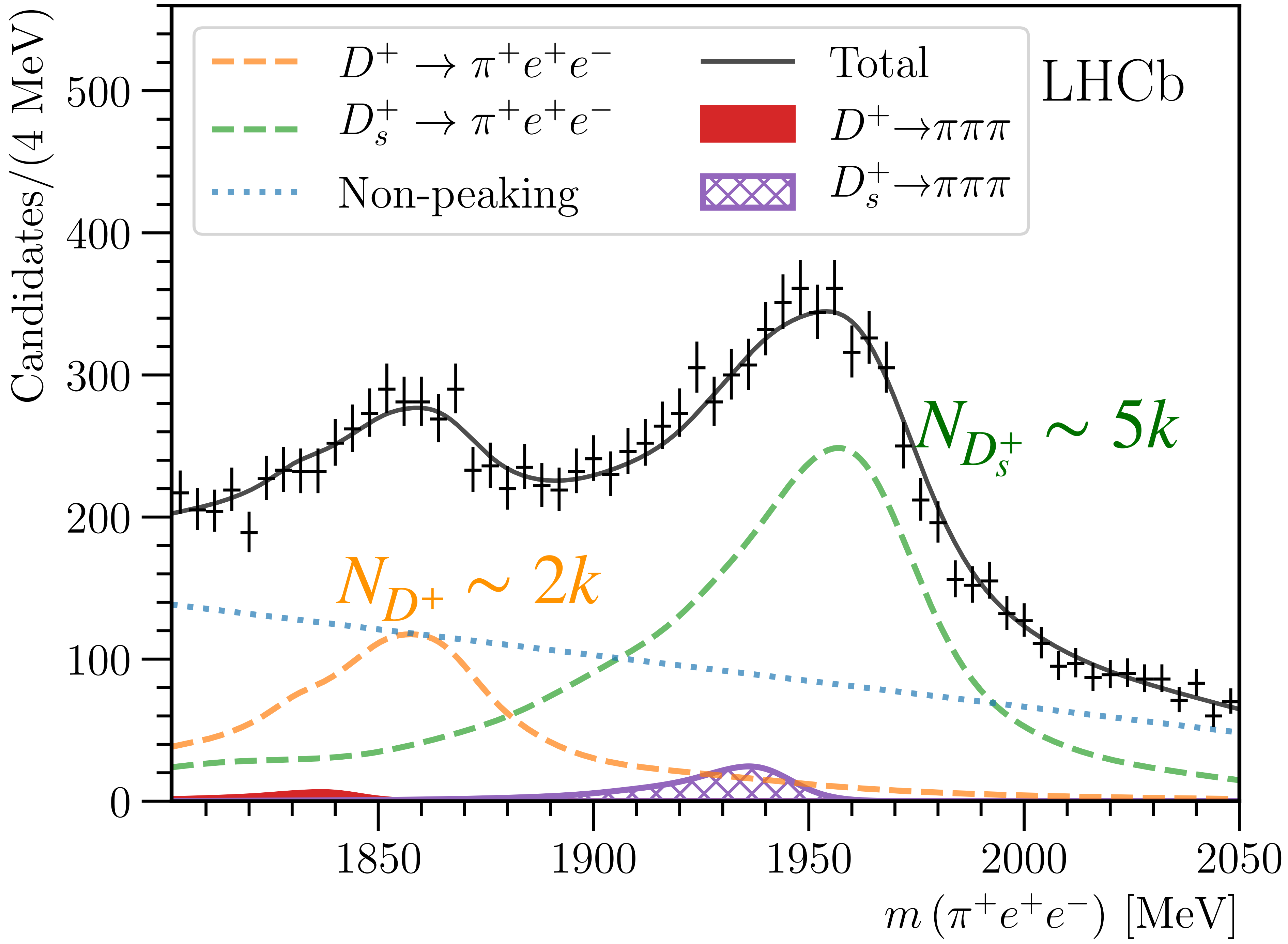
Searches for 25 rare and forbidden decays of D^+ and D_s^+

[JHEP 06 44 (2021)]

- Analysis performed using 2016 dataset, corresponding to 1.7 fb^{-1}.
- Di-muon mass regions corresponding to η and ρ/ω are vetoed.

  $D^+_{(s)} \to (\phi \to \ell^+\ell^-)\,\pi^+$ used as normalisation.
- Signal is extracted by fitting the three body invariant mass,
  - PID selection to suppress the mis-identified background due to hadronic decays.





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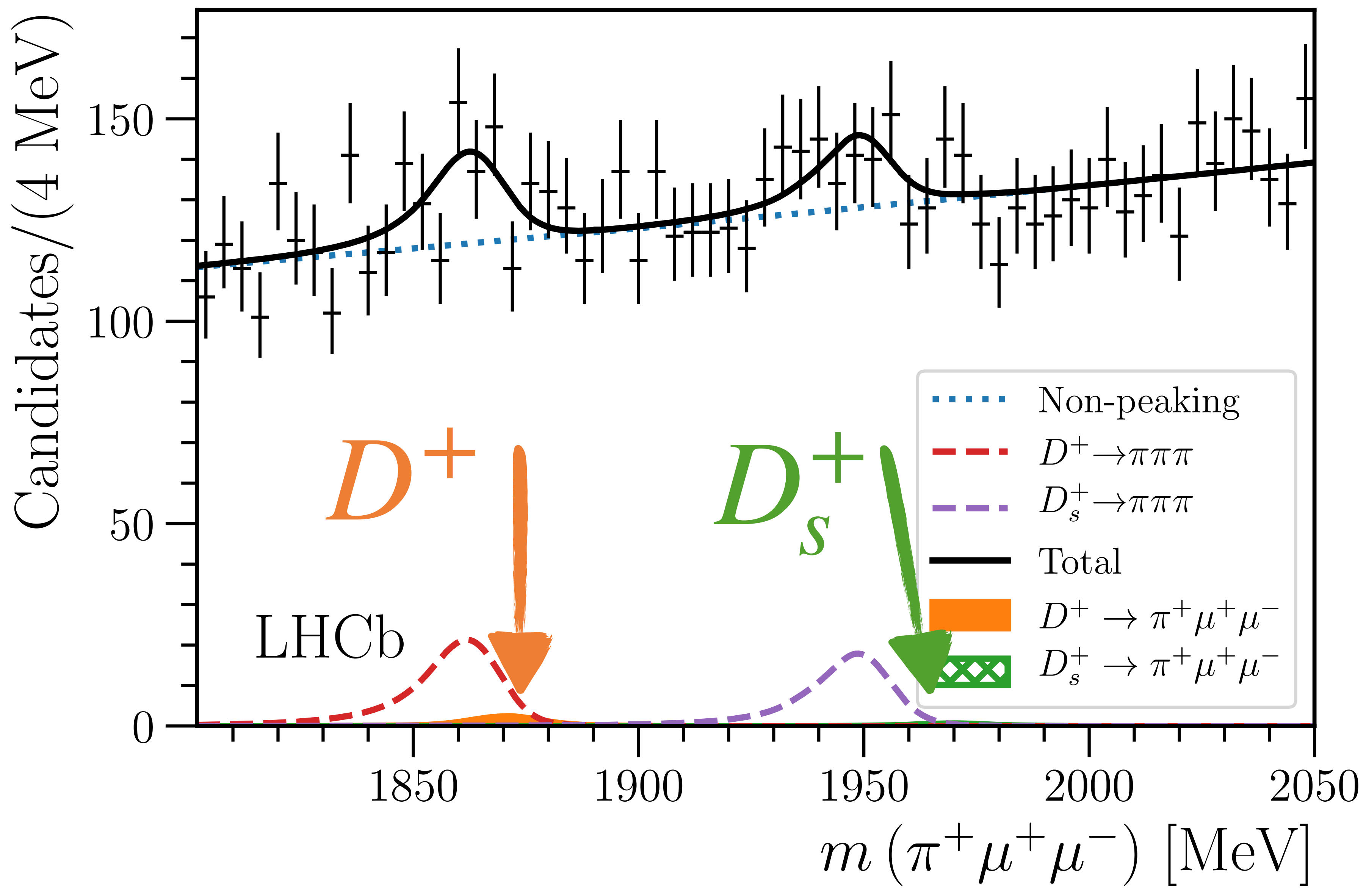
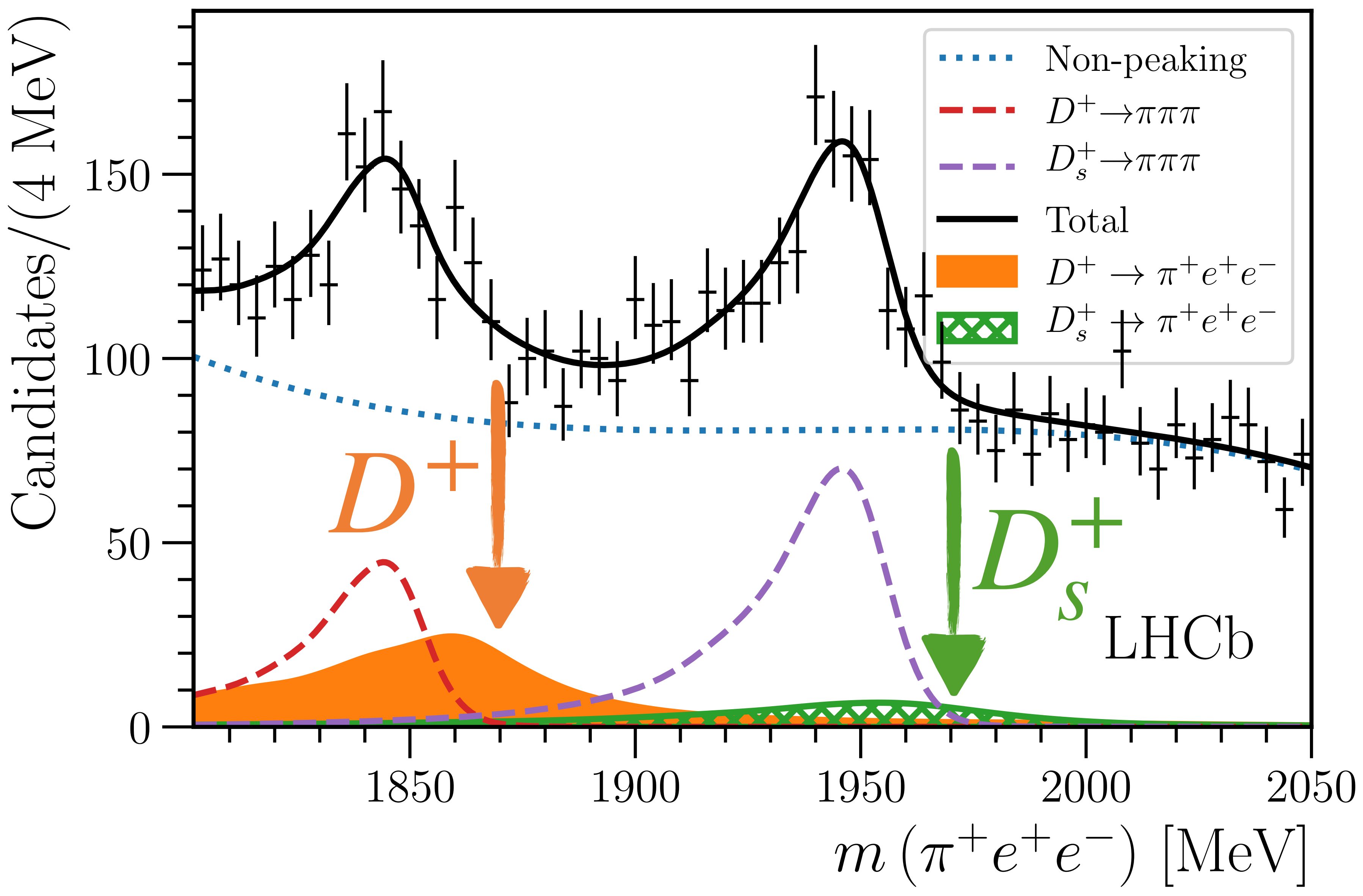
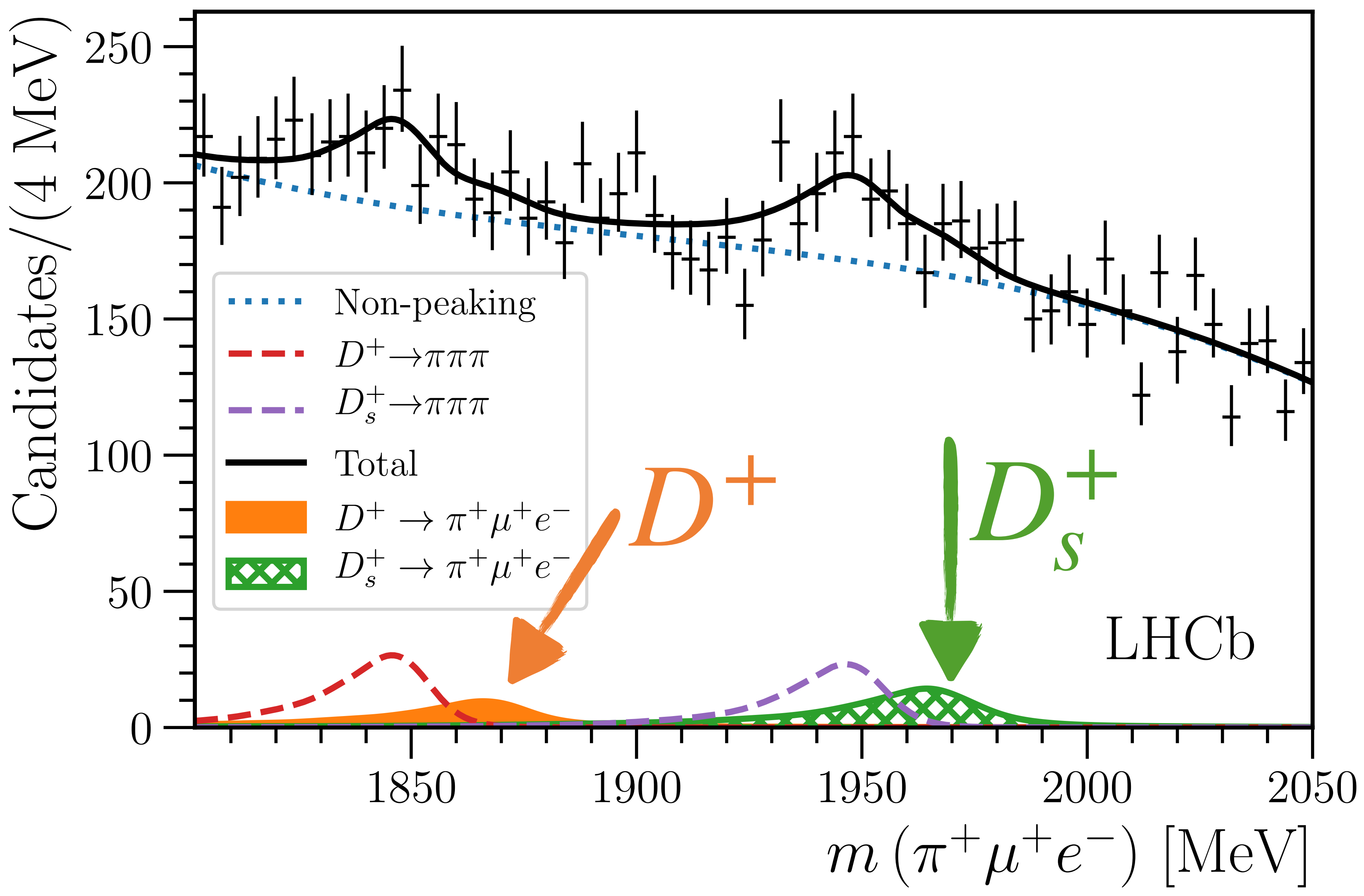
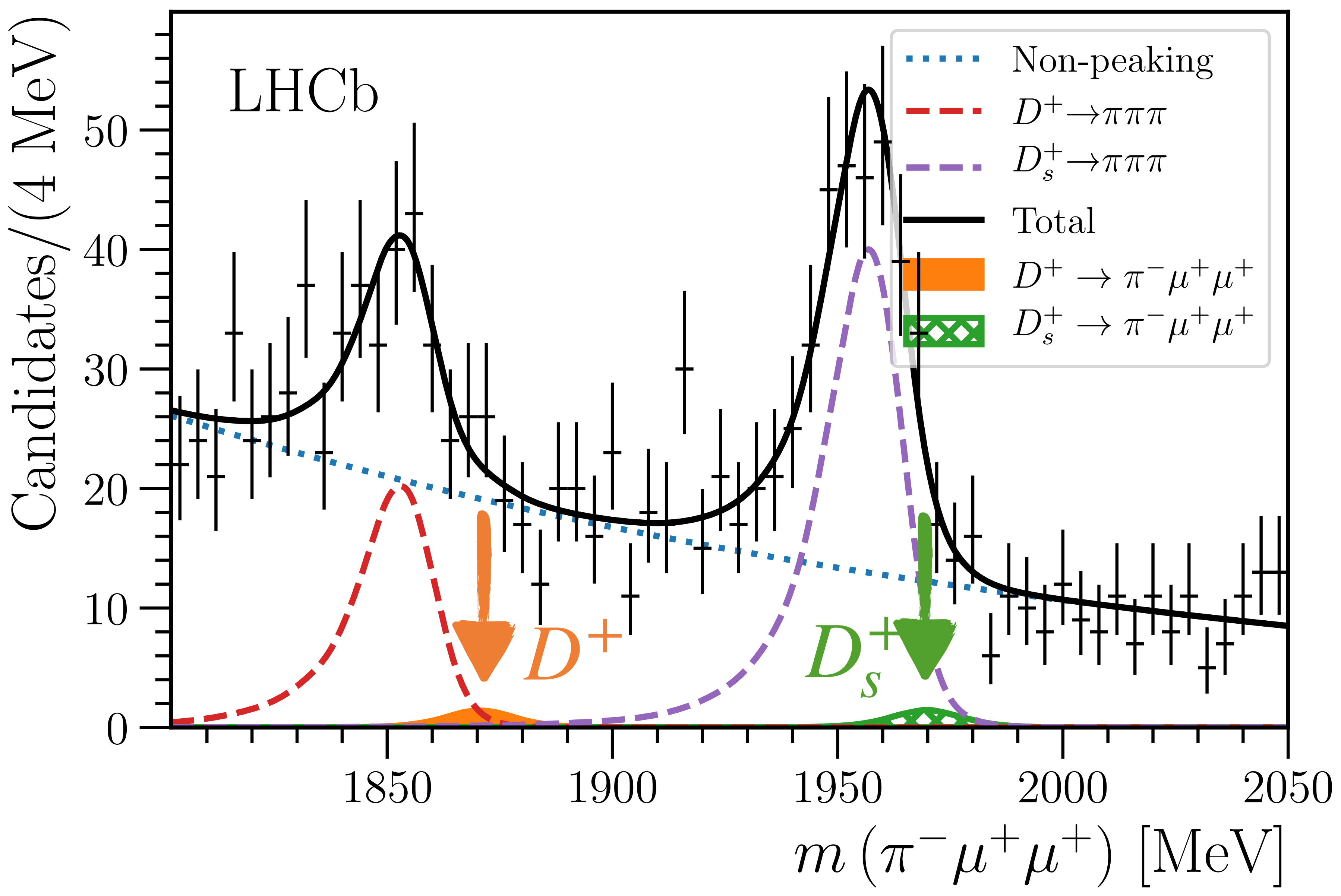




FCNC with muons and electrons in the final state

LFV and LNV decays



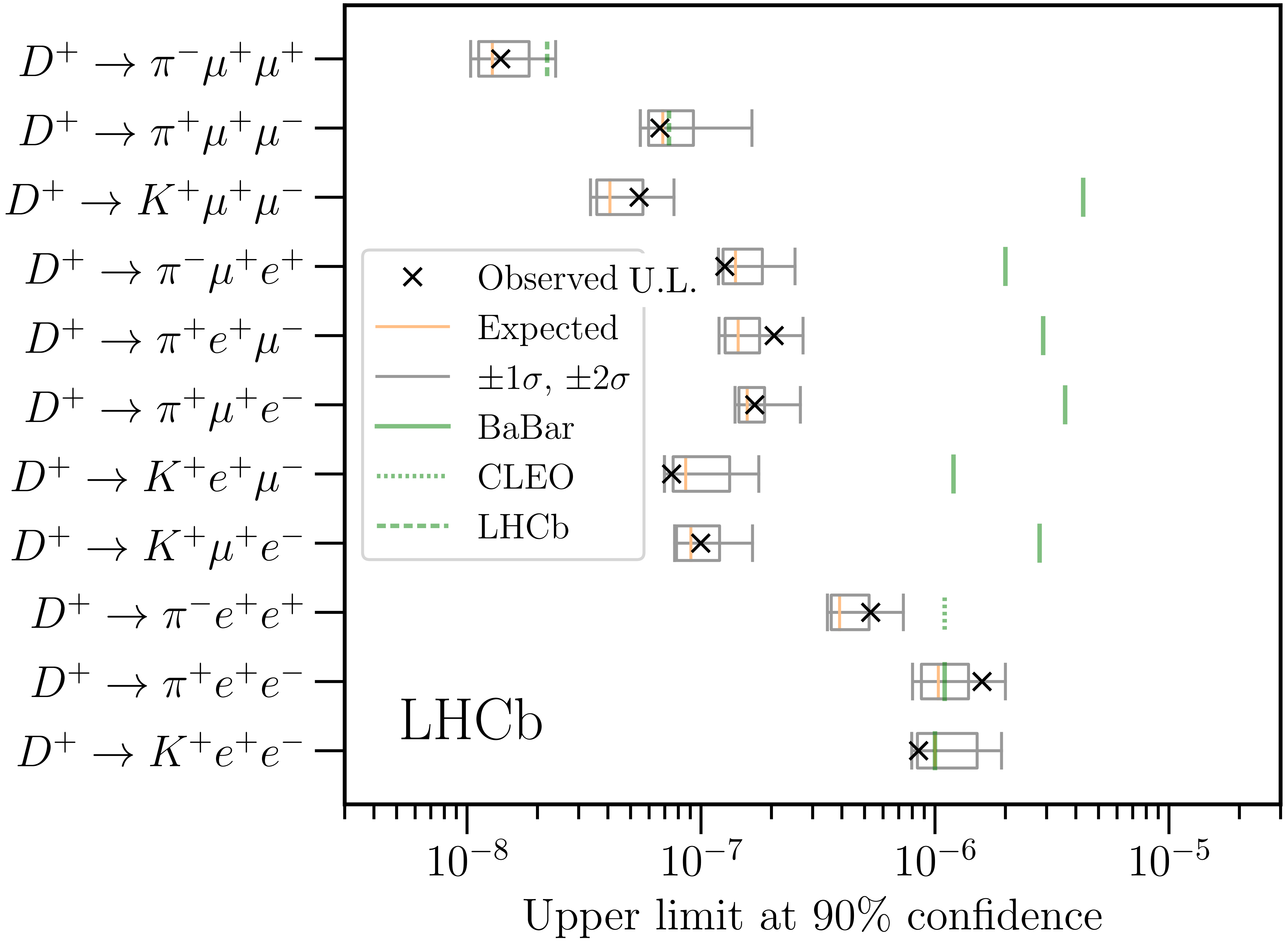


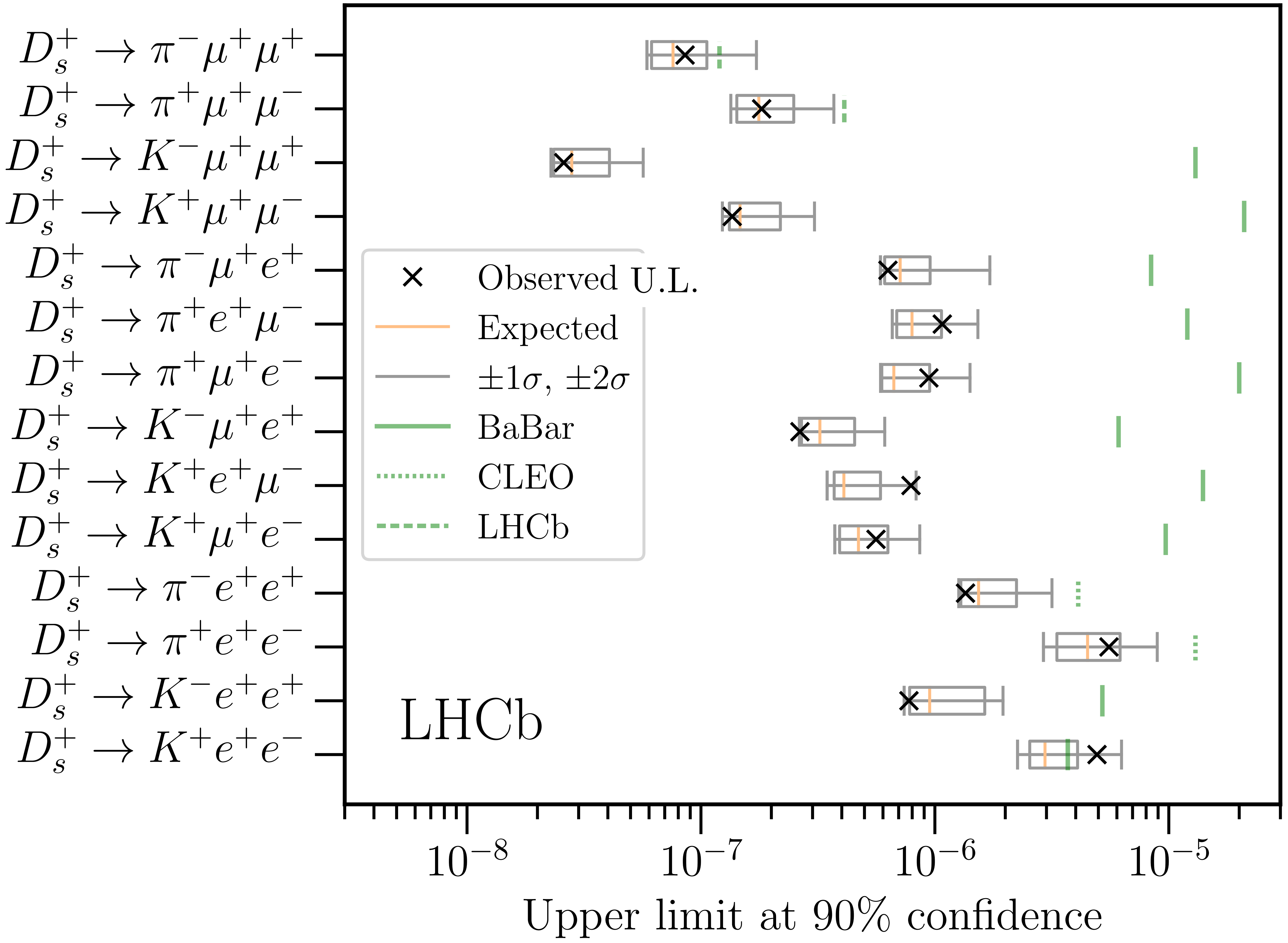
Only decays with pions shown

Searches for 25 rare and forbidden decays of D^+ and D_s^+

[JHEP 06 44 (2021)]

- All results are consistent with the background-only hypothesis,
- Limits on BFs improved by more than 1 order of magnitude (up to a factor of 500).





Angular analysis of $D^0 \to \pi^+\pi^-\mu^+\mu^-$ and $D^0 \to K^+K^-\mu^+\mu^-$ decays and search for CP violation

LHCB-PAPER-2021-035 - arXiv:2111.03327 [hep-ex]

Presented for the first time by D.Mitzel at 11th workshop on “Implications of LHCb measurements and future prospects”

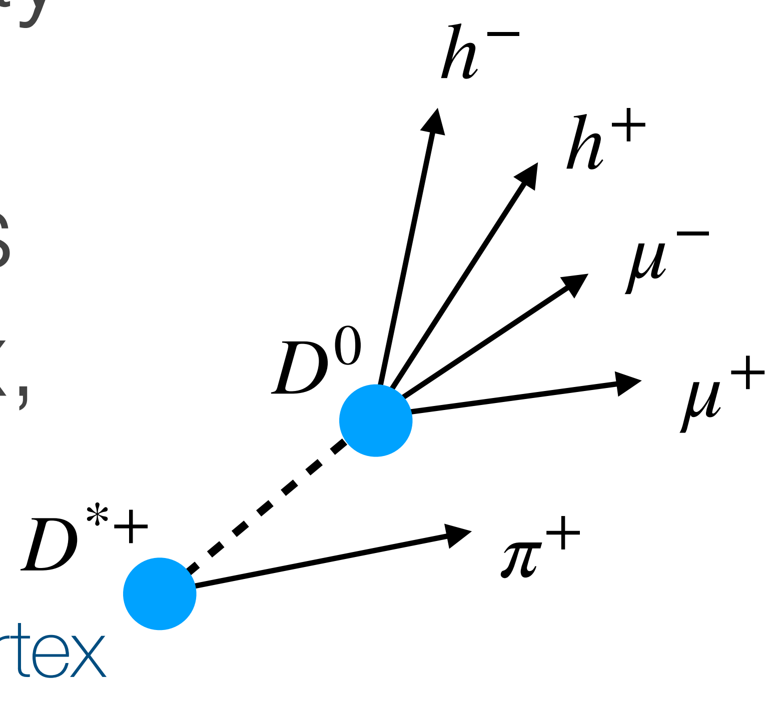
$D^0 \to h^+h^-\mu^+\mu^-$ rare decays

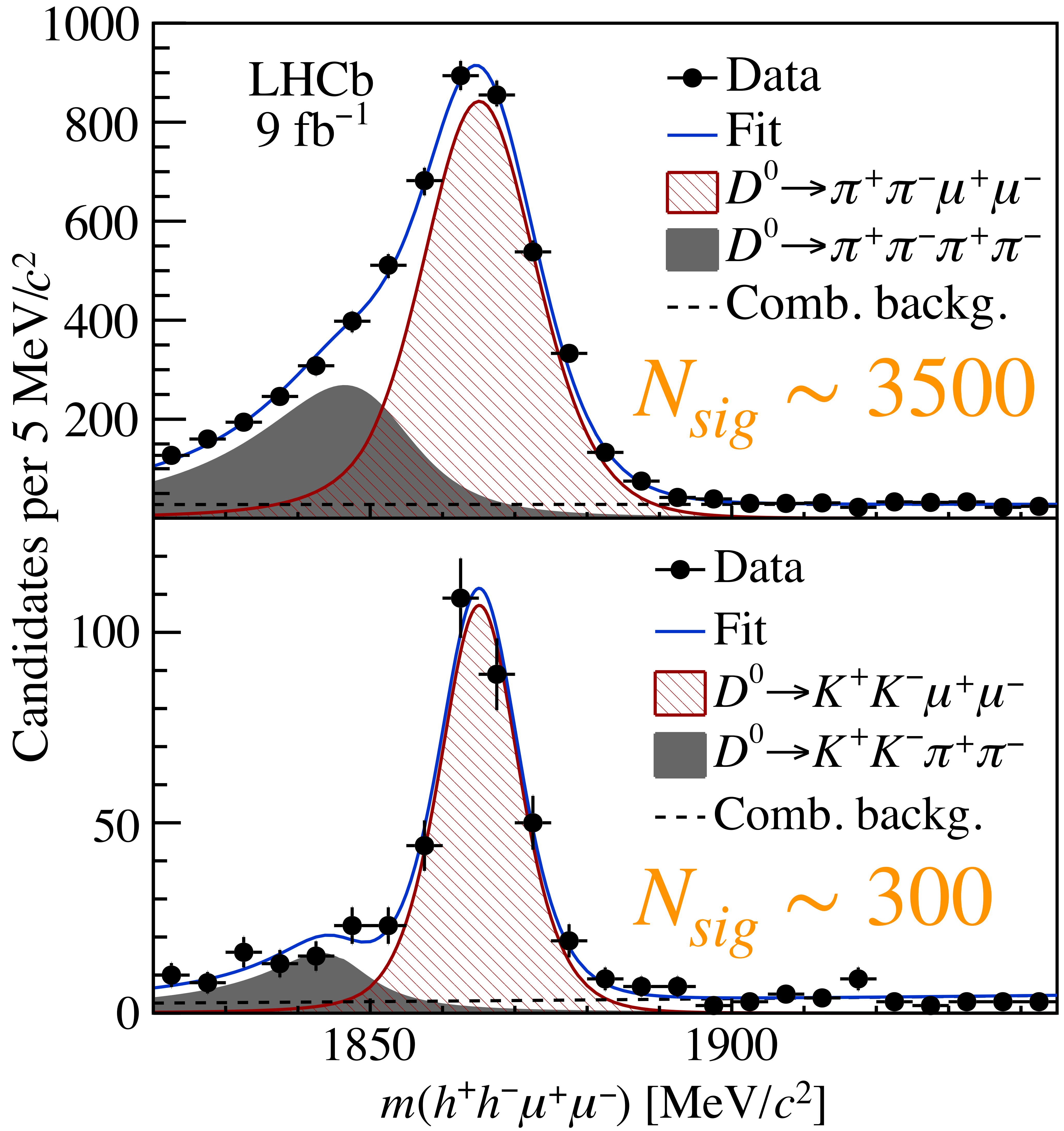
arXiv:2111.03327

- Final state observed by LHCb (rarest charm meson decay observed), compatible with SM [PRL 119 (2017) 181805] , [JHEP 04 (2013) 135]

$$\mathcal{B}(D^0 \to \pi^+\pi^-\mu^+\mu^-) \sim 9.6 \times 10^{-7}$$
$$\mathcal{B}(D^0 \to K^+K^-\mu^+\mu^-) \sim 1.5 \times 10^{-7}$$

- Dominated by resonant contributions in h^+h^- and $\mu^+\mu^-$ systems (mainly ρ/ω and ϕ).
- Analysis performed using 2011-2018 dataset, corresponding to 9 fb^{-1}.
- D^0 selected from $D^{*+} \to D^0\pi^+$ decay ("prompt charm"): well identified kaons/pions and muons forming a displaced secondary vertex, paired with a low-momentum pion to form a D^{*+} vertex.





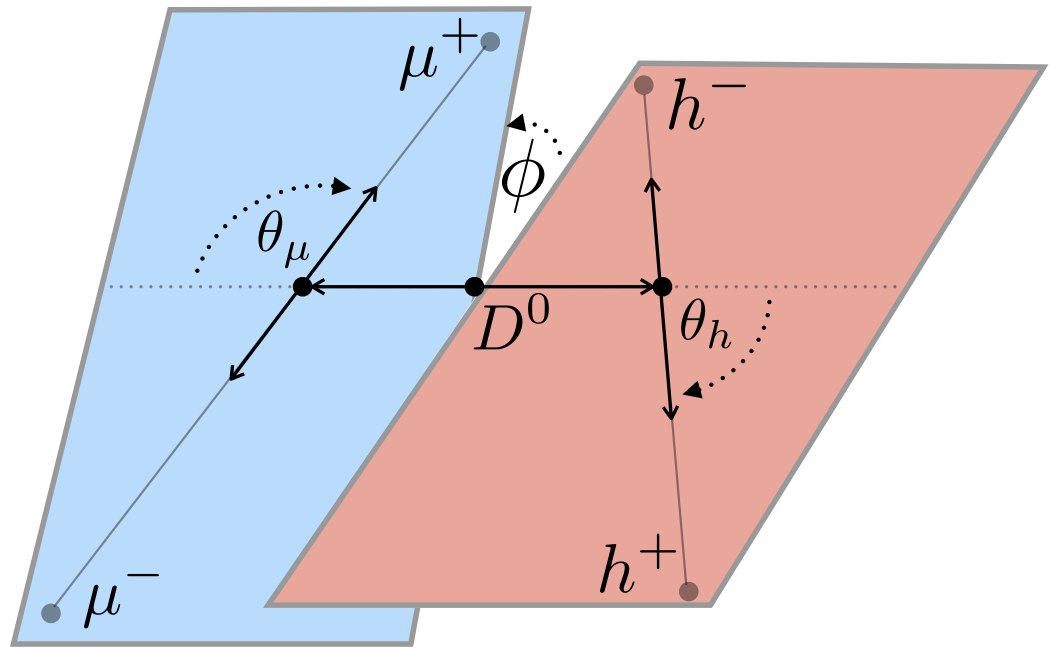
$D^0 \to h^+h^-\mu^+\mu^-$ angular observables

arXiv:2111.03327

- Full angular parametrization [PRD 98 03504 (2018)]

$$\frac{d\Gamma}{dq^2 dp^2 d\Omega} = \frac{1}{2\pi} \sum_{i=1}^{9} I_i(q^2, p^2, \theta_h)\, c_i(\phi, \theta_\mu)$$

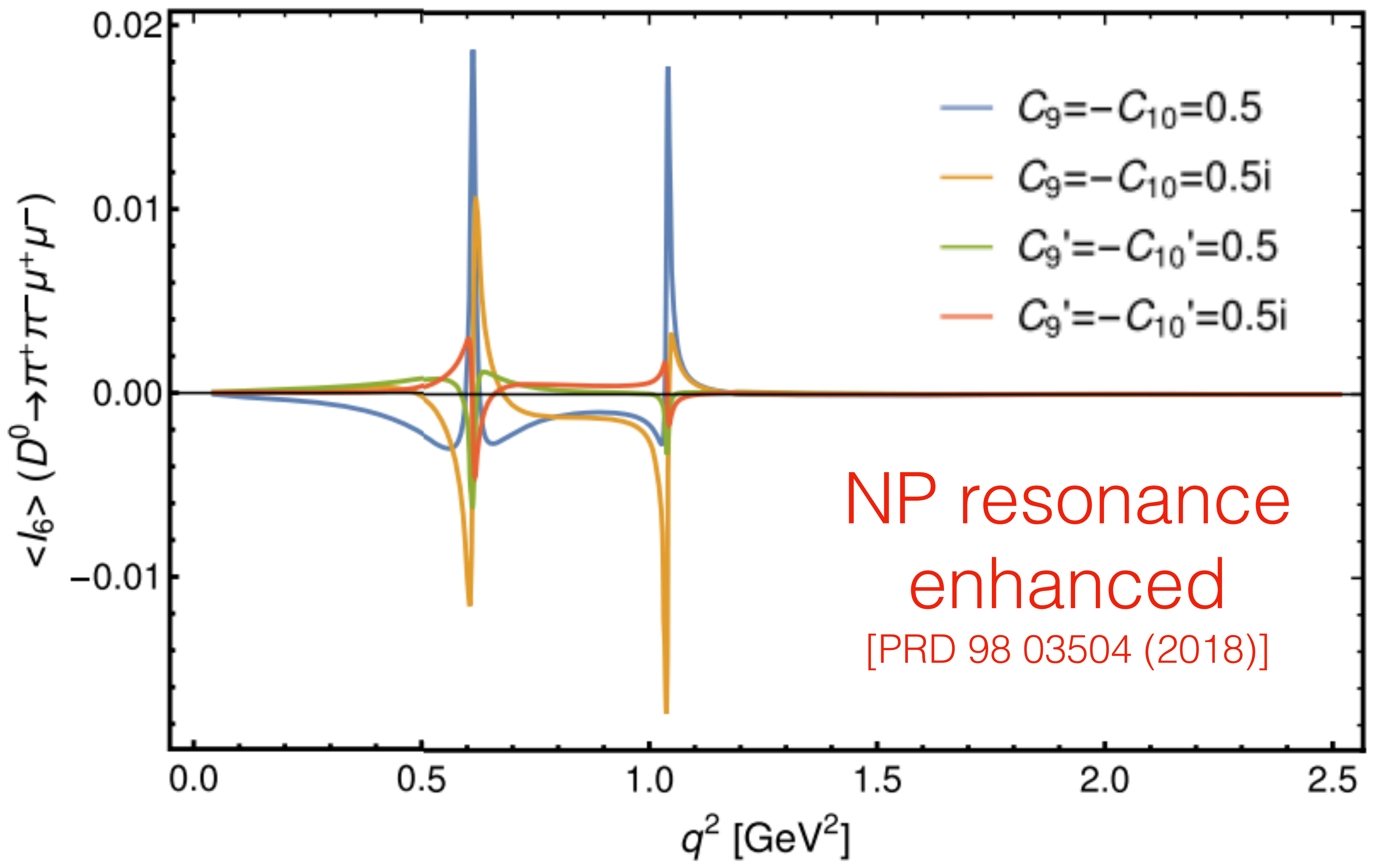
I_i: angular coefficients; c_i: angular basis



Five-dimensional phase-space

$$q^2 = m^2(\mu^+\mu^-),\ p^2 = m^2(h^+h^-)$$

$$d\Omega = d\cos\theta_h\ d\cos\theta_\mu\ d\phi$$



$I_{5,6,7}$ clean
SM null tests

- All coefficients measured integrating out the hadronic system, i.e. integrating on p^2 and $\cos\theta_h$

$$\langle I_{2,3,6,9}\rangle(q^2) = \frac{1}{\Gamma}\int_{4m_h}^{p^2_{max}} dp^2 \int_{-1}^{+1} d\cos\theta_h\ I_{2,3,6,9}$$

$$\langle I_{4,5,7,8}\rangle(q^2) = \frac{1}{\Gamma}\int_{4m_h}^{p^2_{max}} dp^2 \left[\int_{-1}^{0} d\cos\theta_h - \int_{0}^{+1} d\cos\theta_h\right] I_{4,5,7,8}$$

→ Optimal integration of $\cos\theta_h$ giving the p-Wave contributions in h^+h^- system

$D^0 \to h^+h^-\mu^+\mu^-$ angular observables

arXiv:2111.03327

- Coefficients determined independently as differences of the decay rate, with specific integration of the angular variables *("angular tag")*, e.g. for $I_6 \propto A_{FB}$:

$$\langle I_6 \rangle = \frac{\Gamma(\cos\theta_\mu > 0) - \Gamma(\cos\theta_\mu < 0)}{\Gamma(\cos\theta_\mu > 0) + \Gamma(\cos\theta_\mu < 0)} \rightarrow \langle I_6 \rangle = \frac{N(\cos\theta_\mu > 0) - N(\cos\theta_\mu < 0)}{N(\cos\theta_\mu > 0) + N(\cos\theta_\mu < 0)}$$

- Measured separately for D^0 and $\overline{D}^0$, thus flavour-averaged $\langle S_i \rangle$ and CP asymmetries $\langle A_i \rangle$ can be obtained as

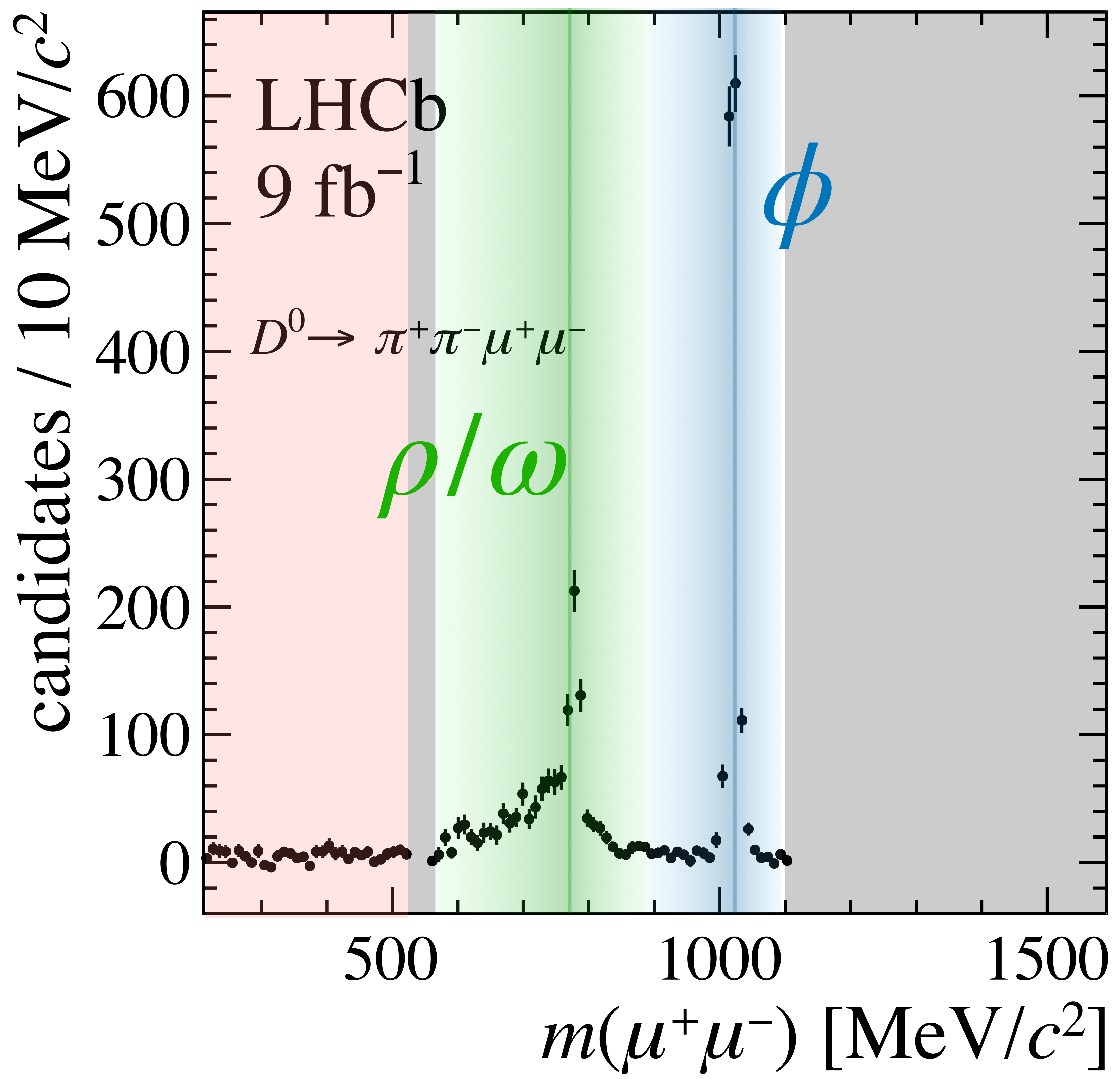
$$\langle S_i \rangle = \frac{1}{2}[\langle I_i \rangle + (-) \langle \bar{I}_i \rangle] \qquad \langle A_i \rangle = \frac{1}{2}[\langle I_i \rangle - (+) \langle \bar{I}_i \rangle]$$

for CP-even (CP-odd) coefficients.

- Access the q^2 dependence by measuring in q^2 regions:

Decay mode	$m(\mu^+\mu^-)$ [MeV/c^2] low mass	η	ρ/ω		ϕ		high mass
$D^0 \to K^+K^-\mu^+\mu^-$	< 525	525-565 (NS)	> 565		-		-
$D^0 \to \pi^+\pi^-\mu^+\mu^-$	< 525	525-565 (NS)	565-780	780-950	950-1020	1020-1100	> 1100 (NS)

NS = No signal



Angular analysis strategy

arXiv:2111.03327

- 16 angular observables (per decay mode, per q^2 region).
- 12 SM-null tests:
  $\langle S_{5,6,7} \rangle_{SM} = 0 \qquad \langle A_{2-9} \rangle_{SM} = 0$
- Correction of acceptance / reconstruction / selection effect across the 5D phase space, by reweighing the D^0 candidates.
- Signal yields extracted by fitting the D^0 invariant mass, splitting by the flavor and angular tag.
- Analysis limited by statistics: systematic uncertainties are typically between 10% and 50% of the statistical one. Main important sources are:
  - Choice of the D^0 invariant mass fitting model,
  - Acceptance correction method.

Updated measurement of A_{CP}

arXiv:2111.03327

- CP asymmetry is measured with a similar strategy, splitting by flavour categories:

$$A_{CP} = \frac{\Gamma(D^0 \to h^+h^-\mu^+\mu^-) - \Gamma(\overline{D}^0 \to h^+h^-\mu^+\mu^-)}{\Gamma(D^0 \to h^+h^-\mu^+\mu^-) + \Gamma(\overline{D}^0 \to h^+h^-\mu^+\mu^-)} \rightarrow A_{CP}^{raw} = \frac{N(D^0 \to h^+h^-\mu^+\mu^-) - N(\overline{D}^0 \to h^+h^-\mu^+\mu^-)}{N(D^0 \to h^+h^-\mu^+\mu^-) + N(\overline{D}^0 \to h^+h^-\mu^+\mu^-)}$$

- Need correction for nuisance asymmetries

$$A_{CP}^{raw} \simeq A_{CP} + A_D + A_P$$

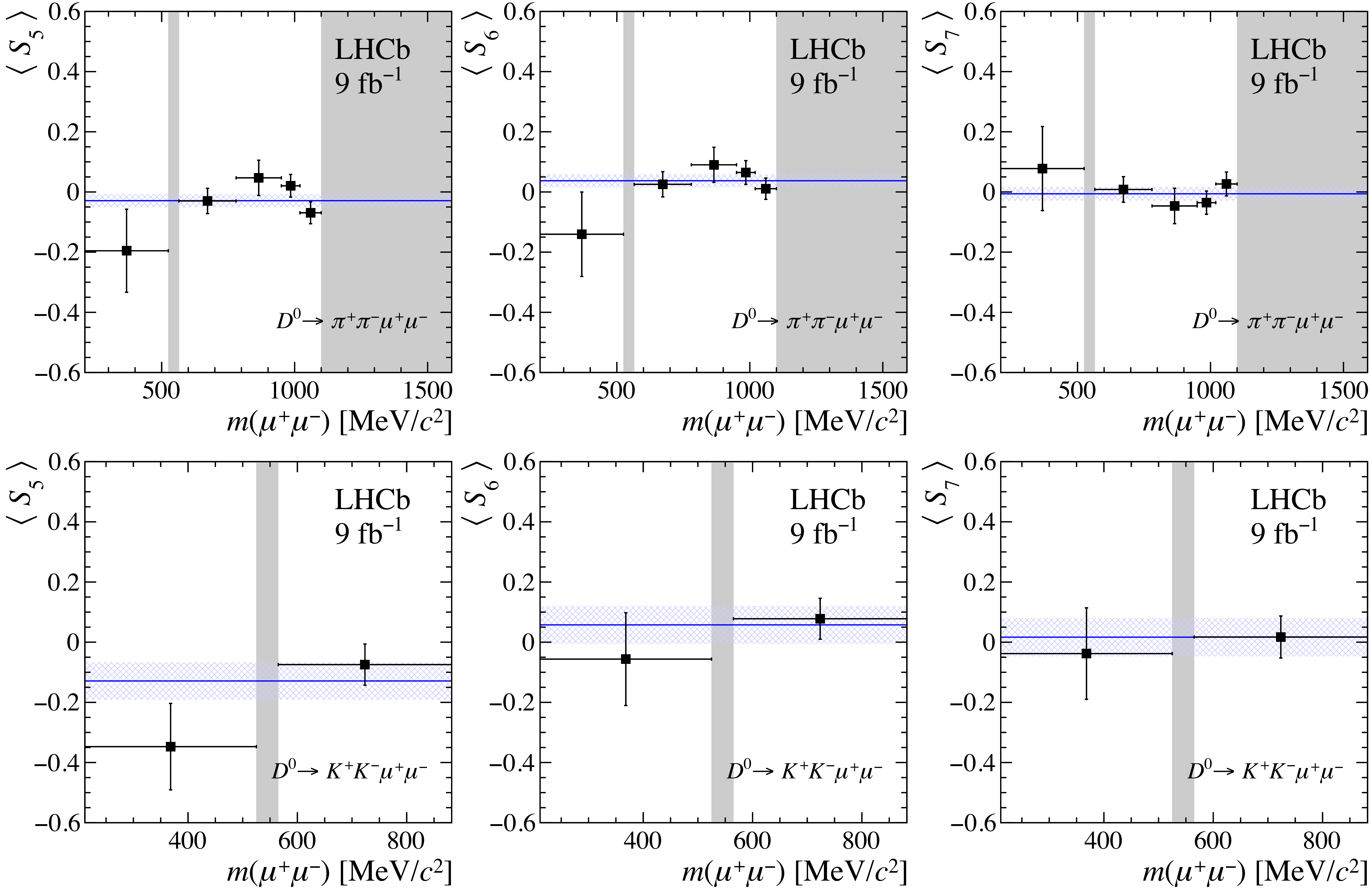
CP asymmetry to measure	Detection asymmetry of tagging particle (π^+)	Production asymmetry of D^{*+}

A sample of $D^{*+} \to D^0(\to K^+K^-)\pi^+$ decay and an independent measurement of A_{CP} are used to subtract the nuisance asymmetries.

- The mass model used in the fit is the main systematic uncertainty.

Results for angular observables $\langle S_i \rangle$

arXiv:2111.03327



SM null tests observables $\langle S_{5,6,7} \rangle$ in the di-muon mass regions.

All in agreement with SM predictions

[JHEP 04 135 (2013)]

[PRD 98 03504 (2018)]

Results for angular observables $\langle A_i \rangle$

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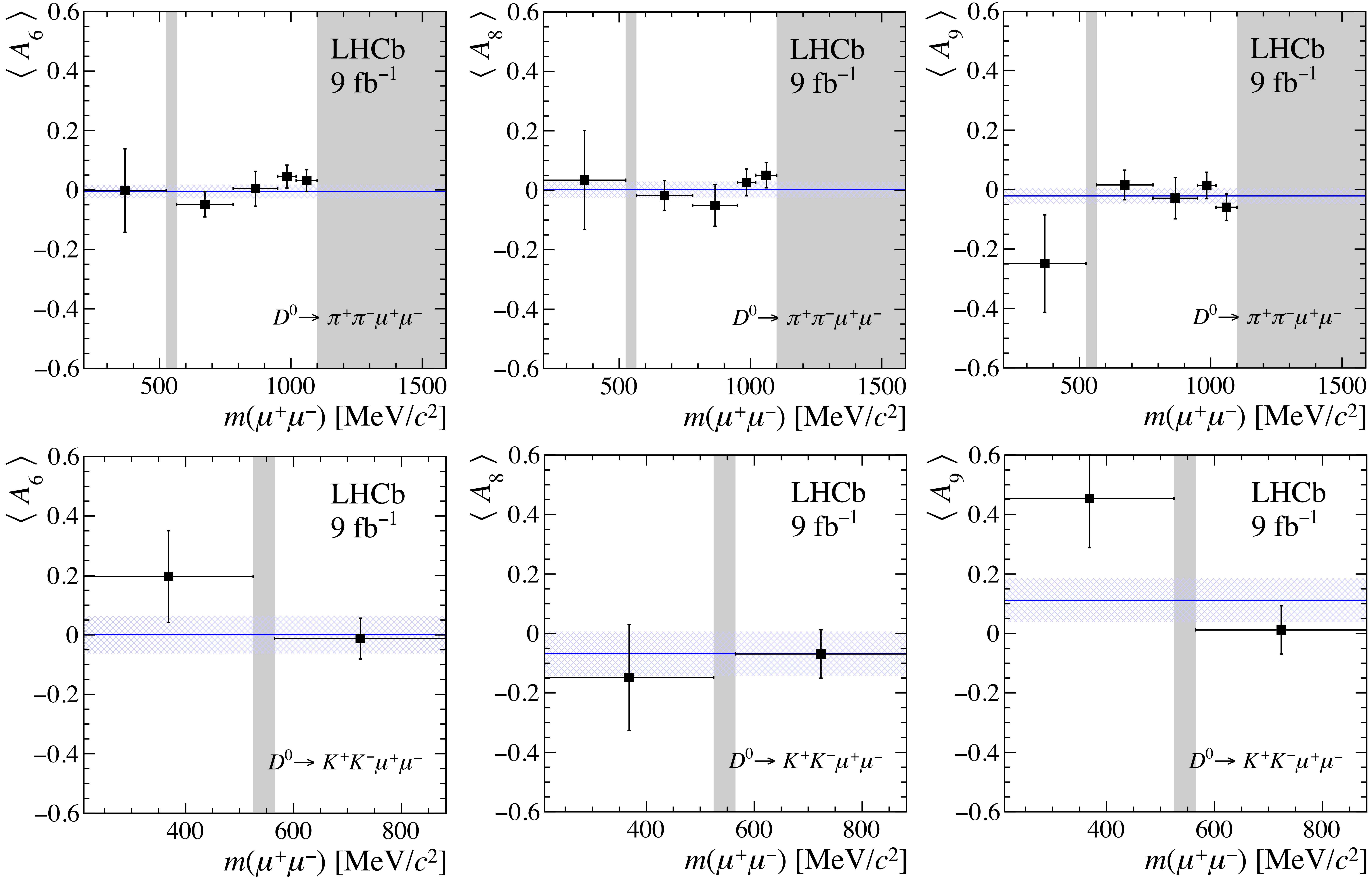


CP asymmetries $\langle A_{6,8,9} \rangle$ in the di-muon mass regions.

All in agreement with SM predictions

[JHEP 04 135 (2013)]

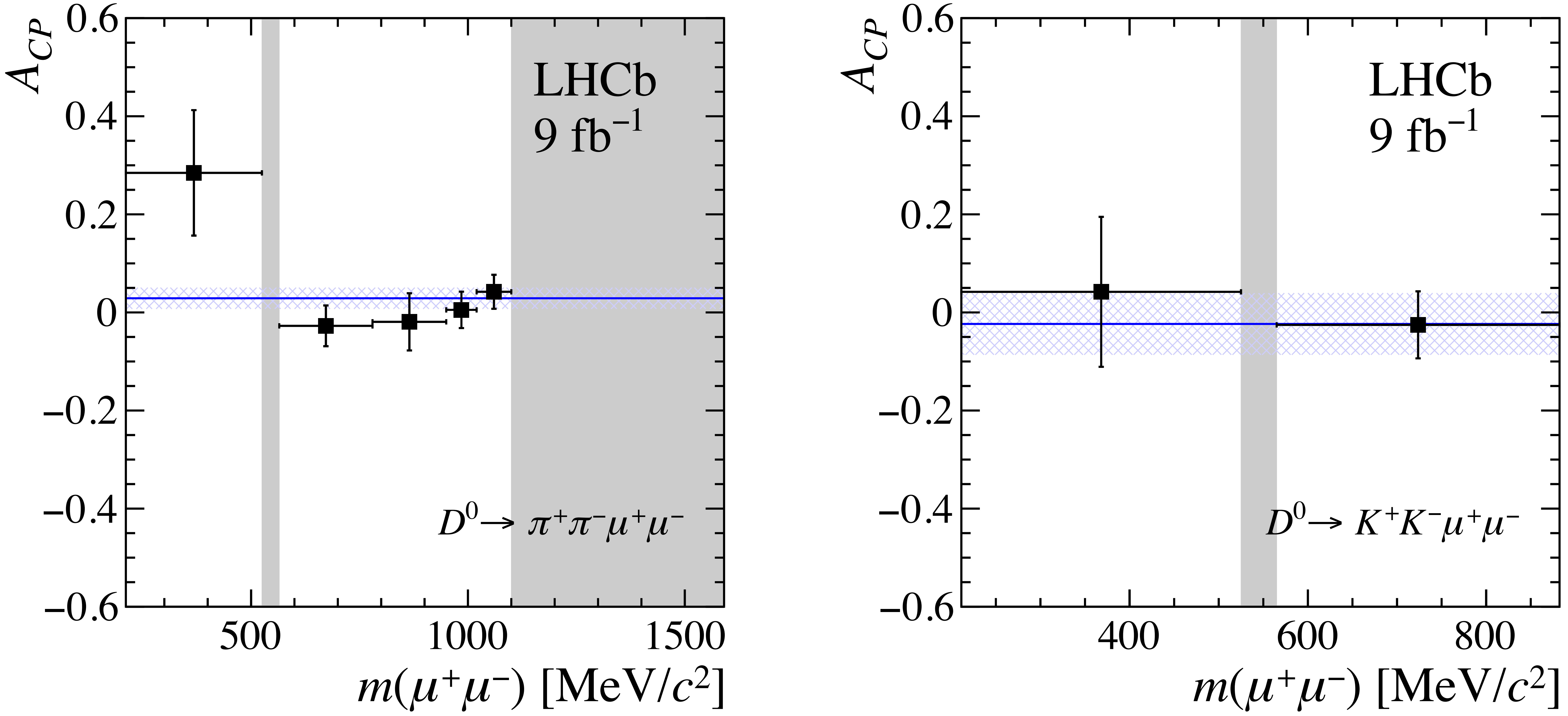
[PRD 98 03504 (2018)]

A_{CP} and final remarks on $D^0 \to h^+h^-\mu^+\mu^-$

arXiv:2111.03327

- CP asymmetry A_{CP} in the di-muon mass regions. Consistent with SM.
- Overall agreement with respect to SM predictions computed for A_{CP}, $\langle A_{2-9}\rangle$, $\langle S_{5,6,7}\rangle$

> - $D^0 \to \pi^+\pi^-\mu^+\mu^-$ p = 79% (0.3 σ)
> - $D^0 \to K^+K^-\mu^+\mu^-$ p = 0.8% (2.7 σ)



$m(\mu^+\mu^-)$ [MeV/c^2]	A_{CP} [%]
$D^0 \to \pi^+\pi^-\mu^+\mu^-$	
< 525	$28 \pm 13 \pm 1$
525–565	–
565–780	$-2.7 \pm 4.1 \pm 0.4$
780–950	$-1.9 \pm 5.8 \pm 0.4$
950–1020	$0.5 \pm 3.7 \pm 0.4$
1020–1100	$4.2 \pm 3.4 \pm 0.4$
> 1100	–
Full range	$2.9 \pm 2.1 \pm 0.4$
$D^0 \to K^+K^-\mu^+\mu^-$	
< 525	$4 \pm 15 \pm 1$
525–565	–
> 565	$-2.5 \pm 6.8 \pm 0.6$
Full range	$-2.3 \pm 6.3 \pm 0.6$

Conclusions

- Steady progress over the years. When probing rates of $\sim 10^{-8}$ we are getting closer the SM regime. Thus rare charm decays is proving itself to be a unique and complementary probe in the field.
- LHCb has collected the largest charm sample and counts many important results:
  - Many world's best measurements,
  - Rarest charm meson decay observed to date,
  - First full angular analysis in a rare charm decay ever performed.
- New updates are coming very soon (D^0 4-body decays with electrons in the final state, update on D^0 3-body decays presented today, update of $D^0 \to \mu^+\mu^-$).
- Given the current limits on the statistics, significative improvements can be expected with LHCb upgrades.

BACKUP

Future measurement precisions

Limit on BF
(far from resonances
for multibody)

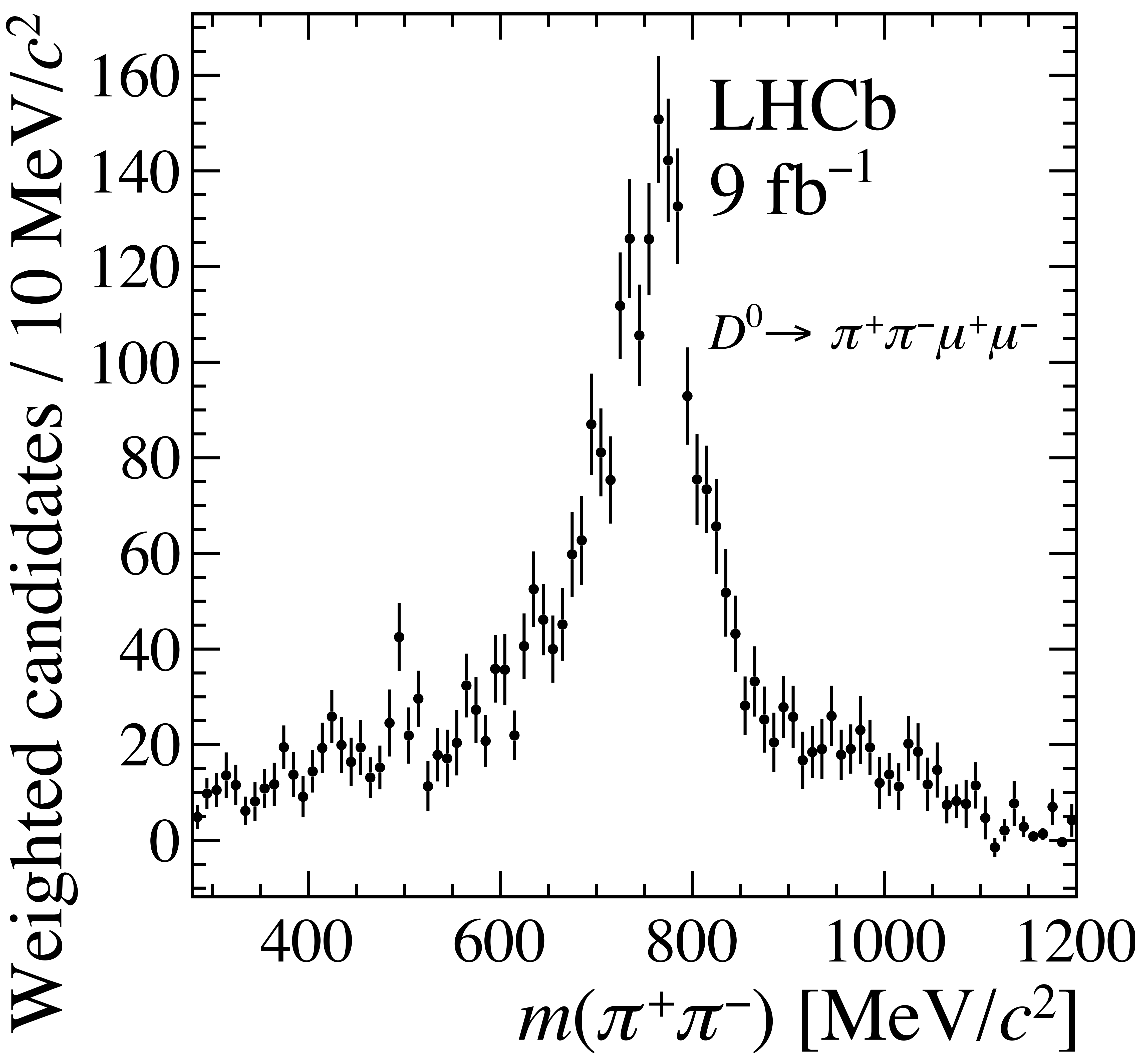
Mode	Upgrade ($50\,\mathrm{fb}^{-1}$)	Upgrade II ($300\,\mathrm{fb}^{-1}$)
$D^0 \to \mu^+\mu^-$	4.2×10^{-10}	1.3×10^{-10}
$D^+ \to \pi^+\mu^+\mu^-$	10^{-8}	3×10^{-9}
$D_s^+ \to K^+\mu^+\mu^-$	10^{-8}	3×10^{-9}
$\Lambda \to p\mu\mu$	1.1×10^{-8}	4.4×10^{-9}
$D^0 \to e\mu$	10^{-9}	4.1×10^{-9}

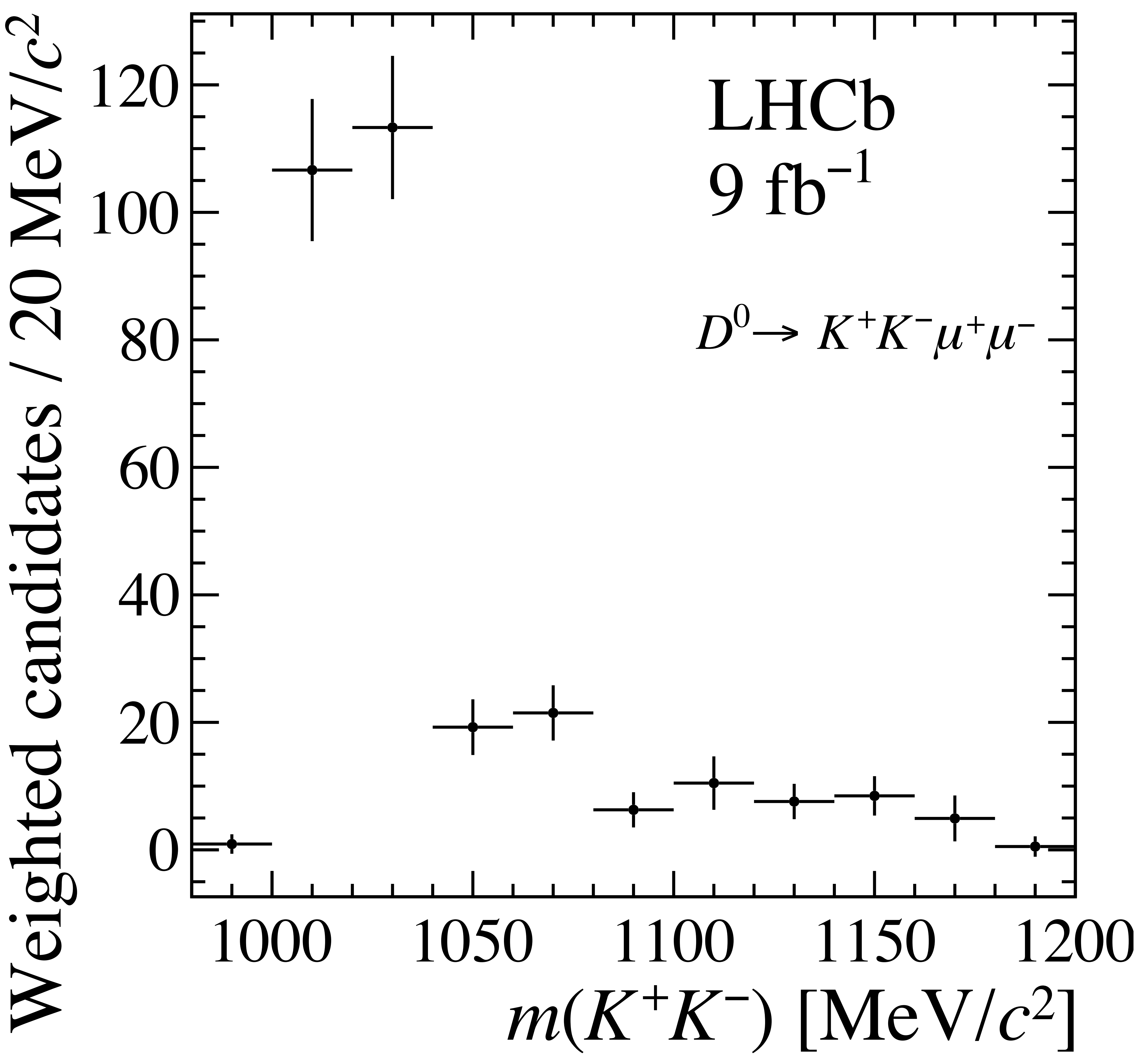
Statistical precision on A_{CP}

Mode	Upgrade ($50\,\mathrm{fb}^{-1}$)	Upgrade II ($300\,\mathrm{fb}^{-1}$)
$D^+ \to \pi^+\mu^+\mu^-$	0.2%	0.08%
$D^0 \to \pi^+\pi^-\mu^+\mu^-$	1%	0.4%
$D^0 \to K^-\pi^+\mu^+\mu^-$	0.3%	0.13%
$D^0 \to K^+\pi^-\mu^+\mu^-$	12%	5%
$D^0 \to K^+K^-\mu^+\mu^-$	4%	1.7%

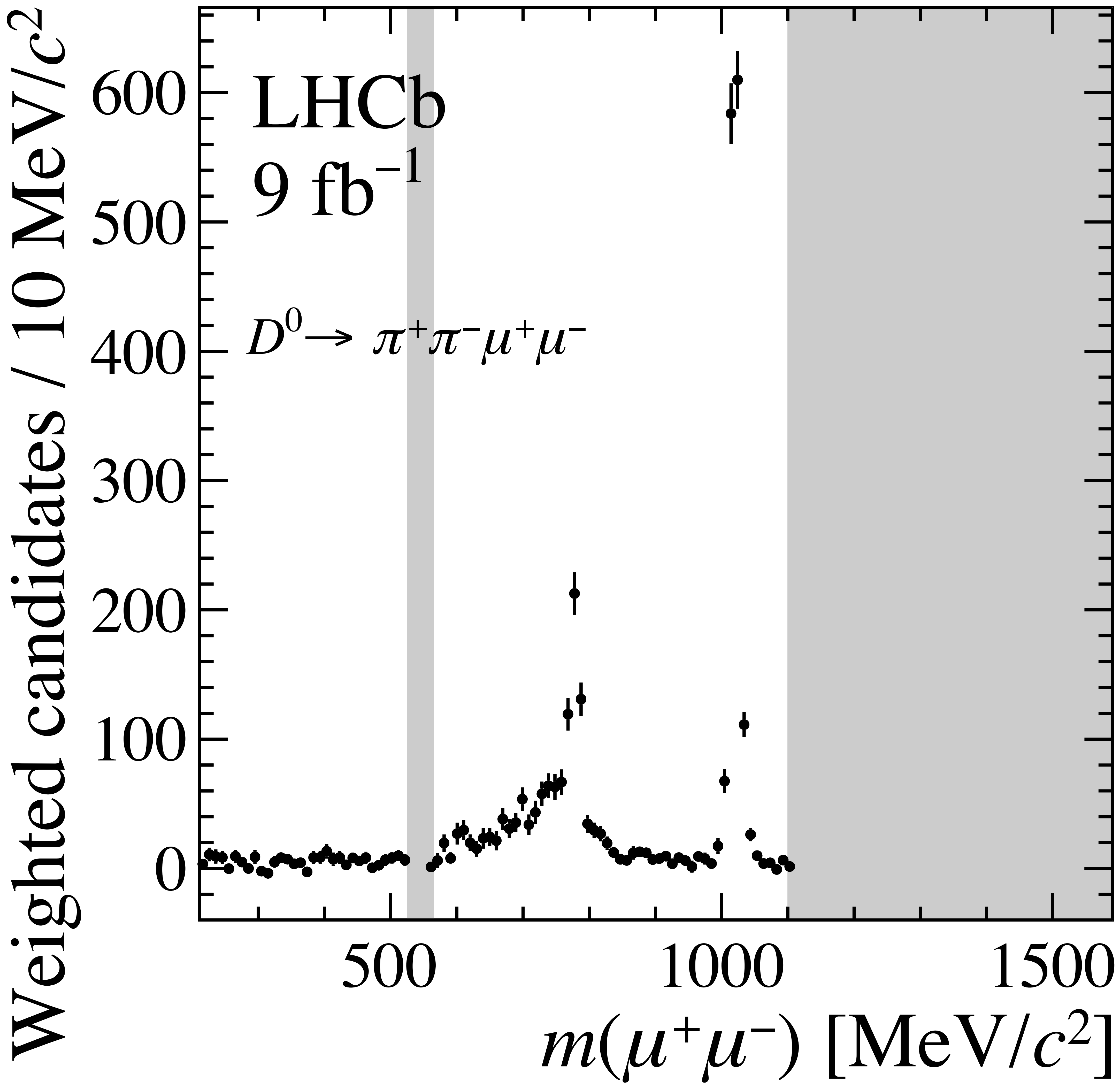
from A.Contu talk “Rare charm Decays and Asymmetries”
Towards Ultimate Precision in Flavour Physics, Durham, UK 2-4 April 2019

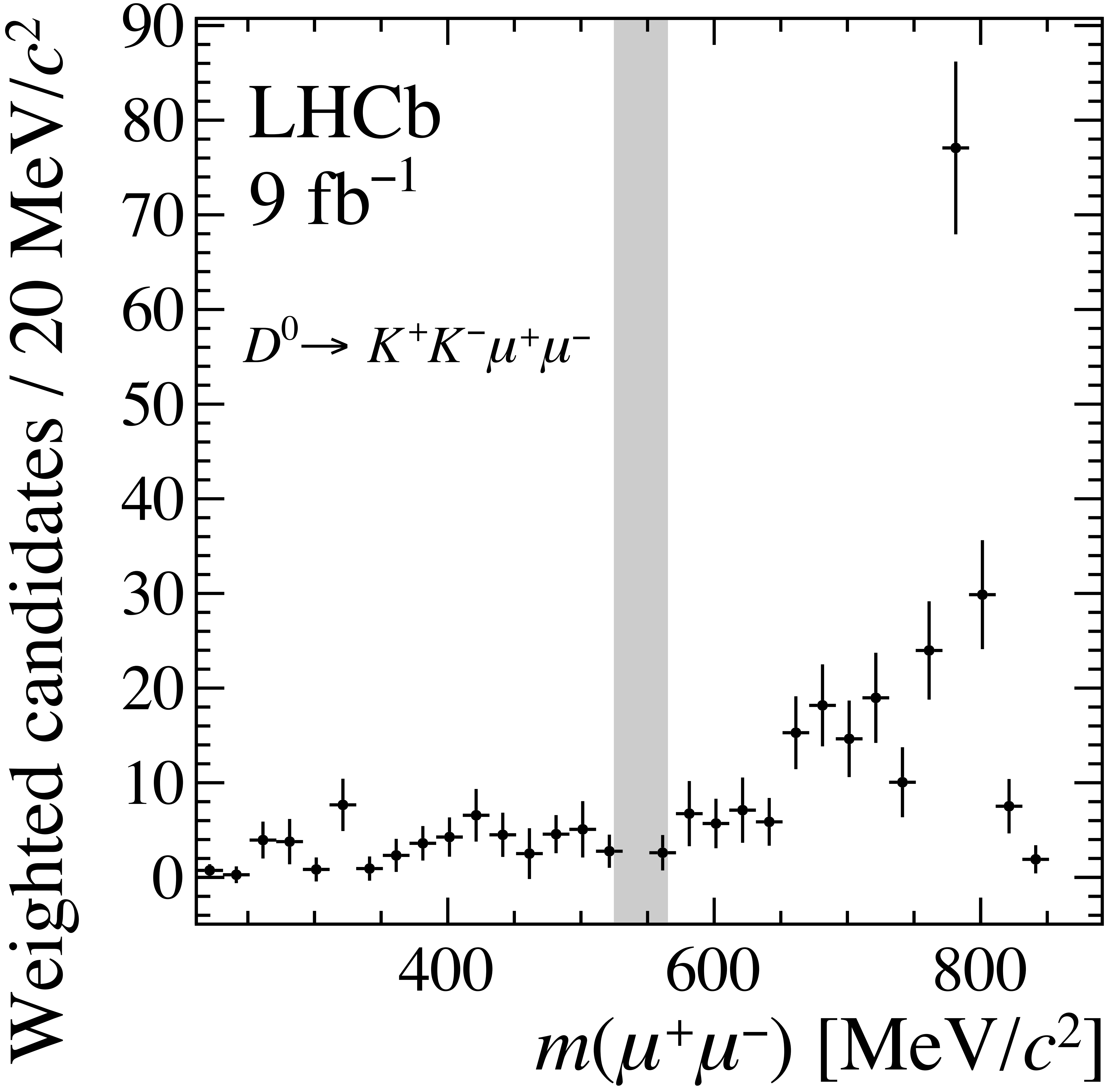
Di-hadron spectra - $D^0 \to h^+h^-\mu^+\mu^-$





Di-muon spectra - $D^0 \to h^+h^-\mu^+\mu^-$



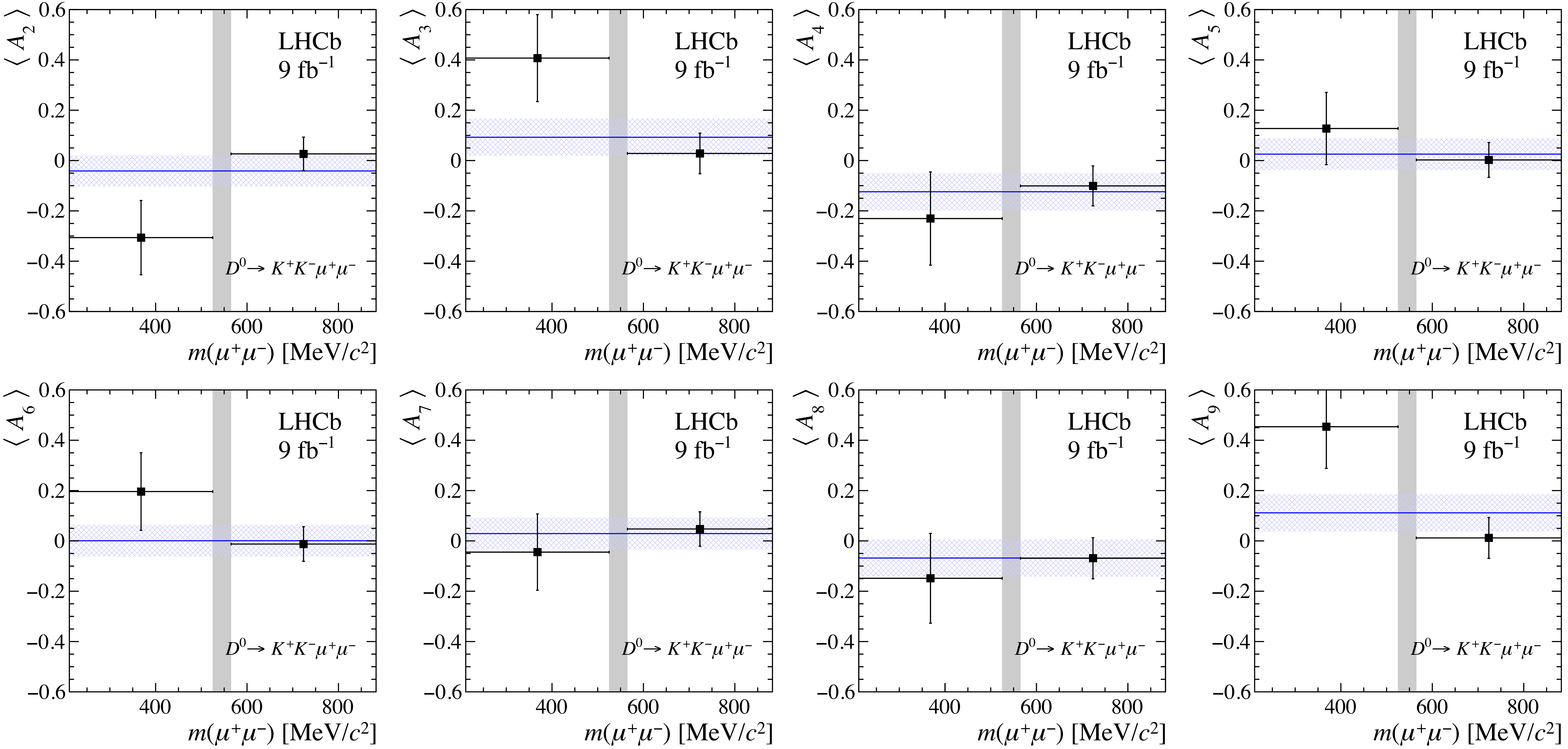


$D^0 \to \pi^+\pi^-\mu^+\mu^-$ observables $\langle S_i \rangle$

$D^0 \to \pi^+\pi^-\mu^+\mu^-$ observables $\langle A_i \rangle$

$D^0 \to K^+K^-\mu^+\mu^-$ observables $\langle S_i \rangle$

$D^0 \to K^+K^-\mu^+\mu^-$ observables $\langle A_i \rangle$



Angular coefficients

$$I_2 = \int_{-\pi}^{\pi} d\phi \left[\int_{-1}^{-0.5} d\cos\theta_\mu + \int_{0.5}^{1} d\cos\theta_\mu - \int_{-0.5}^{0.5} d\cos\theta_\mu\right] \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,,$$

$$I_3 = \frac{3\pi}{8}\left[\int_{-\pi}^{-\frac{3\pi}{4}} d\phi + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\phi + \int_{\frac{3\pi}{4}}^{\pi} d\phi - \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} d\phi - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\phi\right] \int_{-1}^{1} d\cos\theta_\mu \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,,$$

$$I_4 = \frac{3\pi}{8}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi - \int_{-\pi}^{-\frac{\pi}{2}} d\phi - \int_{\frac{\pi}{2}}^{\pi} d\phi\right]\left[\int_{0}^{1} d\cos\theta_\mu - \int_{-1}^{0} d\cos\theta_\mu\right] \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,,$$

$$I_5 = \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi - \int_{-\pi}^{-\frac{\pi}{2}} d\phi - \int_{\frac{\pi}{2}}^{\pi} d\phi\right] \int_{-1}^{1} d\cos\theta_\mu \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,,$$

$$I_6 = \int_{-\pi}^{\pi} d\phi \left[\int_{0}^{1} d\cos\theta_\mu - \int_{-1}^{0} d\cos\theta_\mu\right] \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,,$$

$$I_7 = \left[\int_{0}^{\pi} d\phi - \int_{-\pi}^{0} d\phi\right] \int_{-1}^{1} d\cos\theta_\mu \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,,$$

$$I_8 = \frac{3\pi}{8}\left[\int_{0}^{\pi} d\phi - \int_{-\pi}^{0} d\phi\right]\left[\int_{0}^{1} d\cos\theta_\mu - \int_{-1}^{0} d\cos\theta_\mu\right] \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,,$$

$$I_9 = \frac{3\pi}{8}\left[\int_{-\pi}^{-\frac{\pi}{2}} d\phi + \int_{0}^{\frac{\pi}{2}} d\phi - \int_{-\frac{\pi}{2}}^{0} d\phi - \int_{\frac{\pi}{2}}^{\pi} d\phi\right] \int_{-1}^{1} d\cos\theta_\mu \frac{d^5\Gamma}{dq^2\, dp^2\, d\vec{\Omega}}\,.$$

Angular coefficients

$$I_1 = \frac{1}{16}\left[|H_0^L|^2 + (L \to R) + \frac{3}{2}\sin^2\theta_{P_1}\{|H_\perp^L|^2 + |H_\parallel^L|^2 + (L \to R)\}\right],$$

$$I_2 = -\frac{1}{16}\left[|H_0^L|^2 + (L \to R) - \frac{1}{2}\sin^2\theta_{P_1}\{|H_\perp^L|^2 + |H_\parallel^L|^2 + (L \to R)\}\right],$$

$$I_3 = \frac{1}{16}[|H_\perp^L|^2 - |H_\parallel^L|^2 + (L \to R)]\sin^2\theta_{P_1},$$

$$I_4 = -\frac{1}{8}[\mathrm{Re}(H_0^L H_\parallel^{L*}) + (L \to R)]\sin\theta_{P_1},$$

$$I_5 = -\frac{1}{4}[\mathrm{Re}(H_0^L H_\perp^{L*}) - (L \to R)]\sin\theta_{P_1},$$

$$I_6 = \frac{1}{4}[\mathrm{Re}(H_\parallel^L H_\perp^{L*}) - (L \to R)]\sin^2\theta_{P_1},$$

$$I_7 = -\frac{1}{4}[\mathrm{Im}(H_0^L H_\parallel^{L*}) - (L \to R)]\sin\theta_{P_1},$$

$$I_8 = -\frac{1}{8}[\mathrm{Im}(H_0^L H_\perp^{L*}) + (L \to R)]\sin\theta_{P_1},$$

$$I_9 = \frac{1}{8}[\mathrm{Im}(H_\parallel^{L*} H_\perp^L) + (L \to R)]\sin^2\theta_{P_1}.$$

$$c_1 = 1, \qquad c_2 = \cos 2\theta_l, \qquad c_3 = \sin^2\theta_l \cos 2\phi,$$

$$c_4 = \sin 2\theta_l \cos\phi, \qquad c_5 = \sin\theta_l \cos\phi,$$

$$c_6 = \cos\theta_l, \qquad c_7 = \sin\theta_l \sin\phi,$$

$$c_8 = \sin 2\theta_l \sin\phi, \qquad c_9 = \sin^2\theta_l \sin 2\phi.$$