Stability of Initial Glasma Fields

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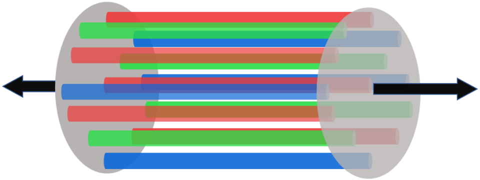
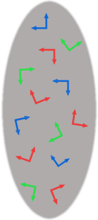
Overview

- Motivation and glasma fields
- Stability analysis in Minkowski coordinates
- Abelian and nonAbelian configurations of glasma fields
- Stability analysis in Milne coordinates
- Summary

Motivation and glasma fields

- The earliest phase of heavy-ion collisions is described in terms of classical fields.





- Early configuration is unstable, but a mechanism of the instabilities is not fully understood
  (P. Romatschke and R. Venugopalan, Phys. Rev. Lett. 96, 062302 (2006),
  P. Romatschke and R. Venugopalan, Phys. Rev. D 74, 045011 (2006).)
- We study the problem systematically starting with the uniform chromoelectric and chromomagnetic fields.

Stability analysis in Minkowski coordinates

Yang-Mills equations in adjoint representation

$$\partial_\mu F_a^{\mu\nu} + g f^{abc} A_\mu^b F_c^{\mu\nu} = j_a^\nu, \qquad F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f^{abc} A_b^\mu A_c^\nu$$

Linearized QCD

$$A_a^\mu(t, \mathbf{r}) = \bar{A}_a^\mu(t, \mathbf{r}) + a_a^\mu(t, \mathbf{r}), \quad \text{where } |\bar{A}(t, \mathbf{r})| \gg |a(t, \mathbf{r})|$$

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Generation of constant fields in one direction is possible in two ways:

- Abelian configuration - single color potential linearly dependent on coordinates,
- nonAbelian configuration - multicolor, constant potential.

Calculations done in SU(2) group: $f^{abc} \to \epsilon^{abc}$.

Stability analysis in Minkowski coordinates

- We considered Abelian and nonAbelian configurations of chromoelectric and chromomagnetic fields.
- Assuming that a_a^μ depends on x^μ through $e^{ik\cdot x}$, the equation of motion is fully algebraic and general solutions were found.
- In each configuration there are unstable modes.

More detailes can be found in:

S. Bazak and S. Mrówczyński, Phys. Rev. D **105**, 034023 (2022)

S. Bazak and S. Mrówczyński, Phys. Rev. D **106**, 034031 (2022)

Glasma fields

Ansatz

- $A^+(x) = \Theta(x^+)\Theta(x^-)x^+\alpha(\tau, x_\perp)$,
- $A^-(x) = -\Theta(x^+)\Theta(x^-)x^-\alpha(\tau, x_\perp)$,
- $A^i(x) = \Theta(x^+)\Theta(x^-)x^+\alpha^i_\perp(\tau, x_\perp) + \Theta(-x^+)\Theta(x^-)\beta^i_1(x_\perp) + \Theta(x^+)\Theta(-x^-)\beta^i_2(x_\perp)$,

where $\beta^i_1(x_\perp)$ and $\beta^i_2(x_\perp)$ - pre-collision potentials,
$\alpha(\tau, x_\perp)$ and $\alpha^i_\perp(\tau, x_\perp)$ - post-collision potentials, $i, j = x, y$.

Boundary conditions

- $\alpha^i_\perp(0, x_\perp) = \beta^i_1(x_\perp) + \beta^i_2(x_\perp)$,
- $\alpha(0, x_\perp) = -\frac{ig}{2}[\beta^i_1(x_\perp), \beta^i_2(x_\perp)]$.

Gauge condition

$x^+A^- + x^-A^+ = 0$

Glasma fields

Non-zero components of E and B fields at $\tau = 0$

- $E(x_\perp) \equiv E^z(0, x_\perp) = -2\alpha(0, x_\perp)$,
- $B(x_\perp) \equiv B^z(0, x_\perp) = -\partial_y \alpha_\perp^x(0, x_\perp) + \partial_x \alpha_\perp^y(0, x_\perp) - ig[\alpha_\perp^y(0, x_\perp), \alpha_\perp^x(0, x_\perp)]$.

Glasma initial potentials

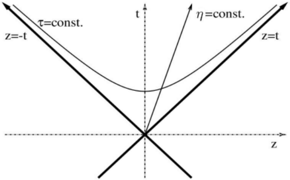
- $A_a^t = \frac{g}{2} z f^{abc} \beta_{1b}^i \beta_{2c}^i$,
- $A_a^z = \frac{g}{2} t f^{abc} \beta_{1b}^i \beta_{2c}^i$,
- $A_a^i = \beta_{1a}^i + \beta_{2a}^i$.

Uniform chromoelectric field is in the Abelian configuration
but uniform chromomagnetic field in the nonAbelian configuration.

Yang-Mills equations in Milne coordinates

- Curvilinear coordinates - Milne coordinates $\tilde{x}^\mu = (\tau, x, y, \eta)$ are defined

$$\begin{cases} \tau = \sqrt{t^2 - z^2}, \\ \eta = \frac{1}{2} \ln \frac{t+z}{t-z}, \end{cases} \qquad \begin{cases} t = \tau \cosh \eta, \\ z = \tau \sinh \eta, \end{cases}$$



- Yang-Mills equations:

$$(\nabla_\mu F^{\nu\mu})_a = \frac{1}{\tau} \partial_\mu (\tau g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma})_a + g g^{\mu\rho} g^{\nu\sigma} f^{abc} A^b_\mu F^c_{\rho\sigma} = j^\nu_a,$$

where $g^{\mu\nu} = \text{diag}(1, -1, -1, -\frac{1}{\tau^2})$, and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$,

- $A^\mu_a = \bar{A}^\mu_a + a^\mu_a$,
- Yang-Mills equations linearized in a^μ_a (not algebraic for $a^\mu \sim e^{ik \cdot x}$).

Stability analysis of Abelian chromoelectric configuration

$$E^z = \frac{1}{\tau}(\partial_\tau \bar{A}_\eta - \partial_\eta \bar{A}_\tau) = -\frac{1}{\tau}(\partial_\tau(\tau^2 \bar{A}^\eta) + \partial_\eta \bar{A}^\tau).$$

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Constant and uniform chromoelectric field

$E^z = -2\bar{A}^\eta - \tau\partial_\tau \bar{A}^\eta$

$\bar{A}^\eta = -\frac{1}{2}E^z$

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Constant and uniform chromoelectric field

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- solutions for special cases

Stability analysis of Abelian chromoelectric configuration

Special case $a_a^x \neq 0$, $a_a^y \neq 0$ and $a_a^\eta = 0$

Assumptions

- mixing: $X^\pm = a_2^x \pm i a_3^x$ and $Y^\pm = a_2^y \pm i a_3^y$,
- $X^\pm(\tau, x, y, \eta) = e^{i(k_x x + k_y y + \nu\eta)} X^\pm(\tau)$

  $Y^\pm(\tau, x, y, \eta) = e^{i(k_x x + k_y y + \nu\eta)} Y^\pm(\tau)$

Final equations

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + k_x^2 + k_y^2 + \left(\frac{\nu}{\tau} \pm \frac{gE\tau}{2}\right)^2\right) X^\pm = 0,$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + k_x^2 + k_y^2 + \left(\frac{\nu}{\tau} \pm \frac{gE\tau}{2}\right)^2\right) Y^\pm = 0.$$

In the short time limit $(\tau^2 \ll 2\nu/(gE))$ - Bessel equation of imaginary order.
In the long time limit $(\tau^2 \gg 2\nu/(gE), \tau^2 \gg 1/(gE))$ - oscillatory running-away solutions.

Stability analysis of nonAbelian chromomagnetic configuration

$$B_1^z = F_1^{yx} = \nabla^y \bar{A}_1^x - \nabla^x \bar{A}_1^y + g\bar{A}_2^y \bar{A}_3^x - g\bar{A}_3^y \bar{A}_2^x$$

Constant homogeneous chromomagnetic field

$$\bar{A}_a^\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda\sqrt{B/g} & 0 \\ 0 & \frac{1}{\lambda}\sqrt{B/g} & 0 & 0 \end{bmatrix}, \quad j_a^\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda}\sqrt{gB^3} & 0 \\ 0 & \lambda\sqrt{gB^3} & 0 & 0 \end{bmatrix}.$$

The chromomagnetic field is independent of λ, but potentials of different λ are physically nonequivalent.

nonAbelian Chromomagnetic configuration

- In special case $a_a^\eta = 0$, $a_a^x \neq 0$, $a_a^y \neq 0$

$$\begin{aligned}
\left(\Box + g^2(\bar{A}_3^x\bar{A}_3^x + \bar{A}_2^y\bar{A}_2^y\right)V_1^\pm - 2g\bar{A}_3^x\partial_x V_2^\pm + 2g\bar{A}_2^y\partial_y V_3^\pm &= 0,\\
\left(\Box + g^2\bar{A}_3^x\bar{A}_3^x\right)V_2^\pm + 2g\bar{A}_3^x\partial_x V_1^\pm \pm 2ig^2\bar{A}_3^x\bar{A}_2^y V_3^\pm &= 0,\\
\left(\Box + g^2\bar{A}_2^y\bar{A}_2^y\right)V_3^\pm - 2g\bar{A}_2^y\partial_y V_1^\pm \mp 2ig^2\bar{A}_2^y\bar{A}_3^x V_2^\pm &= 0,
\end{aligned}$$

  where $\Box = \partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\partial_\eta^2$ and $V_a^\pm = a_a^x \pm i a_a^y$.
- For $k_x = 0$ and $k_y = 0$

$$\begin{aligned}
\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} + b_1^2\right)f_1(\tau) &= 0,\\
\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} \pm b_\pm^2\right)f_\pm(\tau) &= 0.
\end{aligned}$$

- The Bessel equation or the modified Bessel equation.
- For the modified Bessel equation (with "-") there are unstable solutions.
- The fastest mode grows as $e^{\sqrt{gB}\tau}$.

nonAbelian Chromomagnetic configuration

- Generalized case: $k_x \neq 0$ and $k_y \neq 0$
- Solutions found using perturbation theory

$$\Lambda^{(1)} = \Lambda^{(0)} + c_x k_x^2 + c_y k_y^2,$$

where $\Lambda^{(0)}$ is the eigenvalue of matrix for $k_x = k_y = 0$.

-

$$\begin{aligned}
\Big(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} + b_1^2 + (c_x^1 + 1)k_x^2 + (c_y^1 + 1)k_y^2\Big) f_1(\tau) &= 0, \\
\Big(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} \pm b_\pm^2 + (c_x^\pm + 1)k_x^2 + (c_y^\pm + 1)k_y^2\Big) f_\pm(\tau) &= 0.
\end{aligned}$$

- The fastest mode grows as $e^{\sqrt{gB - k_T^2/3}\,\tau}$, $k_T^2 \equiv k_x^2 + k_y^2$.

Temporal evolution of glasma background field

- The glasma fields evolve in time.
- Our stability analysis is reliable if a rate of change of the background is significantly smaller than the one of instabilities, we found.
- In the proper time expansion

$$\begin{aligned}\alpha(\tau,\mathbf{x}_\perp) &= \alpha^{(0)}(\mathbf{x}_\perp)+\tau\alpha^{(1)}(\mathbf{x}_\perp)+\tau^2\alpha^{(2)}(\mathbf{x}_\perp)+\ldots\,,\\ \alpha_\perp(\tau,\mathbf{x}_\perp) &= \alpha_\perp^{(0)}(\mathbf{x}_\perp)+\tau\alpha_\perp^{(1)}(\mathbf{x}_\perp)+\tau^2\alpha_\perp^{(2)}(\mathbf{x}_\perp)+\ldots\,.\end{aligned}$$

For initial uniform chromomagnetic fields one finds

$$\begin{aligned}\alpha_a &= \mathcal{O}(\tau^4),\\ \alpha_{\perp a}^x &= \delta^{a3}\lambda^{-1}\sqrt{B/g}\Big(1-\frac{1}{4}\lambda^2 gB\tau^2+\mathcal{O}(\tau^4)\Big),\\ \alpha_{\perp a}^y &= \delta^{a2}\lambda\sqrt{B/g}\Big(1-\frac{1}{4}\lambda^{-2} gB\tau^2+\mathcal{O}(\tau^4)\Big).\end{aligned}$$

Rate of change of background potential can be smaller than the unstable mode growth rate.

Discussion

- Comparing the growth rate of the unstable mode ($\sqrt{gB}$) with the rate of change of background field ($gB\tau$), one finds that the condition of constancy of the background field is satisfied only marginally.
- Fields are spatially uniform in the transverse plane at a scale L (between Q_s^{-1} and $\Lambda_{\mathrm{QCD}}^{-1}$, $Q_s \approx 2$GeV, $\Lambda_{\mathrm{QCD}} \approx 0.2$GeV).
- Assuming that the domain of uniform field is a square centered in $\mathbf{r}_\perp = 0$ one should replace $(k_x, k_y) \to (2l_x + 1, 2l_y + 1)\frac{\pi}{L}$, where l_x and l_y are integer numbers. The spectrum of eigenmodes becomes discrete.
- From the growth rate of the instability $\sqrt{gB - \frac{k_T^2}{3}}$, one finds that the instability appears if $gB > \frac{2\pi^2}{3L^2}$.
- For $g \approx 1$, $B \approx Q_s^2 \approx 4\mathrm{GeV}^2$, the upper condition is satisfied for $L^{-1} \approx \Lambda_{\mathrm{QCD}} \approx 0.2$GeV.

Summary and conclusions

- We found the spectra of eigenmodes for constant and uniform Abelian chromoelectric and nonAbelian chromomagnetic fields configurations.
- The initial chromomagnetic field is unstable, but the conditions of the space-time uniformity of the field are satisfied only marginally.
- The characteristic growth rate of the unstable mode of initial glasma field is of order Q_s and it is much bigger than the growth rate found in the simulations (P. Romatschke and R. Venugopalan, Phys. Rev. Lett. 96, 062302 (2006) and Phys. Rev. D 74, 045011 (2006)).
- The instability found in the simulations (P. Romatschke and R. Venugopalan, Phys. Rev. Lett. 96, 062302 (2006) and Phys. Rev. D 74, 045011 (2006)) is not the instability of glasma initial field, as claimed by some authors. It is rather the Weibel mode, known in plasma physics.

Thank you for attention!