Initial state and approach to equilibrium

Michał Spaliński
University of Białystok & National Centre for Nuclear Research
The effectiveness of hydrodynamic models at early times could be explained using the concept of hydrodynamic attractors which originated in studies of boost-invariant flows in conformal models. I will review these ideas, attempts at their practical application, and generalisations beyond toy models.
Approaching equilibrium in heavy-ion collisions
The role of boost-invariance

• The origin of attractor behaviour at early times has been linked to the longitudinal expansion and as such may be of kinematic nature.

• This suggests that even though this effect was observed in toy models, it may also occur in QCD given the same kinematics.

• Crucial question: how robust are the features of boost-invariant attractors in conformal models when symmetry assumptions are relaxed?
Bjorken flow in conformal models

\[
(T^{\mu}_\nu) = \text{diag}(-E, P_L, P_T, P_T)
\]

Expressed in terms of 2 functions of proper time:

\[
P_L \equiv \frac{E}{3} \left(1 - \frac{2}{3} \mathcal{A}\right), \quad P_T \equiv \frac{E}{3} \left(1 + \frac{1}{3} \mathcal{A}\right)
\]

\[
E(\tau) \sim T(\tau)^4, \quad w \equiv \tau T
\]

Conservation of the energy-momentum tensor:

\[
\frac{d \log T}{d \log w} = \frac{\mathcal{A} - 6}{\mathcal{A} + 12}
\]

• Solving the conservation equation introduces a single integration constant and the remaining initial data is contained in \( \mathcal{A}(w) \).

• In a number of models there is a \textbf{universal attractor} form of the function \( \mathcal{A}(w) \) which is rapidly approached by generic solutions.
Modelling the attractor in conformal Mueller-Israel-Stewart theory

The Israel-Stewart relaxation equation

\[ C_\tau \left( 1 + \frac{\mathcal{A}}{12} \right) \mathcal{A}' + \frac{C_\tau}{3w} \mathcal{A}^2 = \frac{3}{2} \left( \frac{8C_\eta}{w} - \mathcal{A} \right) \]

Early time asymptotics

\[ \mathcal{A} = 6\sqrt{\frac{C_\eta}{C_\tau \Pi}} + O(w) \quad \text{or} \quad \mathcal{A} \sim \frac{1}{w^4} \]

Information about initial conditions is exponentially “dissipated”

Late time asymptotics

\[ \mathcal{A} = \frac{8C_\eta}{w} + \frac{16C_\eta C_\tau}{3w^2} + \ldots = \sum_{n>0} \frac{a_n^{(0)}}{w^n} + \left( \sigma \frac{c_\eta}{c_\tau} e^{-\frac{3}{2C_\tau}w} \right) \sum_{n \geq 0} \frac{a_n^{(1)}}{w^n} + \ldots \]

Navier–Stokes

2nd order

gradient expansion

transseries sectors
Modelling attractor behaviour in kinetic theory

\[ \frac{\tau}{\tau_R} \]

\[ \frac{\tau}{\tau_R} \]

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Romatchke 1704.08699; Blaizot, Yan 1712.03856, …, 2106.10508; Strickland 1809.01200; 
Almaalol et al 2004.05195

At QM2022: Talks by Almaalol, Plumari.
Origins of attractor behaviour
from the kinetic theory perspective

“The main features of the dynamics of expanding plasmas are determined by the competition between the expansion itself, which is dictated by the external conditions of the collisions, and the collisions among the plasma constituents which generically tend to isotropize the particle momentum distribution functions”.

• In kinetic theory the expansion always wins at early times and leads to free streaming.

• In hydrodynamic models there is fair competition: free streaming is not automatic, details of the early time attractor depend on the dynamics.

• The general answer is not known.

Blaizot, Yan 1712.03856; Kurkela et al. 1907.08101; Kurkela, Moore 1108.4684
Origins of attractor behaviour
In inflationary cosmology

A wide class of initial conditions quickly end up on the inflationary attractor, where the potential gradient competes with the Hubble expansion of the gravitational background.

See e.g. Mukhanov’s “Physical Cosmology”, Cambridge University Press 2005
Origins of attractor behaviour
from the kinetic theory perspective

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There seems to be no unique early-time behaviour, only a late-time an attractor attained at anisotropy $\mathcal{A} \approx 0.6$. 

Kurkela et al. 1907.08101
Other perspectives on very early times

- Classical Yang-Mills
- Adiabatic hydrodynamisation
- Nonthermal attractors

Fu 2110.01540; Carrington et al. 2105.05327; Ipp et al. 2109.05028;
Brewer et al. 1910.00021; Mikheev 2203.02299; Brewer et al. 2203.04027
At QM2022: Talks by Fu; Carrington. Posters by Bazak; Mueller; Scheihing Hitschfeld.
An example application

If we approximate the entire history of a heavy-ion collision with a conformal, boost-invariant attractor, one consequence is the formula

\[
\langle \frac{dN}{dy} \rangle_c \propto \langle \int d^2 x_\perp \mathcal{E}(\tau_0, x_\perp)^{\frac{2}{4-\beta}} \rangle_c
\]

\[
\langle \frac{dN}{dy} \rangle_{c'} \propto \langle \int d^2 x_\perp \mathcal{E}(\tau_0, x_\perp)^{\frac{2}{4-\beta}} \rangle_{c'}
\]

Here \( \beta = 1 \) corresponds to a free-streaming attractor.

Consistency with experiment suggests that the attractor at early times must match the model of the initial energy deposition.

Could be relevant for current Bayesian studies which vary the initial state models using free-streaming pre-hydro evolution.

Giacalone et al. 1908.02866; Jankowski et al. 2012.02184
Moreland et al. 1808.02106; Nijs et al. 2010.15130, 2010.15134
Du, Schlichting 2012.09068, 2012.09079; Coquet et al. 2112.13876

At QM2022: Talk by Nijs. Posters by Du; Heffernan; Jankowski.
Beyond Conformal Bjorken Flow

- With transverse flow
- Without conformal symmetry
  - More degrees of freedom
  - In kinetic theory: an early-time attractor identified in $P_L$
  - Standard 2nd order hydro does not capture this, but a version of aHydro does
- A challenge for hydro-modellers

Kurkela et al. 1907.08101, 2007.06851; Ambrus et al. 2109.03290
Chattopadhyay et al. 2107.05500; Jaiswal et al. 2107.10248; Alqahtani et al. 2203.14968
Modelling attractors with hydrodynamic models

Hydro models (MIS, aHydro, general frame, hydro+, …)

• are engineered to mimic near-equilibrium asymptotics
• propagate initial data in a causal and stable way
• capture (more) information about the initial state
• model nontrivial nonhydrodynamic (transient) sectors

Results coincide only in the asymptotic region, see e.g. the recent comparison of MIS and BDNK.

It looks like matching hydro models to nonconformal kinetic theory presents interesting challenges.

Bantilan et al. 2201.13359; Speranza et al. 2105.01034;
At QM2022: Talk by Yin. Poster by Denicol; Speranza
The Phase Space Approach

Can you see the MIS attractor in this picture?
The hydrodynamic attractor in MIS
A view from phase space

\[ \tau = 0.20 \text{ fm} \]
The hydrodynamic attractor in MIS
A view from phase space

$\tau = 0.22 \text{ fm}$
The hydrodynamic attractor in MIS
A view from phase space

\[ \tau = 0.27 \text{ fm} \]
The hydrodynamic attractor in MIS
A view from phase space

\[ \tau = 0.38 \text{ fm} \]
The hydrodynamic attractor in MIS
A view from phase space

\[ T' = \frac{\tau}{\text{MeV/fm}} = 0.74 \text{ fm} \]
The Phase Space Approach

• A set of solutions spanning a D-dimensional region on the initial time slice ends up in a region of lower dimensionality $d < D$ on a subsequent time slice.

• The attractor phenomenon is identified with this reduction of dimensionality of sets of solutions on slices of phase space.

• No special variables are necessary

• No limitations such as boost-invariance or conformal symmetry

• Natural area for techniques of data science/ML.

Heller et al. 2003.07368
The Phase Space Approach

Examples (still boost-invariant), using PCA for the data analysis:

- MIS-type models with 2d or 3d phase space
- BE-RTA: A 16d coarse-grained representation of phase space
- BE-EKT: A 4d coarse-grained representation of phase space
Outlook

- Approximate boost-invariance at early times may be a key element of why early-time attractors appear beyond toy models
- Early results suggest that some features of toy models persist when idealisations are relaxed
- New approaches may be useful in identifying and making use of attractors in situations with more degrees of freedom
- More elaborate hydrodynamic modelling may be needed
- Recent progress concerning the asymptotics of more general flows may signal progress on attractors in the near future