# Kinetic theory for massive spin-1 particles

David Wagner

in collaboration with

Nora Weickgenannt, Enrico Speranza, and Dirk Rischke

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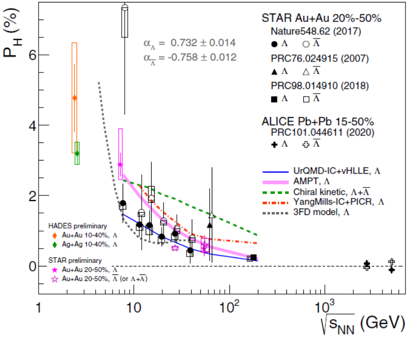
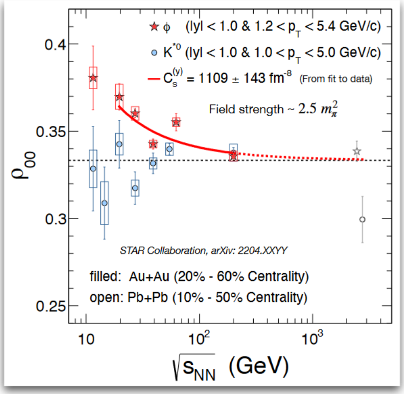
# Motivation

- Heavy-ion collisions provide several polarization observables
  - Spin $1/2$: Decay of $\Lambda$-Hyperons
  - Spin 1: Decay of $\phi/K^{*0}$-Mesons
- Both feature significant polarization




Figures from: T. Niida, EPJ Web Conf. 259, 06002 (2022)
S. Singha (STAR), this conference

# Setup: Charged vector fields

## Maxwell-Proca Lagrangian

$$\mathcal{L} = \hbar\left(-\frac{1}{2}V^{\dagger\mu\nu}V_{\mu\nu} + \frac{m^2}{\hbar^2}V^{\dagger\mu}V_\mu\right) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - iqF_{\mu\nu}V^\mu V^{\dagger\nu} \qquad (1)$$

Field strength tensors $V^{\mu\nu} := D^\mu V^\nu - D^\nu V^\mu \,, \quad F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu .$

- Introduce (gauge-)covariant derivative $D^\mu := \partial^\mu + iq/\hbar A^\mu$
- Take into account magnetic moment $\mu := q\hbar/m$ $(g = 2)$

H. C. Corben, J. Schwinger, Physical Review 58, no. 11, 953-968 (1940)
M. Napsuciale, S. Rodriguez E. G. Delgado-Acosta, M. Kirchbach, Physical Review D 77, no. 1, 430 (2008)

## Constraint equation

$$D^\mu V_\mu = -i\frac{q\hbar}{m^2}J^\mu V_\mu \,. \qquad (2)$$

- Constraint equation removes one degree of freedom

# Wigner function

- Idea: Introduce a quantum-mechanical analogue of the one-particle distribution function H. -W. Lee, Physics Reports 259, no. 3, 147-211 (1995)

## Wigner function for vector fields

$$W^{\mu\nu}(x,k) := -\frac{2}{(2\pi)^4\hbar^5}\int \mathrm{d}^4 v e^{-\frac{i}{\hbar}k^\alpha v_\alpha}\left\langle : V_+^{\dagger\mu} U_{+-} V_-^\nu : \right\rangle \tag{3}$$

with

$$V_\pm^\mu := V^\mu\left(x \pm \frac{v}{2}\right) , \tag{4}$$

$$U_{+-} := \exp\left[-i\frac{q}{\hbar}v^\alpha \int_{-1/2}^{1/2} \mathrm{d}t A_\alpha\left(x+tv\right)\right] . \tag{5}$$

- $U_{+-}$ is the gauge link such that the Wigner function is gauge invariant

Electromagnetic fields are treated as classical

# Decomposition of the Wigner function

- Use dynamics of $V^\mu$ to obtain equations of motion for $W^{\mu\nu}$
- Decompose $W^{\mu\nu}_{S/A} := (W^{\mu\nu} \pm W^{\nu\mu})/2$ to recover nine independent components

## Decomposition with respect to $k^\mu$

$$W_S^{\mu\nu} = E^{\mu\nu} f_E + K^{\mu\nu} f_K + \frac{k^{(\mu}}{2k} F_S^{\nu)} + F_K^{\mu\nu} \quad (6)$$

$$W_A^{\mu\nu} = \frac{k^{[\mu}}{2k} F_A^{\nu]} + \epsilon^{\mu\nu\alpha\beta} \frac{k_\alpha}{m} G_\beta \quad (7)$$

- $F_S^\mu k_\mu = F_A^\mu k_\mu = G^\mu k_\mu = F^\mu_{K,\mu} = 0 \,, \quad k_\mu F_K^{\mu\nu} = 0 \,, \quad F_K^{\mu\nu} = F_K^{\nu\mu}$

- $f_E$, $F_A^\mu$, $F_S^\mu$ are determined by constraint equations

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$E^{\mu\nu} := k^\mu k^\nu / k^2 \,, \quad K^{\mu\nu} := g^{\mu\nu} - E^{\mu\nu} \,,$
$a^{(\mu} b^{\nu)} := a^\mu b^\nu + a^\nu b^\mu \,, \quad a^{[\mu} b^{\nu]} := a^\mu b^\nu - a^\nu b^\mu$

# Boltzmann equation: Zeroth order

- Goal: Solve equations of motion perturbatively
  - $\rightarrow$ Expansion $W^{\mu\nu} = W^{(0),\mu\nu} + \hbar W^{(1),\mu\nu} + \cdots$
- $\mathcal{O}(\hbar^{\mathbf{0}})$: Wigner function on shell, $W^{(0),\mu\nu} \propto \delta(k^2 - m^2)$
  - $f_E^{(0)} = F_A^{(0),\mu} = F_S^{(0),\mu} = 0$
  - Independent components follow Vlasov and Bargmann-Michel-Telegdi (BMT) equations

    V. Bargmann, L. Michel, V. L. Telegdi, Phys. Rev. Lett. 2, 435 (1959)

## Zeroth-order equations

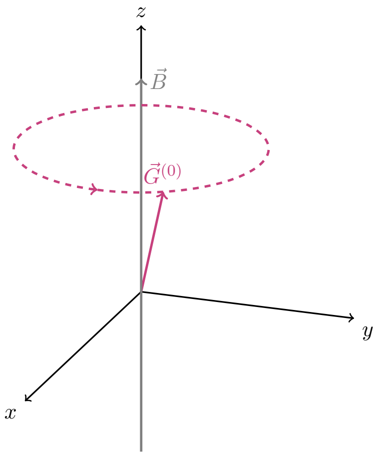
$$0 = k^\mu \left(\partial_\mu - qF_{\mu\nu}\frac{\partial}{\partial k_\nu}\right) f_K^{(0)} \qquad (8)$$

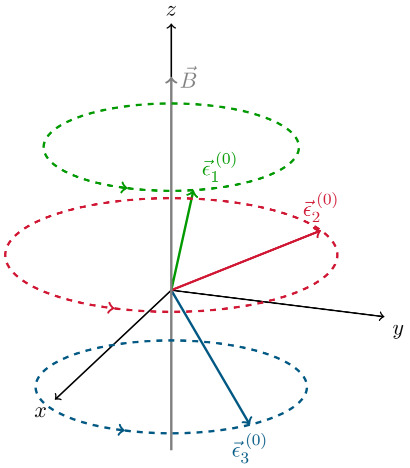
$$0 = k^\mu \left(\partial_\mu - qF_{\mu\nu}\frac{\partial}{\partial k_\nu}\right) G^{(0),\mu} - qF^{\mu\nu}G_\nu^{(0)} \qquad (9)$$

$$0 = k^\mu \left(\partial_\mu - qF_{\mu\nu}\frac{\partial}{\partial k_\nu}\right) F_K^{(0),\mu\nu} - qF_K^{(0),\alpha(\mu}F^{\nu)}{}_\alpha \qquad (10)$$

# Zeroth-order BMT equations

- Consider particle rest frame $\rightarrow (k^\mu) \equiv (m,0,0,0)$
  - Axial Vector: $G^{(0),\mu} \rightarrow \vec{G}^{(0)}$
  - Tensor: $F_K^{(0),\mu\nu} \rightarrow F_K^{(0),ij}$
    - decompose into eigenvectors $\vec{\epsilon}_1^{\,(0)}$, $\vec{\epsilon}_2^{\,(0)}$, $\vec{\epsilon}_3^{\,(0)}$



# Boltzmann equation: First order

- $\mathcal{O}(\hbar^{\mathbf{1}})$: Spin couples to gradients of electromagnetic fields
- Components $f_E^{(1)}$, $F_A^{(1),\mu}$, $F_S^{(1),\mu}$ nonzero, expressed in terms of $f_K^{(0)}$, $G^{(0),\mu}$, $F_K^{(0),\mu\nu}$
  - Enter into first-order Boltzmann equation

    $\longrightarrow$ Formulate in terms of $\mathrm{Tr}W$ and $W_A^{\mu\nu}$

**Scalar Boltzmann equation to order $\mathcal{O}(\hbar)$**

$$0 = k^\mu \left(\partial_\mu - qF_{\mu\nu}\frac{\partial}{\partial k_\nu}\right)\mathrm{Tr}W + \frac{iq\hbar}{2}(\partial^\alpha F^{\mu\nu})\frac{\partial}{\partial k^\alpha}W_{A,\mu\nu} \qquad (11)$$

- Contains free-streaming and Vlasov terms as well as Mathisson force (twice as large as in spin-1/2 case)

N. Weickgenannt, X. -L. Sheng, E. Speranza, Q. Wang, D. Rischke, Physical Review D 100, no. 5, 152 (2019)

- $G^\mu$ and $F_K^{\mu\nu}$ follow more complicated evolution equations

# Global equilibrium

- Two kinds of equilibrium:
  - Local: Collision term vanishes
    - $\rightarrow$ Second part of talk
  - Global: Local equilibrium + solution of the Boltzmann equation
- Use Ansatz for local equilibrium distribution function, Boltzmann equation then determines conditions for global equilibrium
  - Induced vector polarization by thermal vorticity and magnetic fields
    - $\rightarrow$ Analogues of Chiral Vortical Effect (CVE) and Chiral Magnetic Effect (CME) for massive spin-1 particles
  - No induced tensor polarization to first order in $\hbar$

# Including collisions (without electromagnetic fields)

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## General Lagrangian with interactions

$$\mathcal{L} = -\hbar\left(\frac{1}{2}V^{\dagger\mu\nu}V_{\mu\nu} - \frac{m^2}{\hbar^2}V^{\dagger\mu}V_\mu\right) + \mathcal{L}_{\text{int}} \, , \qquad (12)$$

- Obtain field equations with general source $\rho^\mu := -(1/\hbar)\partial\mathcal{L}_{\text{int}}/\partial V^\dagger_\mu$
- Perform Wigner transformation to get equations of motion for Wigner function

## Evolution equations for $W^{\mu\nu}$

$$k^\alpha\partial_\alpha W^{\mu\nu} = \mathcal{C}^{\mu\nu} \qquad (13)$$

$$\mathcal{C}^{\mu\nu} := -\frac{i}{(2\pi\hbar)^4}\int \mathrm{d}^4v e^{-ik^\alpha v_\alpha/\hbar}\left\langle : V^{\dagger\mu}_+\rho^\nu_- - \rho^{\dagger\mu}_+ V^\nu_- :\right\rangle \qquad (14)$$

# Enlarging phase space

- So far: Nine equations of motion in $(x, k)$-phase space
- Idea: Enlarge to $(x, k, \mathfrak{s})$-phase space to account for spin degrees of freedom
- Measure $\mathrm{d}S := [3m/(2\sqrt{2}\pi)]\mathrm{d}^4\mathfrak{s}\,\delta(\mathfrak{s}^2+2)\delta(k^\alpha \mathfrak{s}_\alpha)$ defined such that

$$\int \mathrm{d}S = 3\,, \quad \int \mathrm{d}S\, \mathfrak{s}^\mu \mathfrak{s}^\nu = -2K^{\mu\nu}\,, \quad \int \mathrm{d}S\, K^{\mu\nu}_{\rho\sigma}\mathfrak{s}^\rho \mathfrak{s}^\sigma \mathfrak{s}_\alpha \mathfrak{s}_\beta = (8/5)K^{\mu\nu}_{\alpha\beta}$$

## Distribution function in enlarged phase space

$$\mathfrak{f}(x,k,\mathfrak{s}) := f_K - \mathfrak{s}^\mu G_\mu + \frac{5}{4}\mathfrak{s}^\mu \mathfrak{s}^\nu F_{K,\mu\nu} \qquad (15)$$

$$\mathfrak{C} := C_K - \mathfrak{s}^\mu C_{G,\mu} + \frac{5}{4}\mathfrak{s}^\mu \mathfrak{s}^\nu C_{K,\mu\nu} \qquad (16)$$

- Allows to express evolution equations compactly as $k^\alpha \partial_\alpha \mathfrak{f} = \mathfrak{C}[\mathfrak{f}]$

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$K^{\mu\nu}_{\alpha\beta} := (K^\mu_\alpha K^\nu_\beta + K^\mu_\beta K^\nu_\alpha)/2 - 1/3 K^{\mu\nu}K_{\alpha\beta}\,, \quad K^{\mu\nu} := g^{\mu\nu} - k^\mu k^\nu/m^2$

$C_K := (1/3)K^{\mu\nu}\mathcal{C}_{\mu\nu}\,, \quad C^\mu_G := -[i/(2m)]\epsilon^{\mu\nu\alpha\beta}k_\nu \mathcal{C}_{\alpha\beta}\,, \quad C^{\mu\nu}_K := K^{\mu\nu}_{\alpha\beta}\mathcal{C}^{\alpha\beta}$

- Task: Express collision kernel $\mathfrak{C}$ in terms of distribution function f
  - Long calculation, to be detailed in upcoming publication

    DW, N. Weickgenannt, E. Speranza, D. H. Rischke, in preparation

## Spin-1 collision kernel

$$\begin{aligned}\mathfrak{C}[f] &= \int \mathrm{d}\Gamma_1 \mathrm{d}\Gamma_2 \mathrm{d}\Gamma \mathrm{d}\Gamma' \mathrm{d}S_1'(k)\mathcal{W}\left[f(x+\Delta_1,k_1,\mathfrak{s}_1)f(x+\Delta_2,k_2,\mathfrak{s}_2)\right. \\ &\quad \left. -f(x+\Delta_1',k,\mathfrak{s}_1')f(x+\Delta',k',\mathfrak{s}')\right] \\ &+ \int \mathrm{d}\Gamma_2 \mathrm{d}S_1(k)\mathfrak{W} f(x+\Delta,k,\mathfrak{s}_1)f(x+\Delta_2,k_2,\mathfrak{s}_2) \end{aligned} \tag{17}$$

- Features spin-momentum-exchange and pure spin-exchange term
- First order: Nonlocal shifts $\Delta^\mu := -\hbar/[2m(\hat{t}^\rho k_\rho + m)]\epsilon^{\mu\nu\alpha\beta}k_\nu \hat{t}_\alpha \mathfrak{s}_\beta$ in frame defined by $\hat{t}^\mu$

  N. Weickgenannt, E. Speranza, X.-L. Sheng, Q. Wang, D. H. Rischke, Phys. Rev. D 104, 016022 (2021)

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$\mathfrak{f}(x,k,\mathfrak{s}) := \delta(k^2-m^2)f(x,k,\mathfrak{s})$, $\mathrm{d}\Gamma := \mathrm{d}^4k\,\delta(k^2-m^2)\mathrm{d}S$

# Equilibrium

- Local-equilibrium distribution function has to depend on collisional invariants
  - Four-momentum $k^\mu$
  - Total angular momentum $J^{\mu\nu} := \hbar\Sigma_{\mathfrak{s}}^{\mu\nu} + \Delta^{[\mu}k^{\nu]}$

**Local-equilibrium distribution function**

$$f_{\text{eq}}(x,k,\mathfrak{s}) = \exp\left[-\beta^\mu(x)k_\mu + \frac{\hbar}{2}\Omega_{\mu\nu}(x)\Sigma_{\mathfrak{s}}^{\mu\nu}\right] \tag{18}$$

- Lagrange multipliers $\beta^\mu(x)$, $\Omega^{\mu\nu}(x)$ determined by $\mathfrak{C}[f_{\text{eq}}] = 0$
- Conditions $\partial^{(\mu}\beta^{\nu)} = 0\,,\ \Omega^{\mu\nu} = -(1/2)\partial^{[\mu}\beta^{\nu]}$ identical for local and global equilibrium

N. Weickgenannt, E. Speranza, X.-L. Sheng, Q. Wang, D. H. Rischke, Phys. Rev. Lett. 127, 052301 (2021)

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$\Sigma_{\mathfrak{s}}^{\mu\nu} := -(1/m)\epsilon^{\mu\nu\alpha\beta}k_\alpha\mathfrak{s}_\beta$

Maxwell-Jüttner form due to low-density approximation

# Conclusion and future perspectives

- Work done:
  - Formulated kinetic theory for massive spin-1 particles in electromagnetic fields

    DW, N. Weickgenannt, E. Speranza, D. H. Rischke, in preparation
    - Computed global equilibrium
  - Included collisions
    - Recovered local-equilibrium conditions known from spin-1/2 case

      DW, N. Weickgenannt, E. Speranza, D. H. Rischke, in preparation
- Next steps:
  - Formulate dissipative hydrodynamics
    - Compare to spin-1/2

      N. Weickgenannt, DW, E. Speranza, D. H. Rischke, arXiv: 2203.04766 (2022)
  - Consider gauge fields

    X.-G. Huang, P. Mitkin, A. V. Sadofyev, E. Speranza, J. High Energ. Phys. 2020, 117 (2020)

# Appendix

# Global equilibrium: polarization

- Assumption: polarization at least of order $\mathcal{O}(\hbar)$
  $\rightarrow$ No large initial polarization
- Idea: Split first-order polarization into free and induced parts
  - Free parts follow BMT equation
  - Induced parts determined from force terms in kinetic equations
- Definitions:

$$V^{(0)} := \frac{1}{(2\pi\hbar)^3}\frac{1}{3}\sum_{\lambda,e}\Theta(ek^0)f_\lambda^{(0)e} , \tag{19}$$

$$f_\lambda^{(0)e} := \left(e^{g_\lambda^{(0)e}} - 1\right)^{-1} , \tag{20}$$

$$g_\lambda^{(0)e} := a_\lambda^e + k_\mu\beta^\mu . \tag{21}$$

# Global equilibrium: polarization

## Wigner function in global equilibrium

$$W^{\mu\nu}_{S,\text{eq},\text{on}-\text{shell}} = K^{\mu\nu}\left(V^{(0)} + \hbar V^{(1)}\right) + \hbar\Phi^{\mu\nu} \quad (22)$$

$$W^{\mu\nu}_{A,\text{eq},\text{on}-\text{shell}} = -i\hbar\Bigg[\varpi_{\mu\nu}V^{(0)\prime} + \frac{q}{2k^2}F_{\mu\nu}V^{(0)} + \frac{1}{2}E^{\alpha}{}_{[\mu}\left(\varpi_{\nu]\alpha}V^{(0)\prime} + \frac{2q}{k^2}F_{\nu]\alpha}V^{(0)}\right) + \Xi^{\mu\nu}\Bigg] \quad (23)$$

$$W^{\mu\nu}_{S,\text{eq},\text{off}-\text{shell}} = 0 \quad (24)$$

$$W^{\mu\nu}_{A,\text{eq},\text{off}-\text{shell}} = -i\hbar q F_{\alpha}{}^{[\mu}K^{\nu]\alpha}V^{(0)} \quad (25)$$

- $\Phi^{\mu\nu}$, $\Xi^{\mu\nu}$ follow BMT equations, unconstrained otherwise
- Terms $\propto E^{\alpha}{}_{\mu}$ do not contribute to polarization density $P^{\mu}_{\text{eq}}$

$$P^{\mu}_{\text{eq}}(x,k) \propto \frac{i}{2m}\epsilon^{\mu\nu\alpha\beta}k_{\nu}W_{A,\text{eq},\alpha\beta} \quad (26)$$

F. Becattini, arXiv:2004.04050v2 (2020)

# Kinetic equations to order $\mathcal{O}(\hbar)$

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- Definitions:
  - $\mathcal{V} := g_{\mu\nu} W^{\mu\nu}_{\text{on-shell}}$
  - $\mathcal{S}^{\mu\nu} := i W^{\mu\nu}_{A,\text{on-shell}}$
  - $\mathcal{T}^{\mu\nu} := W^{\mu\nu}_{S,\text{on-shell}} - \frac{1}{4} g^{\mu\nu} \mathcal{V}$

$\rightarrow$ More natural variables than independent quantities at this order

# Kinetic equations to order $\mathcal{O}(\hbar)$

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## Combined kinetic equations

$$0 = k^\alpha \left( \partial_\alpha - qF_{\alpha\beta} \frac{\partial}{\partial k_\beta} \right) \mathcal{V} + \frac{q\hbar}{2} (\partial^\gamma F^{\alpha\beta}) \frac{\partial}{\partial k^\gamma} \mathcal{S}_{\alpha\beta} \tag{27}$$

$$\begin{aligned} 0 &= k^\alpha \left( \partial_\alpha - qF_{\alpha\beta} \frac{\partial}{\partial k_\beta} \right) \mathcal{S}^{\mu\nu} - qF_\rho{}^{[\mu} \mathcal{S}^{\nu]\rho} \\ &\quad - \frac{q\hbar}{2} (\partial^\gamma F_\alpha{}^{[\mu}) \frac{\partial}{\partial k^\gamma} \left( \mathcal{T}^{\nu]\alpha} + \frac{g^{\nu]\alpha}}{4} \mathcal{V} \right) - \frac{q\hbar}{2m^2} J_\alpha \mathcal{T}^{\alpha[\mu} k^{\nu]} \end{aligned} \tag{28}$$

$$\begin{aligned} 0 &= k^\alpha \left( \partial_\alpha - qF_{\alpha\beta} \frac{\partial}{\partial k_\beta} \right) \mathcal{T}^{\mu\nu} + qF_\rho{}^{(\mu} \mathcal{T}^{\nu)\rho} \\ &\quad + g^{\mu\nu}_{\sigma\rho} \frac{q\hbar}{2} (\partial^\gamma F_\alpha{}^\rho) \frac{\partial}{\partial k^\gamma} \mathcal{S}^{\alpha\sigma} - \frac{q\hbar}{2m^2} J_\alpha \mathcal{S}^{\alpha(\mu} k^{\nu)} \end{aligned} \tag{29}$$

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$g^{\mu\nu}_{\alpha\beta} := 1/2 g^{(\mu}_\alpha g^{\nu)}_\beta - 1/4 g^{\mu\nu} g_{\alpha\beta}$

# Collision terms: definitions (1)

## Transition rates

$$\mathcal{W} := \frac{9}{2}\frac{m^2c^2}{2(2\pi)^3}\frac{1}{32}\sum_{\lambda^2,\lambda'^2,\gamma^2,\rho^2,\lambda} \delta^{(4)}\left(k+k'-k_1-k_2\right) h^{(\lambda_1',\lambda_2')}(k',\mathfrak{s}')$$
$$\times h^{(\gamma_1,\rho_1)}(k_1,\mathfrak{s}_1)h^{(\gamma_2,\rho_2)}(k_2,\mathfrak{s}_2)\left[h^{(\lambda_1,\lambda)}(k,\mathfrak{s}_1')H^{(\lambda,\lambda_2)}(k,\mathfrak{s})\right.$$
$$\left.+H^{(\lambda_1,\lambda)}(k,\mathfrak{s})h^{(\lambda,\lambda_2)}(k,\mathfrak{s}_1')\right]$$
$$\times\left\langle k,k';\lambda_2,\lambda_2'\right|\hat{t}\left|k_1,k_2;\gamma_1,\gamma_2\right\rangle\left\langle k_1,k_2;\rho_1,\rho_2\right|\hat{t}^\dagger\left|k,k';\lambda_1,\lambda_1'\right\rangle \ ,$$

$$\mathfrak{W} := -4\frac{9}{2}\frac{m^2c^2}{2(2\pi)^3}\frac{\pi\hbar}{4}\sum_{\gamma^2,\rho^2} h^{(\gamma_2,\rho_2)}(k_2,\mathfrak{s}_2)\left\langle k,k_2;\rho_1,\rho_2\right|\hat{t}+\hat{t}^\dagger\left|k,k_2;\gamma_1,\gamma_2\right\rangle$$
$$\times\frac{i}{2}\epsilon^{*(\lambda')}_{[\alpha}(k)\epsilon^{(\rho)}_{\beta]}(k)\mathfrak{s}^\alpha\mathfrak{s}_1^\beta\left(1+\frac{5}{3}\mathfrak{s}\cdot\mathfrak{s}_1\right)$$

# Collision terms: definitions (2)

## Contractions of polarization vectors

$$H^{(\lambda_2,\lambda_1)}(k,\mathfrak{s}) \quad := \quad \epsilon_\alpha^{(\lambda_1)}(k)\epsilon_\beta^{*(\lambda_2)}(k)\left(\frac{1}{3}K^{\alpha\beta} + \frac{i}{2}\epsilon^{\alpha\beta\rho\sigma}\frac{k_\rho}{mc}\mathfrak{s}_\sigma + \frac{5}{4}\mathfrak{s}^{\langle\alpha}\mathfrak{s}^{\beta\rangle}\right)$$

$$h^{(\lambda_2,\lambda_1)}(k,\mathfrak{s}) \quad := \quad \epsilon_\alpha^{(\lambda_1)}(k)\epsilon_\beta^{*(\lambda_2)}(k)\left(\frac{1}{3}K^{\alpha\beta} + \frac{i}{2}\epsilon^{\alpha\beta\rho\sigma}\frac{k_\rho}{mc}\mathfrak{s}_\sigma + \frac{1}{2}\mathfrak{s}^{\langle\alpha}\mathfrak{s}^{\beta\rangle}\right)$$