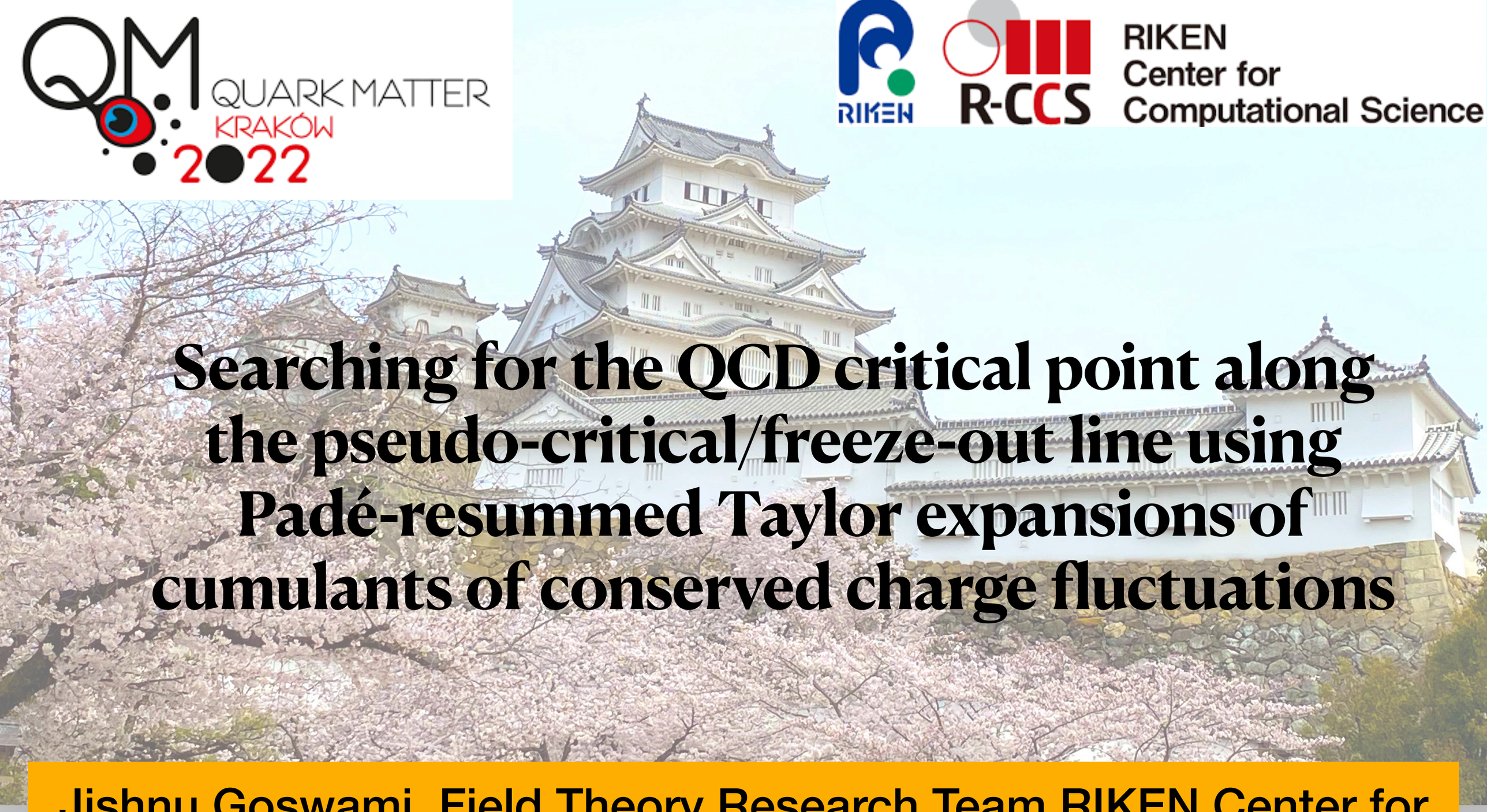


Searching for the QCD critical point along the pseudo-critical/freeze-out line using Padé-resummed Taylor expansions of cumulants of conserved charge fluctuations

Jishnu Goswami, Field Theory Research Team,RIKEN Center for Computational Science

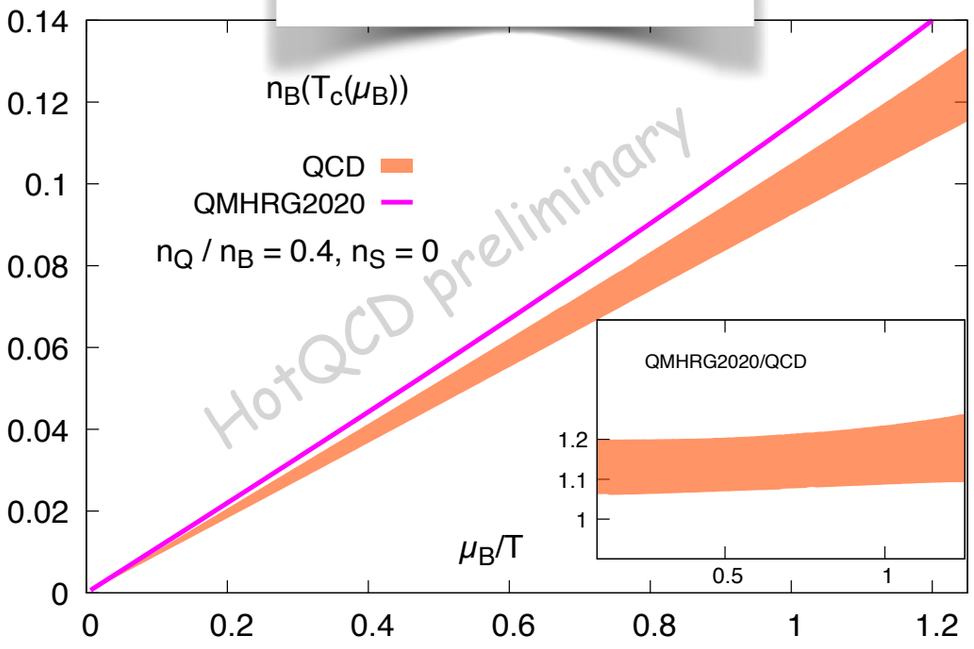
With F. Karsch, Swagato Mukherjee and C. Schmidt

HotQCD collaboration: arXiv:2202.09184

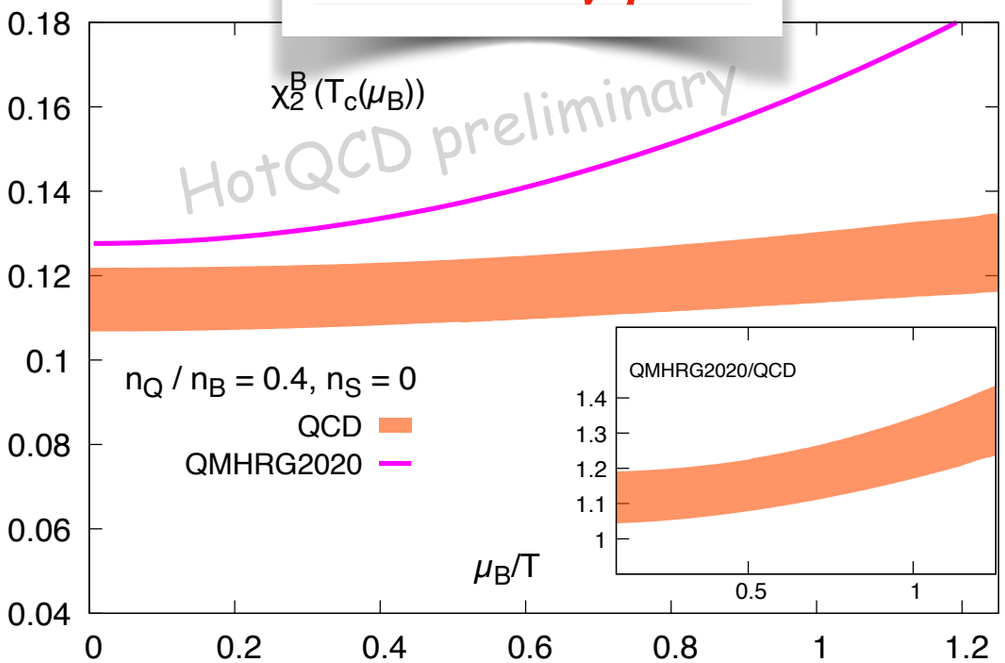
Acknowledgement: To all members of the HotQCD collaboration

Motivation : Lower order cumulants on the pseudo-critical/ freeze-out line

$$n_B/T^3 = \frac{1}{VT^3} M_B$$



$$\chi_2^B/T^2 = \frac{1}{VT^3} \sigma_B^2$$



- Net baryon number density and Second order baryon number fluctuations has $10 - 20\,\%$ deviation from non interacting HRG at $\mu_B = 0$.
- The deviation increases rapidly to almost 30 % at $\mu_B/T = 1$ for second order baryon number fluctuations.
- Obtained μ_B^f from non interacting HRG will be different for these two observables.

Thermodynamics using Lattice QCD

Expansion of the QCD pressure:

$$\frac{P(T,\vec{\mu})}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{1}{i!j!k!} \chi_{ijk}^{BQS} \hat{\mu}_B^i \hat{\mu}_Q^j \hat{\mu}_S^k \ , \ \hat{\mu} = \mu/T,$$

$$\chi_{ijk}^{BQS}(T,0) = \left. \frac{\partial^{i+j+k} P/T^4}{\partial \hat{\mu}_X^{i,j,k}} \right|_{\mu_X=0} , X = B, Q, S$$

Direct Method	Resummation Method
Imaginary chemical potential : M. D'Elia and M. P. Lombardo[hep-lat/0209146], J. Gunther et al [arXiv:1607.02493] Taylor expansion : Bielefeld-Swansea collaboration [hep-lat/0501030], HotQCD collaboration [1701.04325 [hep-lat]] , Datta, Gavai and Gupta[1612.06673 [hep-lat]]…	Padé approximant : Gavai and Gupta[arXiv:hep-lat/0412035, arXiv:0806.2233] , Bielefeld Parma collaboration [2110.15933 [hep-lat]] , HotQCD collaboration [arXiv:2202.09184:This talk] There is more, S. Borsanyi et al arXiv:2102.06660 arXiv:2202.05574. S. Mondal et al arXiv:2106.03165

Conformal maps: V. Skokov arXiv:1008.4549,M. Giordano et al., arXiv:2004.10800 , G. Basar, arXiv:2105.08080

Thermodynamics using Lattice QCD

Expansion of the QCD pressure:

$$\frac{P(T,\vec{\mu})}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{1}{i!j!k!} \chi^{BQS}_{ijk} \hat{\mu}^i_B \hat{\mu}^j_Q \hat{\mu}^k_S \ , \ \hat{\mu} = \mu/T,$$

$$\chi^{BQS}_{ijk}(T,0) = \left. \frac{\partial^{i+j+k} P/T^4}{\partial \hat{\mu}^{i,j,k}_X} \right|_{\mu_X=0} , \ X = B, Q, S$$

Constraint series expansion :

Parametrization of μ_Q, μ_S $[n_Q/n_B = r, n_S = 0]$ **:**

$$\mu_S = s_1 \mu_B + s_3 \mu_B^3 + ...$$

$$\mu_Q = q_1 \mu_B + q_3 \mu_B^3 + ...$$

We will discuss, $n_S = 0, \ \ n_Q/n_B = 0.5, \ \ \Rightarrow \mu_B, \mu_S \neq 0, \ \mu_Q = 0,$

Condition in HIC, $n_S = 0, \ \ n_Q/n_B = 0.4, \ \ \Rightarrow \mu_B, \mu_S \neq 0, \ \mu_Q \ll \mu_B$

Thermodynamics using Lattice QCD

Expansion of the QCD pressure:

$$\frac{P(T,\vec{\mu})}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{1}{i!j!k!} \chi^{BQS}_{ijk} \hat{\mu}_B^i \hat{\mu}_Q^j \hat{\mu}_S^k \ , \ \hat{\mu} = \mu/T,$$

$\mu_Q = 0$

$\mu_S = s_1 \mu_B + s_3 \mu_B^3 + ...$

$$\chi_0^B(T,\hat{\mu}_B) \equiv \frac{P(T,\mu_B) - P(T,0)}{T^4} = \sum_{k=1}^{\infty} P_{2k}(T) \hat{\mu}_B^{2k}$$

$$\chi_1^B(T,\hat{\mu}_B) \equiv \frac{n_B(T,\mu_B)}{T^3} = \sum_{k=1}^{\infty} N^B_{2k-1}(T) \hat{\mu}_B^{2k-1}$$

$$\chi_2^B(T,\mu_B) = \sum_{k=0}^{\infty} \tilde{\chi}_2^{B,k}(T) \hat{\mu}_B^{2k}$$

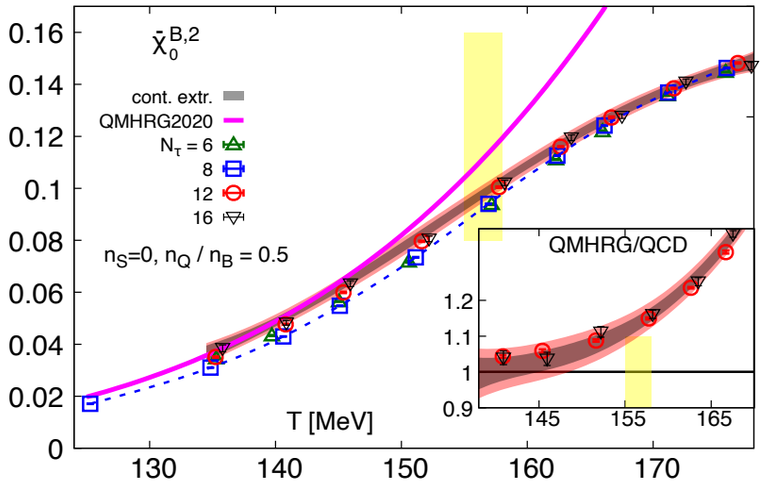
$$P_{2k} \equiv \bar{\chi}_0^{B,2k}/(2k)!$$

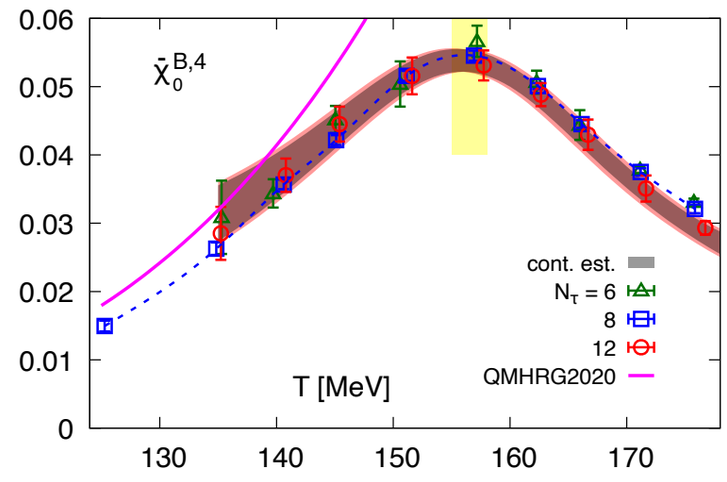
$$N_{2k-1} \equiv \bar{\chi}_0^{B,2k}/(2k-1)!$$

Same set of coefficients

$$\tilde{\chi}_2^{B,k} \equiv \bar{\chi}_2^{B,2k}/(2k)!$$

Expansion coefficients of the Taylor series



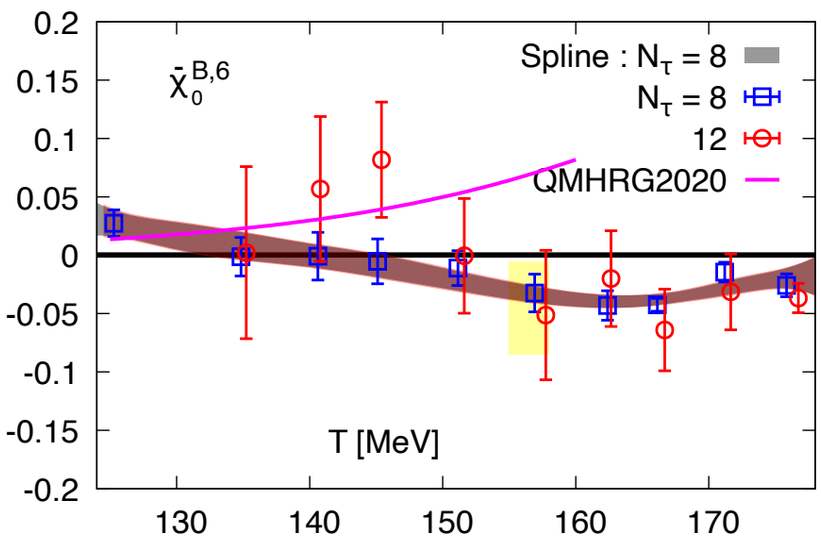


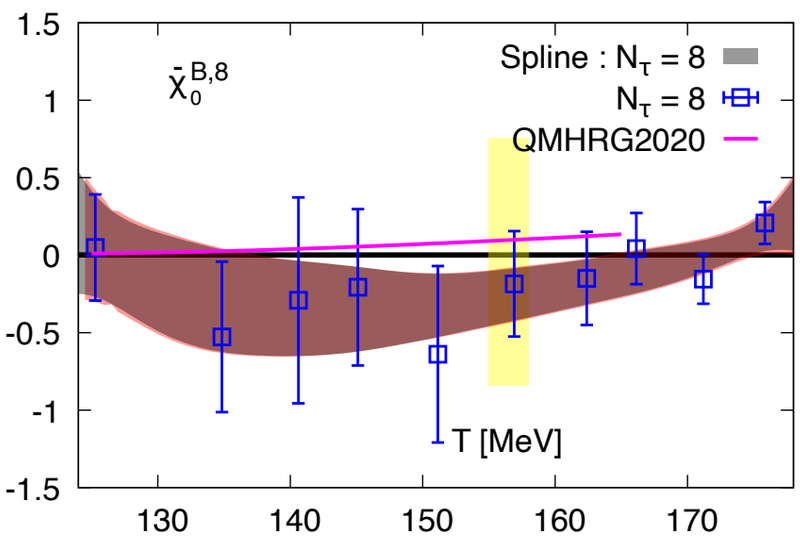
Strictly positive for all temperatures.

- We show the continuum extrapolated results based on $N_\tau \in \{6,8,12,16\}$ and$\{6,8,12\}$ lattices respectively for the first two leading order coefficients.

- For the sixth and the eighth order expansion coefficients we have only used the $N_\tau = 8$ dataset, where we have generated ~ 1.5 million gauge configurations per T.

- Temperature dependence of the expansion coefficients depicts that deviations from the thermodynamics of a non interacting HRG rapidly become larger for higher order cumulants at non-zero μ_B.





$\bar{\chi}_0^{B,8}$ is strictly negative for temperature range, $T \in [135 : 165]$ MeV.

Radius of convergence of Taylor series upto $\hat{\mu}_B^8$

$$\chi_0^B(T,\mu_B) \equiv \frac{P(T,\mu_B)-P(T,0)}{T^4} = \sum_{k=1}^{4} P_{2k}(T)\hat{\mu}_B^{2k}, \text{ Where, } P_{2k} = \frac{\bar{\chi}_0^{B,2k}}{2k!}$$

$$\chi_1^B(T,\hat{\mu}_B) \equiv \frac{n_B(T,\mu_B)}{T^3} = \sum_{k=1}^{4} N_{2k-1}^B(T)\hat{\mu}_B^{2k-1}, \text{ Where, } N_{2k-1}^B = 2kP_{2k} = \frac{\bar{\chi}_0^{B,2k}}{(2k-1)!}$$

- **Reliability of Taylor Series** : Good agreement between subsequent orders.

- Reliability of the Taylor expansion will be smaller for the number density than the pressure series. [$P_8^B = \bar{\chi}_0^{B,8}/8!,\ N_7^B = \bar{\chi}_0^{B,8}/7!$]

- **Radius of convergence of Taylor series** : The true radius of convergence can be written as , $r_c = \lim_{k\to\infty} r_{2k}^P = \lim_{k\to\infty} r_{2k}^{nB}$.

Radius of convergence of Taylor series upto $\hat{\mu}_B^8$

$$\chi_1^B(T,\hat{\mu}_B) \equiv \frac{n_B(T,\mu_B)}{T^3} = \sum_{k=1}^{4} N_{2k-1}^B(T)\hat{\mu}_B^{2k-1}, \text{ Where, } N_{2k-1}^B = \frac{\chi_{2k}^B}{(2k-1)!}$$

$$\chi_0^B(T,\mu_B) \equiv \frac{P(T,\mu_B) - P(T,0)}{T^4} = \sum_{k=1}^{4} P_{2k}(T)\hat{\mu}_B^{2k}, \text{ Where, } P_{2k} = 2kN_{2k-1} = \frac{\chi_{2k}^B}{2k!}$$

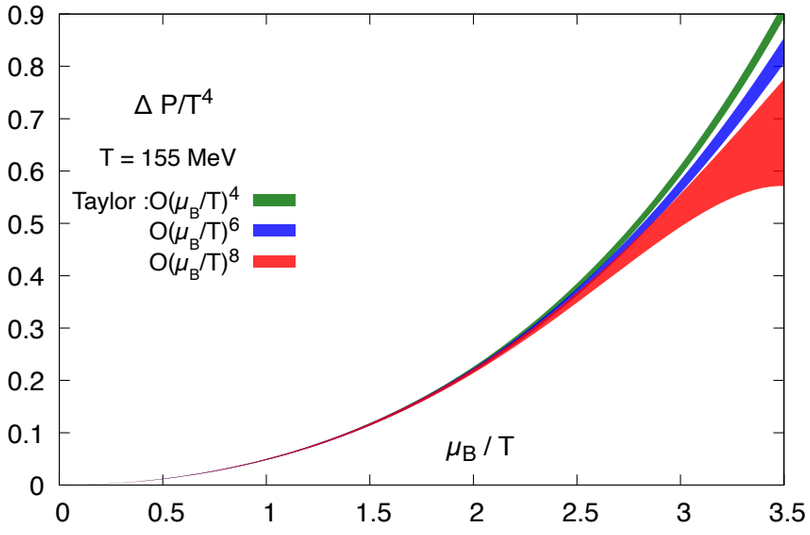
- The usual ratio estimator of radius of convergence can be described as, $r_{2k}^P = |P_{2k-2}/P_{2k}|^{1/2}$ and $r_{2k}^{nB} = |N_{2k-3}^B/N_{2k-1}^B|^{1/2}$ for pressure and number density respectively.

$$r_{2k}^P/r_{2k}^{nB} = \sqrt{[2k/(2k-2)]} = 1 + 1/2k + O(k^{-2})$$

$$r_c = \lim_{k\to\infty} r_{2k}^P = \lim_{k\to\infty} r_{2k}^{nB}$$

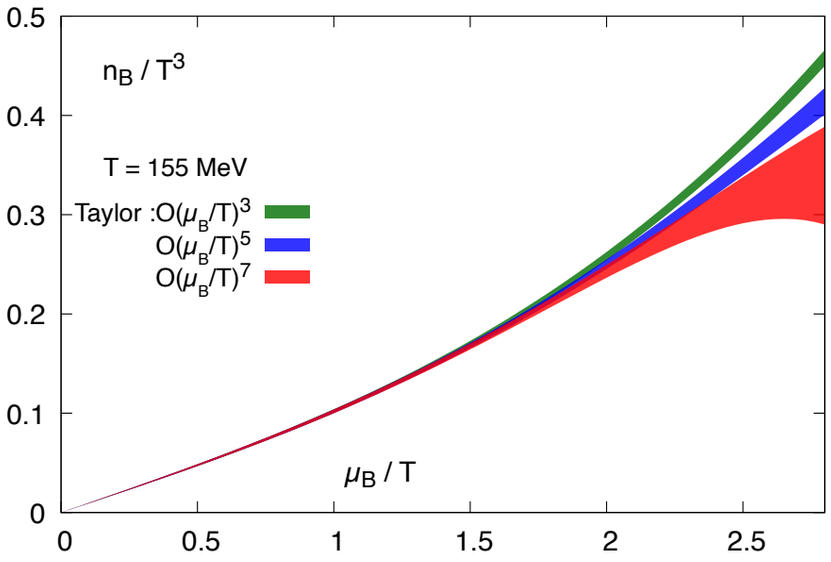
Reliability of the Taylor series for any observable can be improved by including more higher order coefficients as long as we are inside the disc of radius , $|\hat{\mu}_B| < r_c$

Reliability of the expansion



Reminder : $\chi_0^{B,8}$ is strictly negative for temperature range, $T \in [135 : 165]$ MeV.

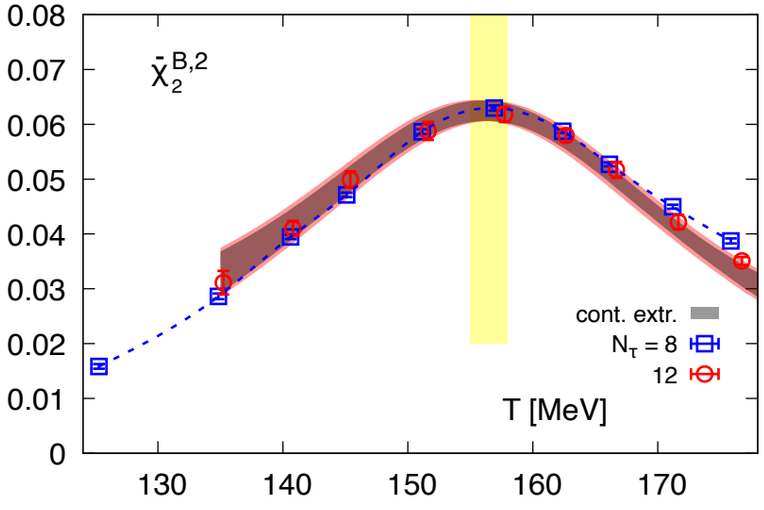
- Taylor expansion of the pressure, net baryon-number density are shown for different orders.

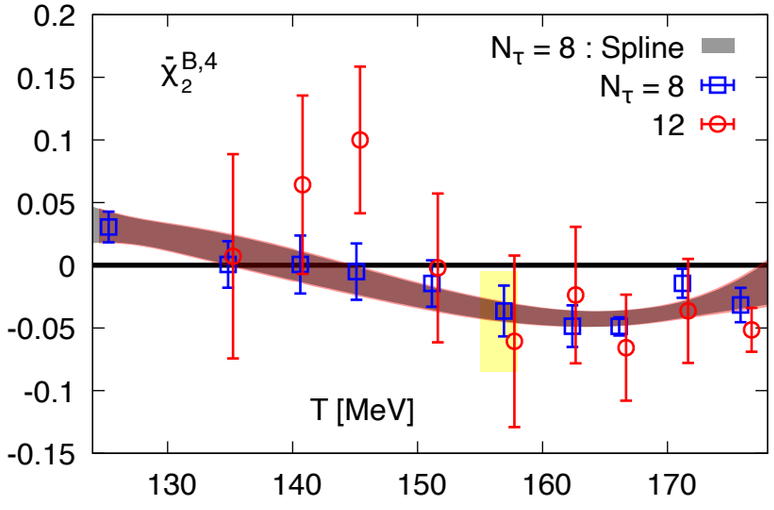


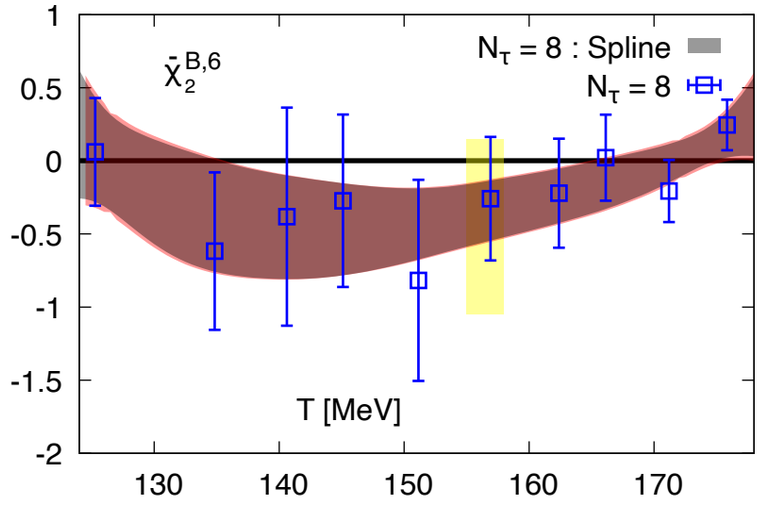
- The convergence between subsequent orders shifted to smaller values of the chemical potential for higher order cumulants.

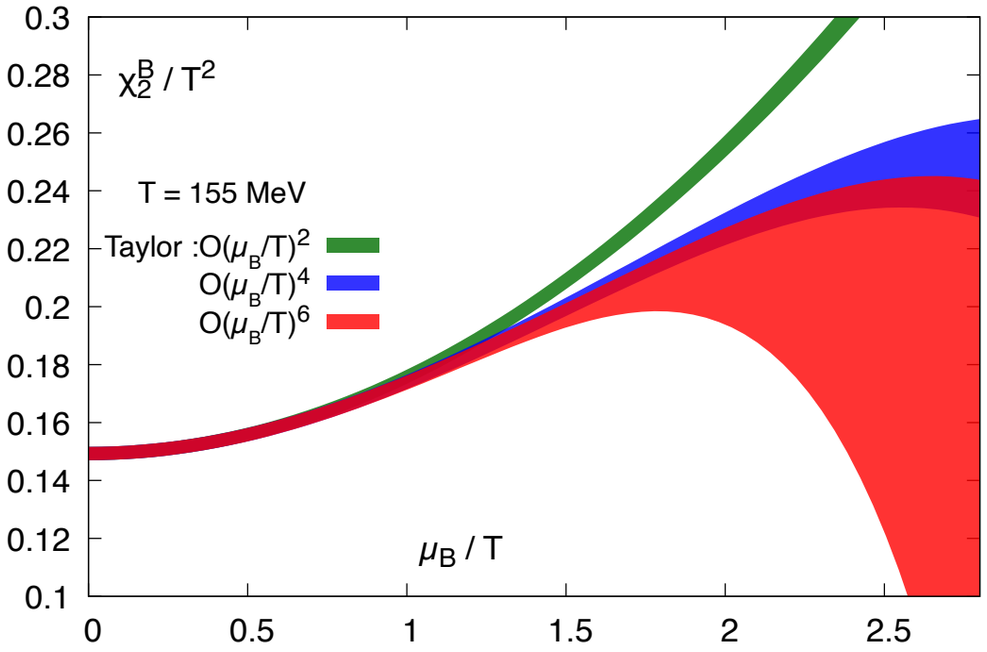
Reliability of the expansion

For χ_2^B, three different expansion coefficients are needed for NLO , N^2LO and N^3LO.









- Second order cumulant of net baryon-number fluctuations are shown for different orders.
- The convergence of subsequent orders shifted to smaller value of the chemical potential than pressure and number density series.

Searching for CEP using Padé approximants

In practice we work with finite number of coefficients!!

$$f(x) = \sum_{i=0}^{n} c_i x^i$$

Expectation : CEP will exist at a lower temperature than T_{chiral} (~ 130 MeV)

Halasz et al,arXiv:hep-ph/9804290

- **Lee Yang** : Phase transitions are related to singularities of the Taylor series on the real axis.

- If all the expansion coefficients are of same sign, could be an indication that the singularity of the series is on the real axis and hence is an indication of a critical point.

- Alternatively, one could construct Padé approximants which are rational functions of the form, $f(x) = \dfrac{\sum_{i=0}^{a} c_i x^i}{1 + \sum_{j=0}^{b} d_j x^j}$, and evaluate its singularities.

- Furthermore, singularities in the complex plane can show some universal scaling behaviour related to LYEs close to a critical point (this will not be discussed here).

See talk by C. Schmidt , T03 : Wed

Swagato Mukherjee , Vladimir Skokov, arXiv:1909.04639,
P. Dimopoulos et al, 2110.15933 [hep-lat]

Padé approximants from $\hat{\mu}_B^8$ Taylor series

$$\Delta P/T^4 = \sum_{k=1}^{4} P_{2k}(T)\hat{\mu}_B^{2k} = (\bar{x}^2 + \bar{x}^4 + c_{6,2}\bar{x}^6 + c_{8,2}\bar{x}^8)P^2/P_4,\ \bar{x} = \sqrt{\frac{P_4}{P_2}}\ \hat{\mu}_B$$

Reminder : P_2 and P_4 are strictly positive for all temperatures.

- One can construct various $[m,2]$ and $[n,4]$ Padé's from the above series.
  $[m \in \{2,4,6\}$ and $n \in \{2,4\}]$

- The convergence of Padé approximants will be unaffected by a singularity in the complex plane contrary to the Taylor series.

- The poles of the Padé approximants closest to the origin determine the radius of convergence.

- The poles of a general $[m,2]$ and $[n,4]$ Padé's are the usual ratio estimator (r_c^n) and Mercer Roberts estimator (r_c^{MR})[*For two pair of complex poles] of radius of convergence of the Taylor series.

$$r_c^{MR} = \left| \frac{c_{n+2}\, c_{n-2} - c_n^2}{c_{n+4}\, c_n - c_{n+2}^2} \right|^{1/4}, \ n \text{ even}$$

G. N. Mercer and A. J. Roberts, SIAM J. Appl. Math. 50, 1547 (1990)

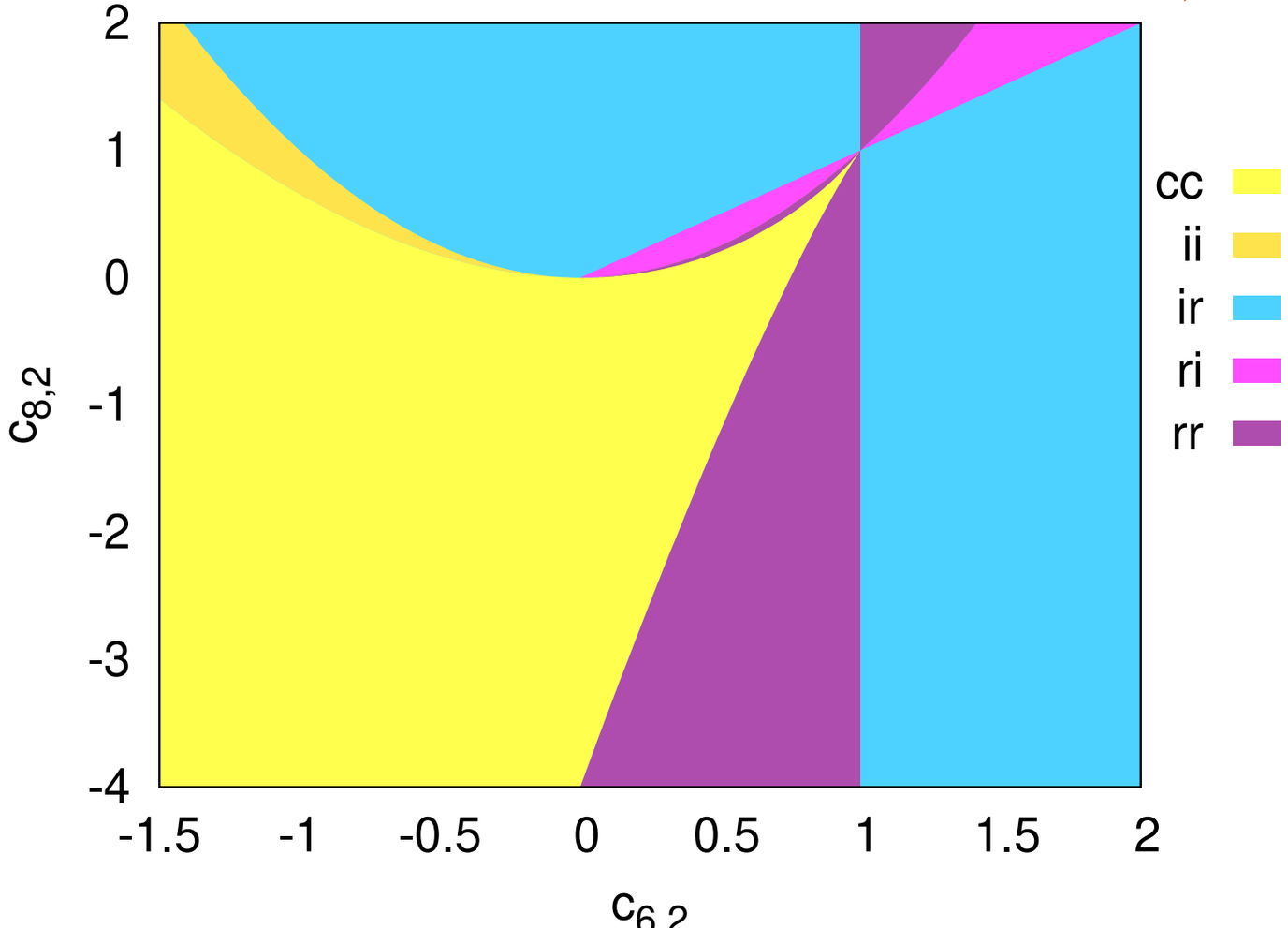
Constraints for a real pole of [4,4] Padé

Reminder : Critical points are related to the singularities on the real axis.
All expansion coefficients have to be positive for a singularity in the real axis.
$\chi_0^{B,6}, \chi_0^{B,8}$ are negative for temperature range, $T \in [135 : 165]$ MeV.

$$P[4,4] = \frac{(1-c_{6,2})\bar{x}^2 + \left(1 - 2c_{6,2} + c_{8,2}\right)\bar{x}^4}{(1-c_{6,2}) + (c_{8,2}-c_{6,2})\bar{x}^2 + (c_{6,2}^2 - c_{8,2})\bar{x}^4}$$

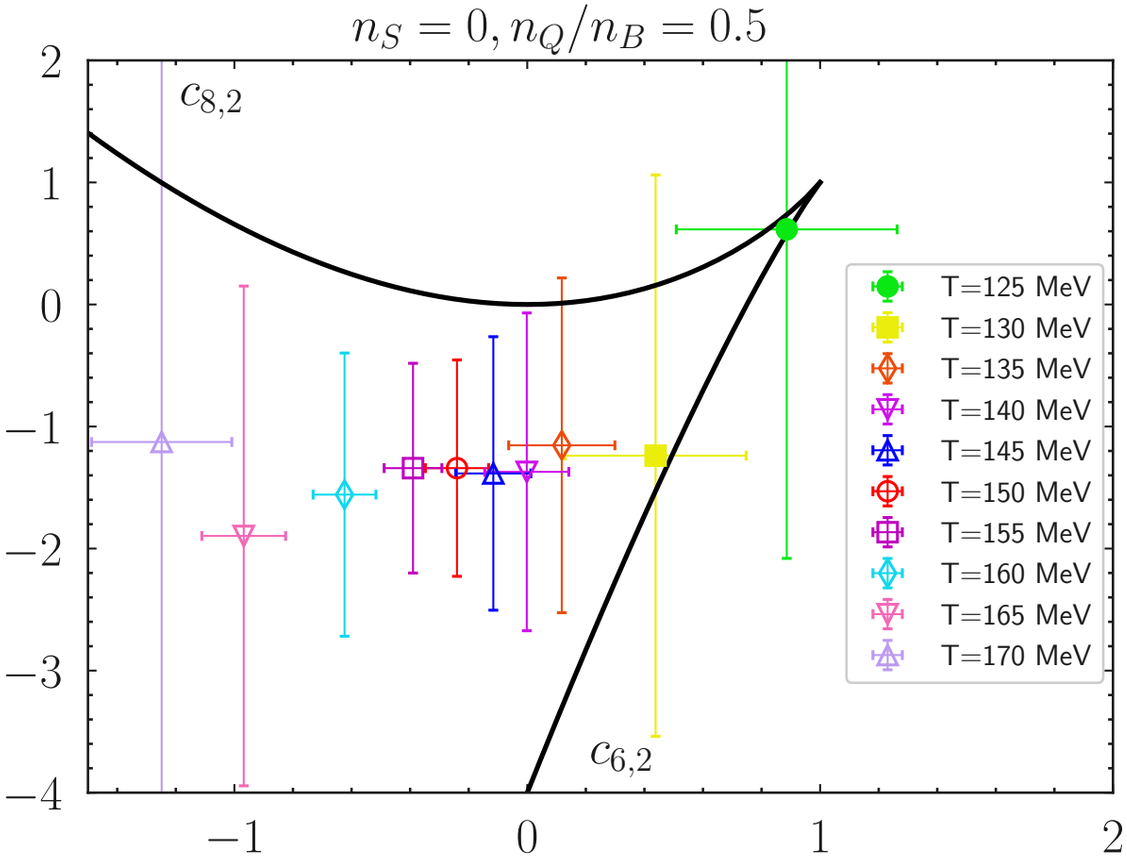
Poles can be written as,

$$z \equiv \bar{x} \qquad z^{\pm} = \frac{c_{8,2} - c_{6,2} \pm \sqrt{(c_{8,2} - c_{8,2}^{+})(c_{8,2} - c_{8,2}^{-})}}{2(c_{8,2} - c_{6,2}^2)}; \;\; c_{8,2}^{\pm} = -2 + 3c_{6,2} \pm 2(1-c_{6,2})^{3/2}$$

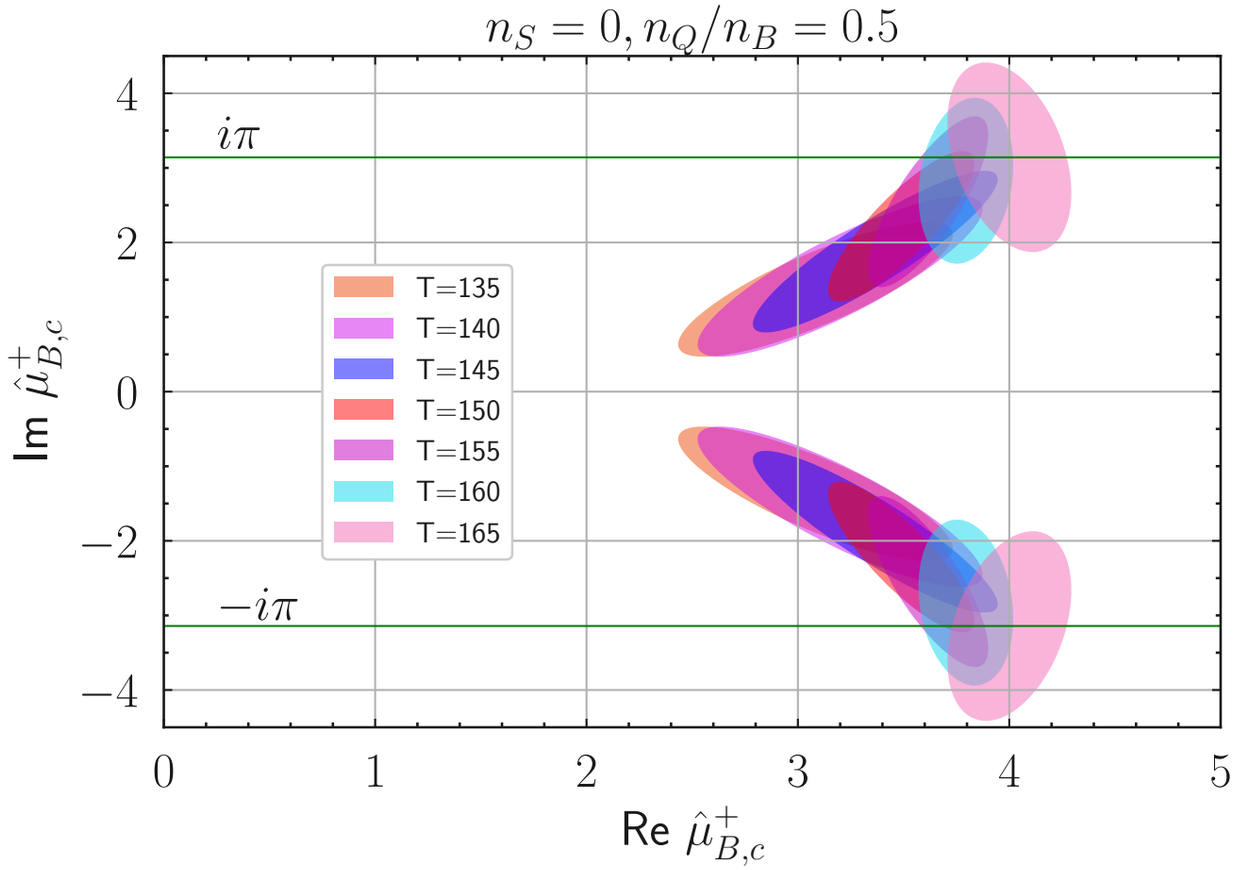


Values(including sign) of $c_{6,2}$ and $c_{8,2}$ which are related to χ_6^B and χ_8^B are crucial to have a pole in the real axis.

Location of the poles from [4,4] Padé approximants for QCD



Poles are complex for the temperature range,
$T \in [135 : 165]$ MeV.
The possibility of occurrence of a real pole cannot be ruled out for $T < 135$ MeV



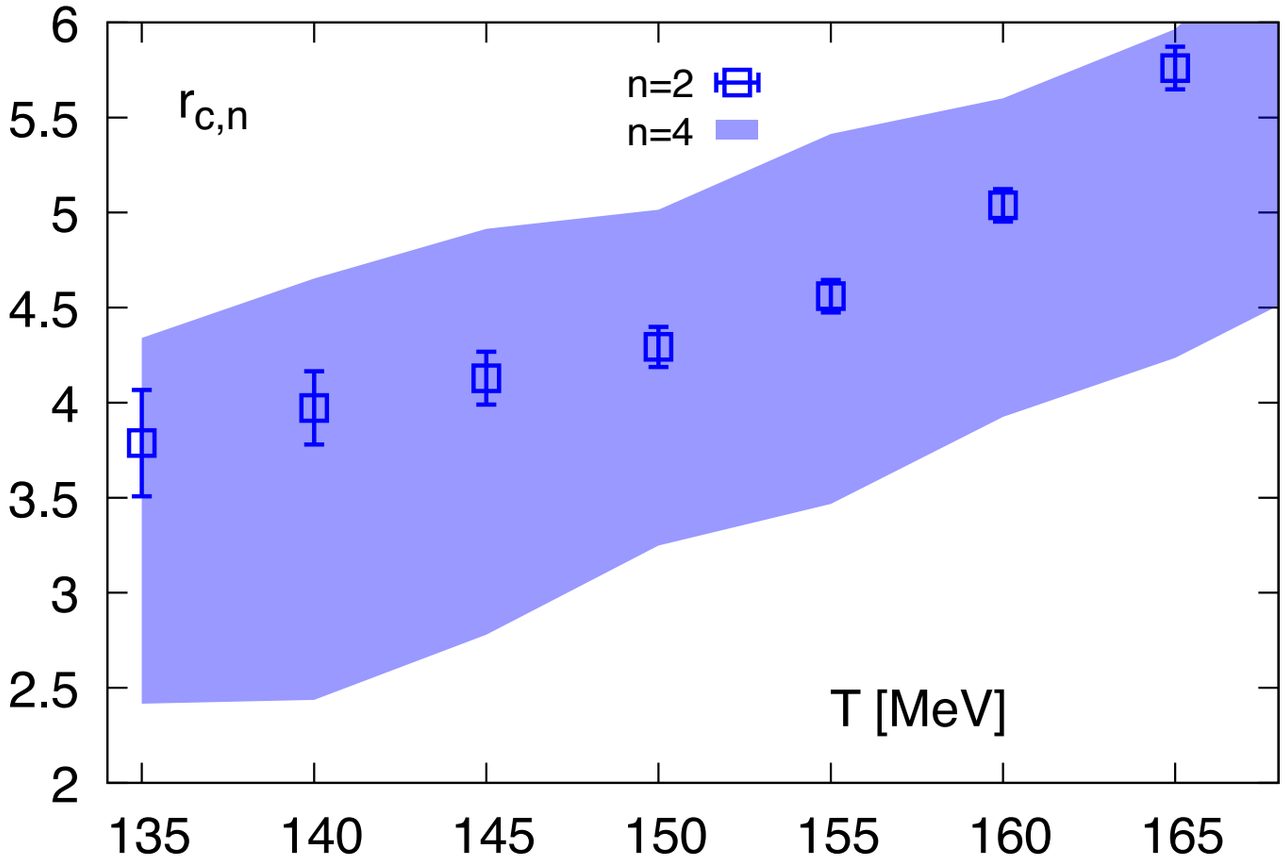
Only complex poles having a positive real part are shown.
The poles show a tendency to move to real axis for $T < 135$ MeV

Bound for CEP is,
$T^{CEP} < 135$ MeV.
Consistent with $T^{CEP} < T_c^{chiral}$ (~ 130 MeV) [arXiv:1903.04801]

Radius of convergence from diagonal Padé approximants

$$r_{c,2} = \sqrt{12\bar{\chi}_2^B/\bar{\chi}_4^B}$$

$$r_{c,4} = r_{c,2}\,|z^+z^-|^{1/4} = \sqrt{\frac{12\bar{\chi}_0^{B,2}}{\bar{\chi}_0^{B,4}}}\left|\frac{1-c_{6,2}}{c_{6,2}^2-c_{8,2}}\right|^{1/4}$$



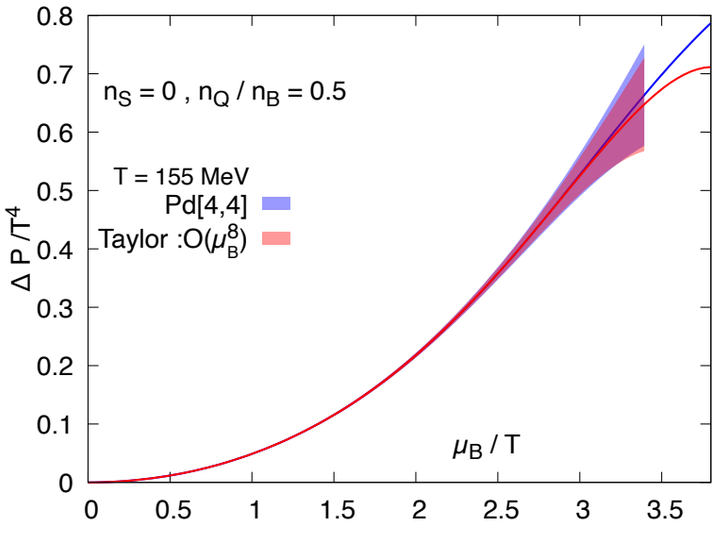
The radius of convergence of $[2,2]$ and $[4,4]$ Padé in the temperature range $T \in [135 : 165]$ MeV obtained as, $\left|\hat{\mu}_B^c\right| \sim [2.5 : 5.5]$.

This is also the current updated estimate for the radius of convergence (r_c) from a $\hat{\mu}_B^8$ Taylor series.

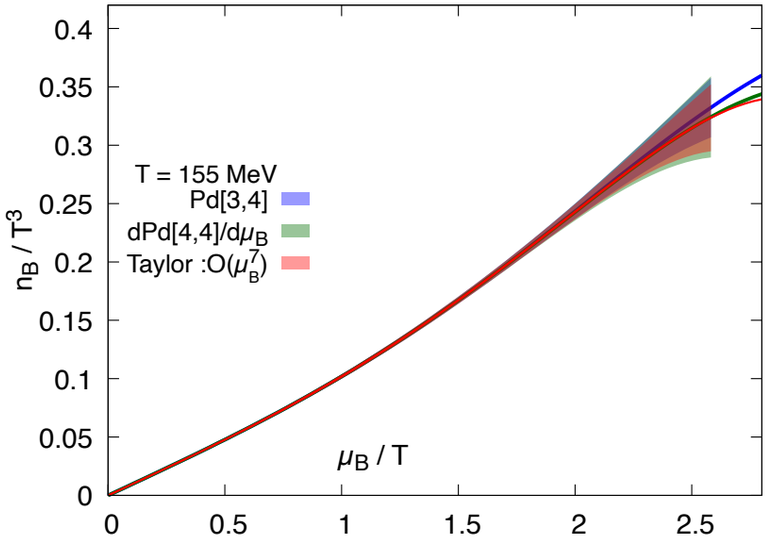
Bound for CEP is,

$$\hat{\mu}_B^{CEP} > 2.5.$$

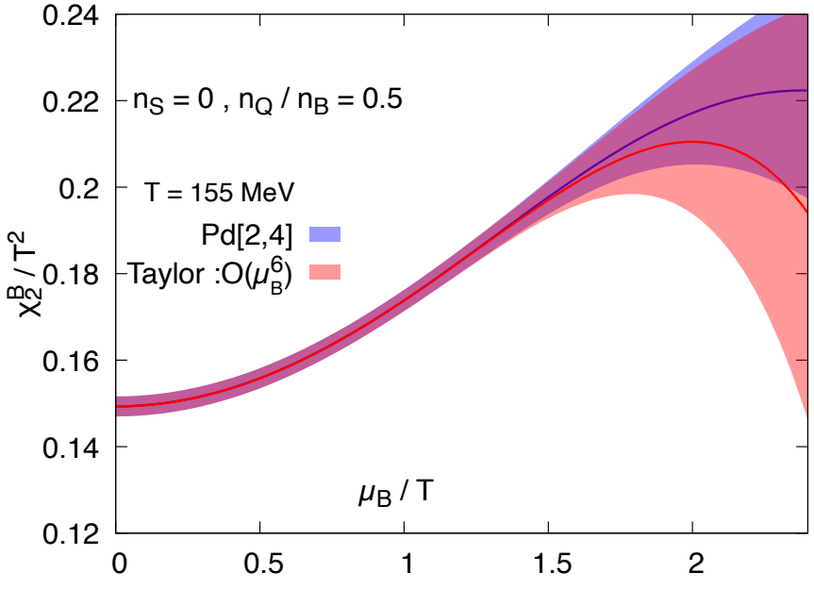
Radius of convergence and reliability of expansion



Padé approximant and Taylor series show good agreement for , $\hat{\mu}_B \leq 3$



Padé approximant and Taylor series show good agreement for , $\hat{\mu}_B \leq 2.5$

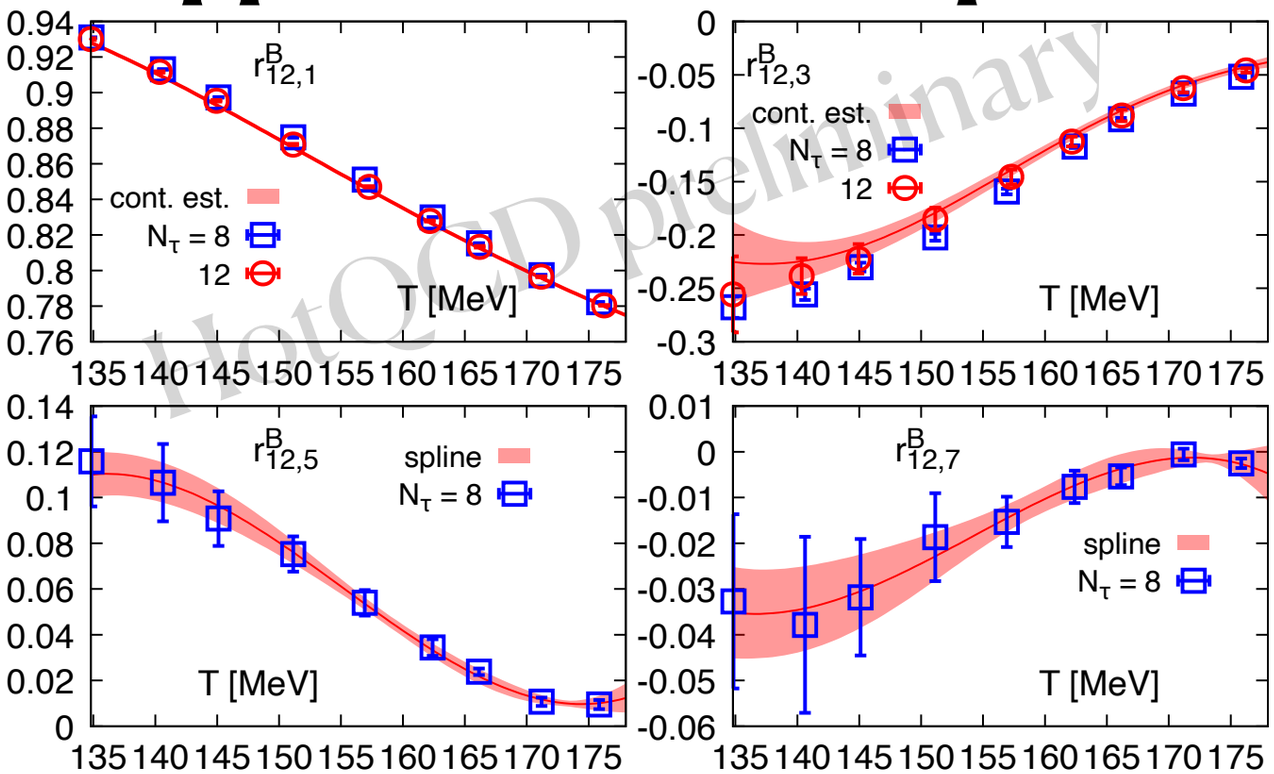


Padé approximant and Taylor series show good agreement for , $\hat{\mu}_B \leq 1.5$

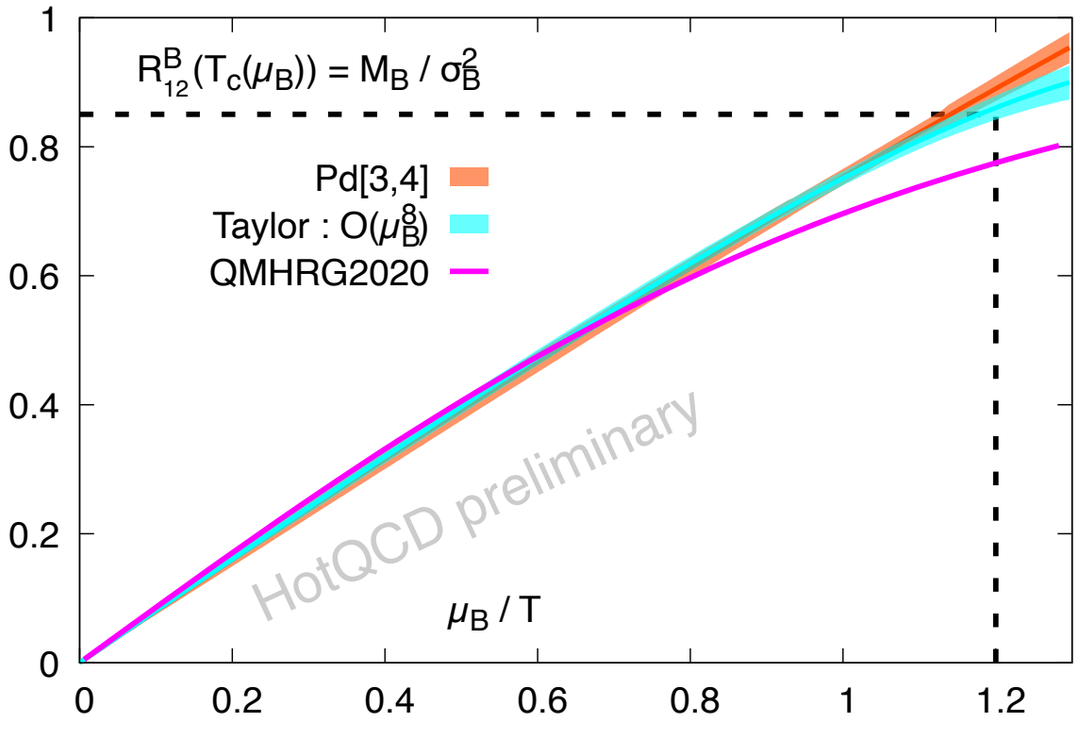
- ★ The current range of reliability of the expansions are different for different observables. Which is , $\hat{\mu}_B/T \sim 3,\ 2.5$ and 1.5 for pressure, net-baryon number density and second order baryon number fluctuations close to the pseudo-critical temperature.
- ★ All the observables have same radius of convergence. The current updated estimate is $\left|\hat{\mu}_B^c\right| \sim 3$, close to the pseudo-critical temperature.

Similar qualitative results can be obtained for $n_S = 0, n_Q/n_B = 0.4$, the condition that is met in the relativistic Heavy Ion collision experiments

Outlook : Padé approximant in the pseudo-critical line



$$R^B_{12} = r^B_{12,1}\hat{\mu}_B + r^B_{12,3}\hat{\mu}^3_B + r^B_{12,5}\hat{\mu}^5_B + r^B_{12,7}\hat{\mu}^7_B$$



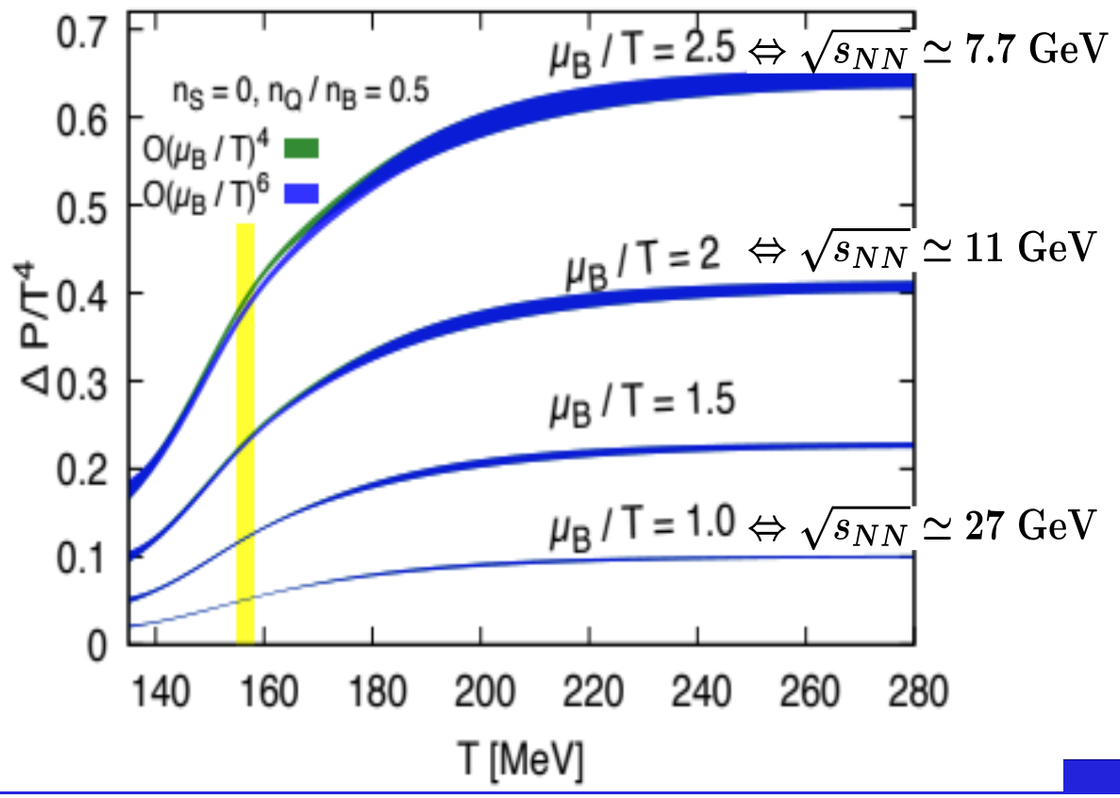
✦Close to CEP, σ^2_B will diverge while M_B stays finite.

That will imply that, $R^B_{12}(T_c(\mu_B)) \to 0$ close to the CEP.

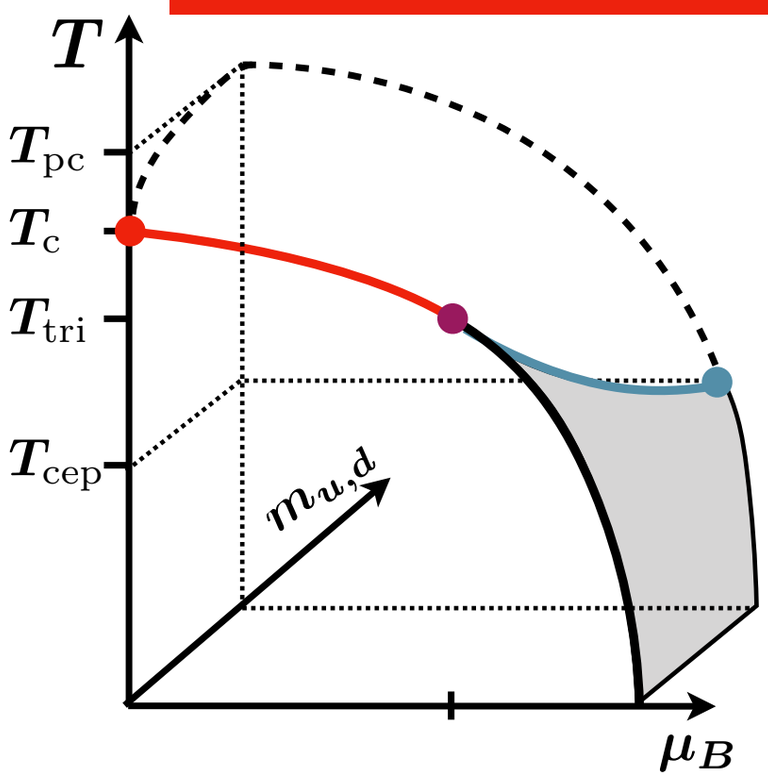
Hence, for $n_Q/n_B = 0.4$ and $n_S = 0$, R^B_{12} doesn't show any hint for a CEP, for $\hat{\mu}_B < 1.2$.

Conclusions

- Good convergence of Taylor series of pressure, number density and second order baryon number fluctuation for $\hat{\mu}_B/T \sim 3,\ 2.5$ and 1.5 respectively.
- We resum the Taylor series using Padé approximant method and determine the current estimate of the radius of convergence for the Taylor series as, $|\hat{\mu}_B^c| \sim 2.5$ close to our lowest temperature.
- The current bound for the QCD critical point is,



$$T_c^{CEP} \leq 135 \text{ MeV},\ \hat{\mu}_B^{\text{CEP}} > 2.5$$



T_c^{CEP} is consistent with , $T_c^{CEP} < T_c^{chiral}$

F. Karsch, arXiv:1905.03936

Thank you

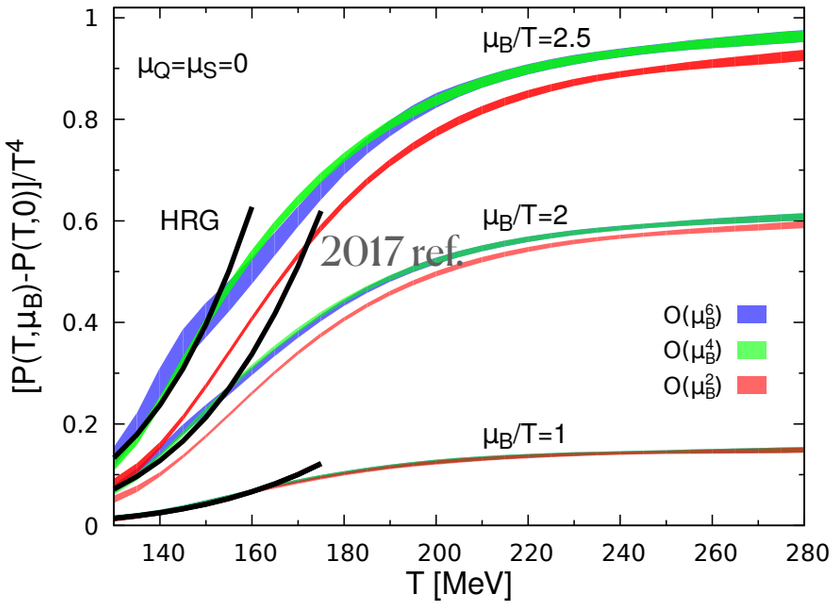
Back Up Slides

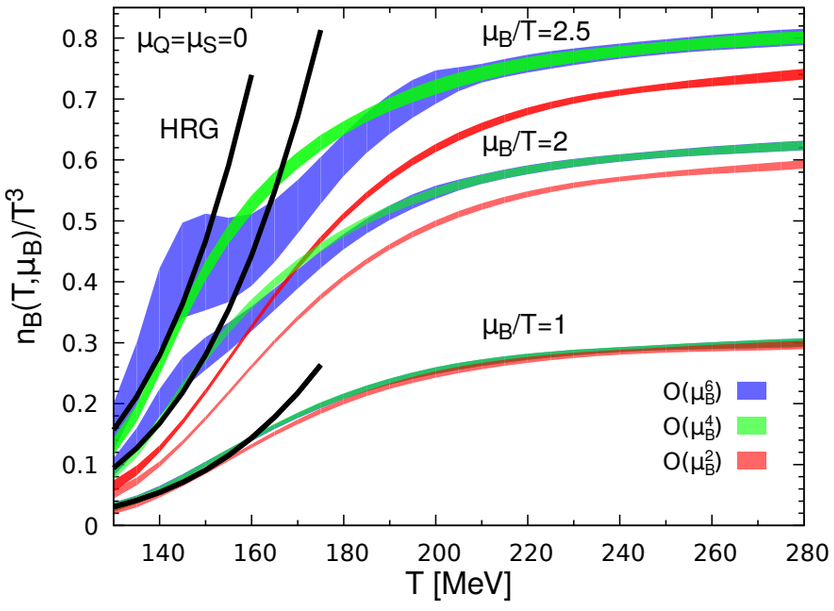
Taylor series of pressure and number density

$$\chi_0^B(T,\mu_B) \equiv \frac{P(T,\mu_B)-P(T,0)}{T^4} = \sum_{k=1}^{3} P_{2k}(T)\hat{\mu}_B^{2k}$$

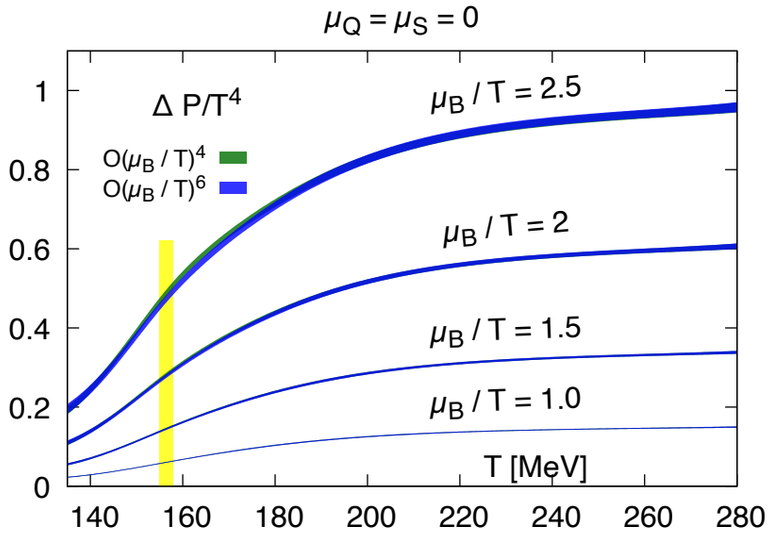
$$\chi_1^B(T,\hat{\mu}_B) \equiv \frac{n_B(T,\mu_B)}{T^3} = \sum_{k=1}^{3} N_{2k-1}^B(T)\hat{\mu}_B^{2k-1}$$

$\mu_B \neq 0,\ \mu_Q = \mu_S = 0$

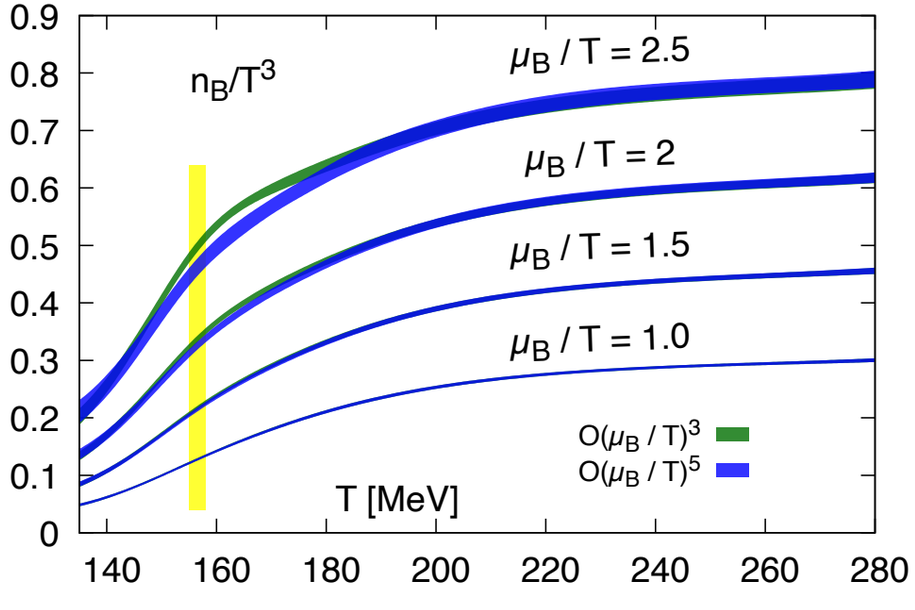




Increase of statistics: 20 times —> Sixth order is in under more control.



2022 ref.



Searching for CEP using Padé approximants

In practice we work with finite number of coefficients!!

All expansion coefficients are of same sign!!

$$f(x) = \sum_{i=0}^{n} c_i x^i$$

(Possibility) Singularity in the real axis!!

(Possible) existence of critical point !!

Taylor Series

Padé approximant

$$f(x) = \frac{\sum_{i=0}^{a} c_i x^i}{1 + \sum_{j=0}^{b} c_j x^j}$$

Complex singularity:

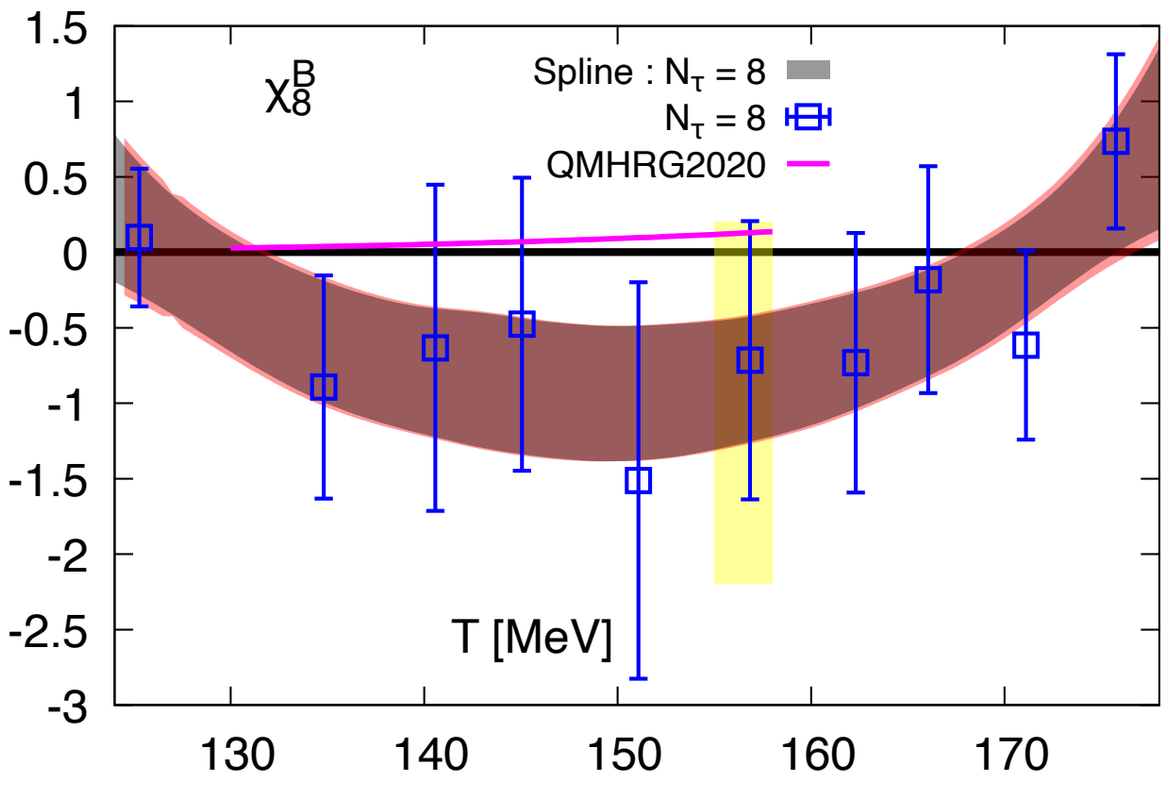
- Irrelevant for QCD thermodynamics ??
- Universal LYEs (SM, CS ref.) ??

Not related to any phase transition as function of real mu_B.

Taylor series including $\hat{\mu}_B^8$

$$\chi_0^B(T,\mu_B) \equiv \frac{P(T,\mu_B) - P(T,0)}{T^4} = \sum_{k=1}^{3} P_{2k}(T)\hat{\mu}_B^{2k} \; + \; P_8(T)\hat{\mu}_B^8$$

$$\chi_1^B(T,\hat{\mu}_B) \equiv \frac{n_B(T,\mu_B)}{T^3} = \sum_{k=1}^{3} N_{2k-1}^B(T)\hat{\mu}_B^{2k-1} + N_7^B(T)\hat{\mu}_B^7$$



Eight Order result is negative for
$T \in [135 : 165]$ MeV

$$P_8 = \frac{\chi_8^B}{8!}$$

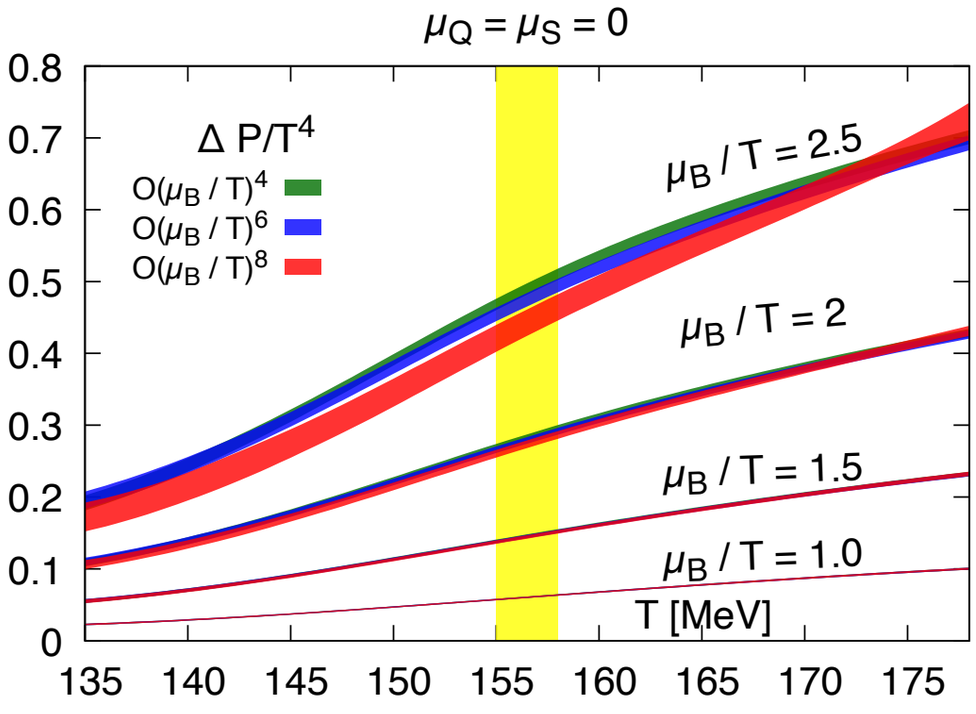
$$N_7^B = \frac{\chi_8^B}{7!}$$

Convergence of Taylor series ??

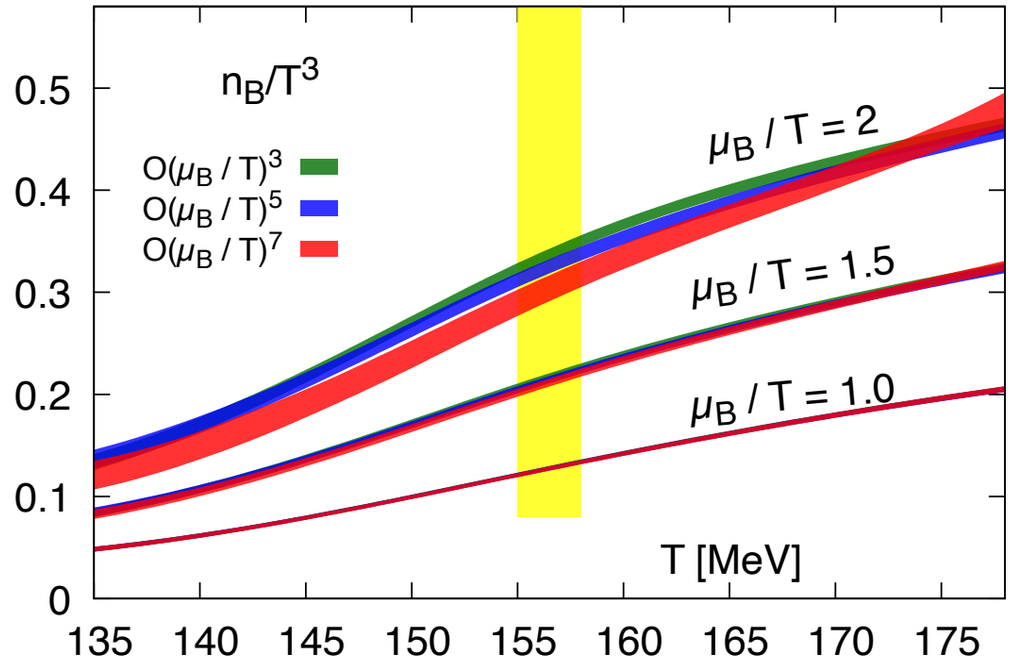
Taylor series including $\hat{\mu}_B^8$

$$\chi_0^B(T,\mu_B) \equiv \frac{P(T,\mu_B) - P(T,0)}{T^4} = \sum_{k=1}^{3} P_{2k}(T)\hat{\mu}_B^{2k} \; + \; P_8(T)\hat{\mu}_B^8$$

$$\chi_1^B(T,\hat{\mu}_B) \equiv \frac{n_B(T,\mu_B)}{T^3} = \sum_{k=1}^{3} N_{2k-1}^B(T)\hat{\mu}_B^{2k-1} + N_7^B(T)\hat{\mu}_B^7$$



Good convergence of Taylor series , $\hat{\mu}_B \leq 2.5$



Good convergence of Taylor series , $\hat{\mu}_B \leq 2.0$