Analytic continuation and the equation of state
Rescaling and expansion - the analysis
Results at $n_S = 0$ and $\mu_Q = 0$
Beyond strangeness neutrality

The equation of state form Lattice QCD with finite $\mu_B$ and $\mu_S$

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The sign problem

The QCD partition function:

$$Z(V, T, \mu) = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_F(U, \psi, \bar{\psi}) - \beta S_G(U)}$$

$$= \int \mathcal{D}U \text{det} M(U) e^{-\beta S_G(U)}$$

- For Monte Carlo simulations $\text{det} M(U) e^{-\beta S_G(U)}$ is interpreted as Boltzmann weight
- If there is particle-antiparticle-symmetry $\text{det} M(U)$ is real
- If $\mu^2 > 0 \text{det} M(U)$ is complex
Analytic continuation from imaginary chemical potential

Common technique:
- [deForcrand:2002hgr]
- [Bonati:2015bha]
- [Cea:2015cya]
- [DElia:2016jqh]
- [Bonati:2018nut]
- [Borsanyi:2018grb]
- [Borsanyi:2020fev]
- [Bellwied:2021nrt]
- ...
Trouble with the equation of state

[Borsanyi:2021sxv], [Borsanyi:2018grb], $N_t = 12$
Trouble with the equation of state

[Borsanyi:2021sxv], [Borsanyi:2018grb], $N_t = 12$

Taylor method

[Bazavov:2017dus]

[Bollweg:2022rps]
Trouble with the equation of state

[Borsanyi:2021sxv]

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Taylor method

[Bazavov:2017dus]

[Borsanyi:2021sxv]

[Bollweg:2022rps]
Results at $\mu_S = 0$

Find a different extrapolation scheme for extrapolating to higher $\mu_B$.

- [Borsanyi:2021sxv]
- $N_t = 10, 12, 16$
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Strangeness Neutrality

Enforcing the conditions $\mu_Q = 0$ and $\chi_1^S = 0$:

$$\frac{d\mu_S}{d\mu_B} = -\frac{\chi_{11}^{BS}}{\chi_2^S}.$$ 

On this line, total derivatives with respect to the baryochemical potential read

$$\frac{d}{d\hat{\mu}_B} = \frac{\partial}{\partial \hat{\mu}_B} + \frac{d\hat{\mu}_S}{d\hat{\mu}_B} \frac{\partial}{\partial \hat{\mu}_S} = \frac{\partial}{\partial \hat{\mu}_B} - \frac{\chi_{11}^{BS}}{\chi_2^S} \frac{\partial}{\partial \hat{\mu}_S}. $$

For the pressure we get:

$$c^n_B(T, \hat{\mu}_B) \equiv \left. \frac{dn(\hat{\mu}_B)}{d\hat{\mu}_B} \right\|_{\mu_Q=0}^{\chi_1^S=0}. $$

The net baryon density is given by:

$$c_1^B(T, \hat{\mu}_B) = \chi_1^B - \frac{\chi_{11}^{BS}}{\chi_2^S} \chi_1^S = \chi_1^B.$$
This rescaling will break down at large $T \rightarrow$ rescaling with SBL
This rescaling will break down at large \( T \rightarrow \) rescaling with SBL
Why does the rescaling work?

- It is an observation that it works
- It could be related to the critical scaling in the chiral limit
- If the universal contribution to EoS is large $\rightarrow$ single scaling variable
- If strength of transition is strongly influenced by light quark masses $\rightarrow$ curves keep their shape
- Fits with the observation of constant width of the transition

![Graph showing crossover and physical point]
Measuring the shift

\[ \frac{c_1^B(T, \hat{\mu}_B)}{c_1^B(\hat{\mu}_B)} \]

- **lattice**: \( 48^3 \times 12 \)
- **Spline**
- **measured difference**
- **data**

\[ \hat{\mu}_B = \frac{0\pi}{8} \]
\[ \hat{\mu}_B = \frac{4\pi}{8} \]
\[ \hat{\mu}_B = \frac{6\pi}{8} \]

- \( c_1^B \): net baryon density
- \( \overline{c_1^B} \): SBL of net baryon density

\[ \Pi(T, \hat{\mu}_B, N_T) = \frac{T'(T, \hat{\mu}_B, N) - T}{T \hat{\mu}_B} \]
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Measuring the shift

$c_B^1$: net baryon density

$\overline{c_B^1}$: SBL of net baryon density

$$\Pi(T, \hat{\mu}_B, N_\tau) = \frac{T'(T, \hat{\mu}_B, N) - T}{T \hat{\mu}_B}$$
Measuring the shift

\[ \frac{c_1^B(T, \hat{\mu}_B)}{c_1^B(\hat{\mu}_B)} \]

- **lattice**: $48^3 \times 12$

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Measuring the shift

\[ c_1^B : \text{net baryon density} \]

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Measuring the shift

- $\mu_B$: net baryon density
- $\overline{c_1 B}$: SBL of net baryon density

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Measuring the shift

$c_B^B$: net baryon density

$\overline{c_B^B}$: SBL of net baryon density

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\Pi(T, \hat{\mu}_B, N_\tau) = \frac{T'(T, \hat{\mu}_B, N) - T}{T \hat{\mu}_B}
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Measuring the shift

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Measuring the shift

$c_1^B$: net baryon density

$\overline{c_1^B}$: SBL of net baryon density

$$\Pi(T, \hat{\mu}_B, N_T) = \frac{T'(T, \hat{\mu}_B, N) - T}{T \hat{\mu}_B}$$
Lattice Setup

- Action: tree-level Symanzik improved gauge action, with four times stout smeared staggered fermions
- 2+1+1 flavour, on LCP with pion and kaon mass
- Simulation at $\langle n_S \rangle = 0$
- Continuum estimate from lattice sizes: $32^3 \times 8$, $40^3 \times 10$, $48^3 \times 12$ and $64^3 \times 16$
- $\mu_B T = i \frac{j \pi}{8}$ with $j = 0, 3, 4, 5, (5.5), 6$ and $6.5$
- Two methods of scale setting: $f_\pi$ and $w_0$, $L m_\pi > 4$
Systematic Errors

- 3 different sets of spline node points at \( \mu_B = 0 \)
- 2 different sets of spline node points at finite imaginary \( \mu_B \)
- \( \omega_0 \) or \( f_\pi \) based scale setting
- 2 different chemical potential ranges in the global fit: \( \hat{\mu}_B \leq 5.5 \) or \( \hat{\mu}_B \leq 6.5 \)
- 2 functions for the chemical potential dependence of the global fit: linear or parabola
- including the coarsest lattice, \( N_\tau = 8 \), or not, in the continuum extrapolation.

In total we perform 96 Fits. We weight every result with a \( Q > 0.01 \) uniformly.
The expansion coefficients

\[ \Pi(T, \hat{\mu}_B, N_T) = \frac{T'(T, \hat{\mu}_B, N) - T}{T \hat{\mu}_B} \]

\[ \Pi(T, \hat{\mu}_B, N_T) = \lambda_2^A + \lambda_4^A \hat{\mu}_B^2 + \lambda_6^A \hat{\mu}_B^4 \]

\[ + \frac{1}{N_T^2} (\alpha^A + \beta^A \hat{\mu}_B^2 + \gamma^A \hat{\mu}_B^4) \]

We make a fit to calculate derivatives and constrain it with the HRG.
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More strangeness

Two more observables:

\[ \hat{\mu}_B = i \frac{6\pi}{8} \]
\[ \hat{\mu}_B = i \frac{5\pi}{8} \]
\[ \hat{\mu}_B = i \frac{4\pi}{8} \]
\[ \hat{\mu}_B = i \frac{3\pi}{8} \]
\[ \hat{\mu}_B = 0 \]
More strangeness

Two more expansion:
Beyond strangeness neutrality

\[ \Delta \hat{\mu}_S \equiv \hat{\mu}_S - \hat{\mu}_S^* , \]

the dimensionless strangeness and baryon densities become:

\[ \chi_S^1(\hat{\mu}_S) \approx \chi_S^2(\hat{\mu}_S^*) \Delta \hat{\mu}_S \]

\[ \chi_B^1(\hat{\mu}_S) \approx \chi_B^1(\hat{\mu}_S^*) + \chi_{11}^{BS}(\hat{\mu}_S^*) \Delta \hat{\mu}_S, \]

where we only kept the linear leading order terms in \( \Delta \hat{\mu}_S \). We will express thermodynamic quantities in terms of the strangeness-to-baryon fraction:

\[ R = \frac{\chi_S^1}{\chi_B^1} = \frac{\chi_S^2(\hat{\mu}_S^*) \Delta \hat{\mu}_S}{\chi_B^1(\hat{\mu}_S^*) \Delta \hat{\mu}_S + \chi_{11}^{BS}(\hat{\mu}_S^*)}. \]

Inverting this equation we get:

\[ \Delta \hat{\mu}_S = \frac{R \chi_B^1(\hat{\mu}_S^*)}{\chi_S^2(\hat{\mu}_S^*) - R \chi_{11}^{BS}(\hat{\mu}_S^*)}. \]
Beyond strangeness neutrality

\[ \Delta \hat{\mu}_S = \frac{R \hat{\chi}_1^B(\hat{\mu}_S^*)}{\hat{\chi}_2^S(\hat{\mu}_S^*) - R \hat{\chi}_1^{BS}(\hat{\mu}_S^*)} \]

\[ R = \frac{\chi_S^1}{\chi_B^1} \]
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Strange Baryon density

Expanding the baryon density:

\[
\frac{\chi_B^1(T, \hat{\mu}_B, R)}{\chi_B^1(T, \hat{\mu}_B, R = 0)} \approx 1 + R \frac{\chi_{BS}^1(T, \hat{\mu}_B, R = 0)}{\chi_S^2(T, \hat{\mu}_B, R = 0)}
\]

where all quantities on the right hand side are along the strangeness neutral line.
At the strangeness neutral line the $O(R)$ correction of the pressure vanishes. The leading order correction gives:

$$\hat{p}(T, \hat{\mu}_B, R) \approx \hat{p}(T, \hat{\mu}_B, R) + \frac{1}{2} \frac{d^2\hat{p}}{dR^2}(T, \hat{\mu}_B) R^2,$$

where

$$\frac{d^2\hat{p}}{dR^2}(T, \hat{\mu}_B) = \frac{(\chi_1^B(T, \hat{\mu}_B))^2}{\chi_2^S(T, \hat{\mu}_B)}.$$
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$\mu_Q$

![Graph showing $\mu_Q$ vs. $T$ with data points and error bars]
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$k$ vs. $\lambda$

![Graph showing $\lambda^2(T)$ at $N_t=12$ and $\kappa^2(T)$ at $N_t=12$ vs. $T$ in MeV. The graph includes error bars.](image-url)
Thermodynamics

\[
\frac{p(T, \mu_B)}{T^4} = \frac{p(T, 0)}{T^4} + \int_0^{\mu_B} c_1^B(T, \mu_B') d\mu_B',
\]

with

\[
c_1^B(T, \mu_B) = c_2^B(T', 0) \frac{c_1^B(\mu_B)}{c_2^B(0)},
\]

and \(\frac{p(T, 0)}{T^4}\) from [Borsanyi:2013bia] The entropy density is defined as

\[
s = \left. \frac{\partial p}{\partial T} \right|_{\mu_B, \mu_S},
\]

which can be rewritten in terms of dimensionless quantities as:

\[
\hat{s} = 4\hat{\rho} + T \frac{\partial \hat{\rho}}{\partial T} \bigg|_{\mu_B} = 4\hat{\rho} + T \frac{\partial \hat{\rho}}{\partial T} \bigg|_{\mu_B} - \hat{\mu}_B \chi_1^B,
\]

where \(\hat{s} \equiv \frac{s}{T^3}\) and we took into account the difference between derivatives at fixed \(\mu_B\) versus at fixed \(\mu_B\).
By noticing that on the strangeness neutral line

\[
\frac{d\hat{p}(T, \hat{\mu}_B, \hat{\mu}_S(T, \hat{\mu}_B))}{dT} = \chi_1 \frac{\partial \hat{\mu}_S}{\partial T} + \frac{\partial \hat{p}}{\partial T} = \frac{\partial \hat{p}(T, \hat{\mu}_B, \hat{\mu}_S(T, \hat{\mu}_B))}{\partial T},
\]

we can write the logarithmic temperature derivative of the pressure as:

\[
T \left. \frac{\partial \hat{p}(T, \hat{\mu}_B)}{\partial T} \right|_{\hat{\mu}} = T \left. \frac{\partial \hat{p}(T, 0)}{\partial T} \right|_{\hat{\mu}}
\]

\[
+ \frac{1}{2} \int_0^{\hat{\mu}_B^2} T \left. \frac{dc_2^B(T', 0)}{dT'} \right|_{T'} \times \left[ 1 + \lambda_2^{BB} y + \lambda_4^{BB} y^2 + T \left( \frac{d\lambda_2^{BB}}{dT} y + \frac{d\lambda_4^{BB}}{dT} y^2 \right) \right] dy
\]

where \( \frac{dc_2^B(T)}{dT} \) is calculated at \( \mu_B = 0 \) and \( T' = T (1 + \lambda_2^{BB} y + \lambda_4^{BB} y^2) \)

Given the pressure and the entropy, the dimensionless energy density is given by:

\[
\hat{\epsilon} = \hat{s} - \hat{p} + \hat{\mu}_B \chi_1^B,
\]

where \( \hat{\epsilon} = \frac{\epsilon}{T^4} \).