Far-from-equilibrium attractor in non-conformal plasmas

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29th International Conference on Ultra–Relativistic Nucleus-Nucleus Collisions

April 7, 2022

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Motivation

- What is the domain of applicability of hydrodynamics?

- Textbooks: Close to local equilibrium, $\lambda_{\text{mfp}} \ll L$ or $|\nabla^{\mu}u^\nu|/T \ll 1$. Hence, hydrodynamics is formulated as an expansion in velocity gradients. 

- The issue: Navier-Stokes equations, i.e., 1st—order equations in Landau or Eckart frame, imposes instantaneous response of dissipative fluxes to dissipative forces $\Rightarrow$ acausality. 
  Hiscock and Lindblom (1983, 1985)

- Two possible resolutions:
  
  
  2) 1st—order theory with a different choice of hydrodynamic frame.
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• Two possible resolutions:

     2) 1st—order theory with a different choice of hydrodynamic frame.  
Modern hydrodynamic theories

- Promote dissipative fluxes to independent degrees of freedom.
  

\[ \tau_\pi \dot{\pi}^{\mu\nu} + \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu} + \cdots \]

so that it relaxes to Navier-Stokes over a time scale \( \sim \tau_\pi \).

- This introduces ‘non-hydrodynamic’ modes \((\omega \neq 0 \text{ as } k \to 0)\).

- This approach has led to development of variety of theories:

  Denicol-Niemi-Molnar-Rischke  
  Denicol et.al., 1202.4551 (2012)

  Chapman-Enskog approach  
  Jaiswal, 1305.3480 (2013)

  anisotropic hydro  
  Romatschke and Strickland; Martinez and Strickland; Florkowski and Ryblewski; McNelis, Bazow and Heinz; and others..

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- Are ‘modern’ formulations of hydrodynamics applicable over a broader domain than traditional hydrodynamics?
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Attractor in ‘Modern’ hydrodynamics

**Bjorken flow** J. D. Bjorken, PRD 27, 140 (1983)

- Boost-invariant longitudinal expansion, \( v^z = \frac{z}{t}, v^x = v^y = 0 \).

- Switch to Milne coordinate system \((\tau, x_\perp, \phi, \eta_s)\).
  - Fluid appears static, \( u^\mu = (1, 0, 0, 0) \), However, finite expansion rate, \( \partial_\mu u^\mu = 1/\tau \).
  - Shear tensor is diagonal, \( \pi^{\mu\nu} = \text{diag}(0, \pi/2, \pi/2, -\pi/\tau^2) \).

**Hydrodynamic attractor** Heller and Spalinski, 1503.07514 (2015)

- Consider Muller-Israel-Stewart hydrodynamics.
- \( f \equiv \frac{2}{3} + \frac{\pi}{4e}; \quad w \equiv \tau T \propto \tau/\tau_\pi \)
- Navier-Stokes: \( f_{\text{NS}} = \frac{2}{3} + \frac{4C_\eta}{9w} \)
- Second-order: \( f_2 = f_{\text{NS}} + \frac{8C_\tau C_\eta}{27w^2} \)
- Attractor: Stable FP: \( f_0 = \frac{2}{3} + \frac{\sqrt{C_\eta}}{3\sqrt{C_\tau}} \) at \( w \approx 0 \)
Early and late time attractors

- Late-time or near equilibrium attractor: Navier Stokes limit.
  - Expected. Not interesting.

- Early-time attractor: Universal far-from-equilibrium.

Kurkela, van der Schee, Wiedemann, and Wu 1907.08101 (2019)

Power law decay of different initializations on attractor in $\tau/\tau_R \ll 1$ regime.

- Question: How general is this emergent ‘early-time attractor’?
Early and late time attractors

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- Question: How general is this emergent ‘early-time attractor’?
Consider Boltzmann equation in relaxation-time approximation (RTA):

\[ p^\mu \partial_\mu f = -\frac{(u \cdot p)}{\tau_R} (f - f_{\text{eq}}) \]

Exact solution for Bjorken:

\[ f(\tau; p_T, w) = D(\tau, \tau_0) f_{\text{in}}(\tau_0; p_T, w) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_R(\tau')} D(\tau, \tau') f_{\text{eq}}(\tau'; p_T, w) \]

Relaxation time: \( \tau_R(\tau) = \frac{5C}{T(\tau)} \)

Damping function: \( D(\tau_2, \tau_1) = \exp\left( -\int_{\tau_1}^{\tau_2} \frac{d\tau'}{\tau_R(\tau')} \right) \)

Initial distribution:

\[ f_{\text{in}} \equiv \frac{1}{\alpha_0} \exp\left( -\frac{\sqrt{m^2 + p_T^2} + (1 + \xi_0)(w/\tau_0)^2}{\Lambda_0} \right) \]

Three parameters: \( \alpha_0, \Lambda_0, \xi_0 \). Can be tuned for initial \( \epsilon, P_L, P_T \).

Allows to create a system with large initial bulk and/or shear stress.
Kinetic bounds on viscous stresses

- Since \( f(x, p) \geq 0 \implies \)
  
  (i) \( \epsilon = \langle p_0^2 \rangle \geq 0 \)
  
  (ii) \( P_L = \langle w^2 \rangle / \tau^2 \geq 0 \)
  
  (iii) \( P_T = \frac{1}{2} \langle p_T^2 \rangle \geq 0 \)
  
  (iv) \( P + \Pi = \frac{1}{3} \langle p_T^2 + (w/\tau)^2 \rangle \geq 0 \)
  
  (v) \( T_\mu^\mu = m^2 \langle (1) \rangle = \epsilon - 3(P + \Pi) \geq 0 \)

Here, \( \langle (\ldots) \rangle \equiv \int d^2p_T dw / [(2\pi)^3 \tau p_T] (\ldots) f \)

\[ w \equiv p_\eta = tp^z - zE_p \]

- Therefore, bound on shear and bulk stress: \( \begin{cases} \Pi = \Pi / P, & \pi = \pi / P \end{cases} \)

\[ \Pi + \frac{1}{2} \pi \geq -1, \quad \Pi - \pi \geq -1, \quad \Pi \geq -1, \quad \Pi \leq \frac{\epsilon}{3P} - 1. \]

Chattopadhyay, SJ, Du, Heinz, Pal, 2107.05500 (2021)

For conformal gas of massless particles (\( \Pi = 0 \)), allowed region shrinks to \( \Pi / P = 0 \);
discontinues jump in allowed region for arbitrary small mass.
Collisionless regime: Free streaming

- Initial conditions:
  \( m = 50 \text{ MeV} \). \( T_0 = 500 \text{ MeV} \) at \( \tau_0 = 0.1 \text{ fm/c} \).

- Free streaming evolution:
  \( f_{fs}(\tau; p_T, p^z) = f_{in}(p_T, p^z(\tau/\tau_0)) \)
  become sharply peaked in \( p_z \) as time evolves.

- Attractive fixed line: \( P_L = 0 \)
  All curves approach this line of vanishing longitudinal pressure.

- Repulsive fixed point: \( \{P_T/P = 0, \Pi/P = 0\} \)
  Backward evolution of all curves approach this point.

- Saddle point: \( \{P_T/P = 0, P_L/P = 0\} \)
  Spherically symmetric distribution: \( f \propto \delta^{(3)}(p) \propto \delta(|p|)/|p|^2 \).
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With interactions: Large Bulk, small shear

\[ m = 50 \text{ MeV}. \quad T_0 = 500 \text{ MeV at } \tau_0 = 0.1 \text{ fm}/c \]

- Absence of early-time attractor in KT in both bulk and shear inverse Reynolds number.
  - Convergence of trajectories seen only at \((\tau/\tau_R) \gtrsim 3\) with the Navier-Stokes attractor.
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\[ \text{Re}_{\Pi}^{-1} \equiv \frac{\Pi}{\epsilon + P} \]

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\[ \tau_R = \frac{5C}{T} \]

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- Above statement holds for 2\textsuperscript{nd} order non-conformal hydrodynamics.
  Denicol, Jeon and Gale, 1403.0962 (2014), Jaiswal, Ryblewski and Strickland, 1407.7231 (2014)

- Can one recover the early-time attractor in shear stress as seen in conformal system for small bulk viscous pressure?
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- In shear channel: initial power law decay for non-conformal KT is lost quickly when compared to conformal KT evolution.
  - Conformal KT converges \(\approx (\tau/\tau_R) \gtrsim 0.5\); No ETA in non-conformal KT in shear.
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- ETA also absent in non-conformal hydrodynamics in both channels.
Is there an early-time attractor?

Yes!

- $\overline{P_L} \equiv \frac{P_L}{P}$, $\tau_R = 5C/T$.
- Attractor reappears in $P_L/P$!
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- $\overline{P}_L \equiv PL/P$, $\tau_R = 5C/T$.
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- Early time attractor is absent in non-conformal hydrodynamics.

- Reason for ETA: Fast expansion + weak coupling $\implies$ Approximate free-streaming at early times $\implies P_L = 0$ line acts as an attractor.
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Evolution of the distribution function

\[ \frac{\tau}{\tau_R} = \left( \frac{\tau}{\tau_R} \right)_0 \]

\[ \frac{\tau}{\tau_R} = 0.5 \]

\[ \frac{\tau}{\tau_R} = 1 \]

\[ \frac{\tau}{\tau_R} = 5 \]
Some remarks..

Early-time attractor: feature of free-streaming, not interactions.

- Possible confusion: ETA exists for normalized shear for conformal systems. Governed by interactions?
  - For conformal systems undergoing Bjorken flow, $P_L$ and $\pi$ are equivalent quantities: $P_L/P = 1 - \pi/P$. Share a common attractor.
  - Not the case for non-conformal system: $P_L/P = 1 + (\Pi - \pi)/P$.

- Is there a relation between the divergence of hydrodynamic gradient series and far-from-equilibrium hydrodynamic attractor?
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**Modified aHydro**

- Hydrodynamics derived from KT: expansion of the distribution function around the thermal distribution.
  - Does not capture well the effects of strong longitudinal expansion.

- **anisotropic hydrodynamics** captures the free-streaming effects.
  
  Romatschke and Strickland, hep-ph/0304092 (2003); Martinez and Strickland, 1007.0889 (2010); Florkowski and Ryblewski; McNelis, Bazow and Heinz; and others.

\[
f = \frac{1}{\alpha(\tau)} \exp \left( - \frac{\sqrt{m^2 + p_T^2} + (1 + \xi(\tau))(w/\tau)^2}{\Lambda(\tau)} \right)
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SJ, Chattopadhyay, Du, Heinz, Pal, 2107.10248 (2021)
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SJ, Chattopadhyay, Du, Heinz, Pal, 2107.10248 (2021)
Consider a system with massive particle and vanishing chemical potential undergoing Bjorken expansion.

- **Goal**: Study transition from free-streaming to hydrodynamic regime.

\[ \mathcal{L}_n \equiv \int_p p^2_0 P_{2n}(p_z/p_0) f(\tau, p), \quad \mathcal{M}_n \equiv m^2 \int_p P_{2n}(p_z/p_0) f(\tau, p) \]

where \( \int_p \equiv \frac{d^3 p}{(2\pi)^3 p_0} \) and \( P_{2n} \) is the Legendre polynomial of order \( 2n \).

- **Coupled equations for** \( \mathcal{L}_n \) **and** \( \mathcal{M}_n \)-**moments**:

\[
\begin{align*}
\frac{\partial \mathcal{L}_n}{\partial \tau} &= -\frac{1}{\tau} \left( a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1} \right) - (1 - \delta_{n,0}) \frac{\left( \mathcal{L}_n - \mathcal{L}_n^{eq} \right)}{\tau_R} \\
\frac{\partial \mathcal{M}_n}{\partial \tau} &= -\frac{1}{\tau} \left( a'_n \mathcal{M}_n + b'_n \mathcal{M}_{n-1} + c'_n \mathcal{M}_{n+1} \right) - \frac{\left( \mathcal{M}_n - \mathcal{M}_n^{eq} \right)}{\tau_R}
\end{align*}
\]

where the coefficients \( a_n, b_n, c_n, a'_n, b'_n, c'_n \) are pure numbers.

- **Only three moments are hydro quantities**: \( (\mathcal{L}_0 = \epsilon, \mathcal{L}_1, \mathcal{M}_0 = T^\mu_\mu) \)

\[ \epsilon = \mathcal{L}_0, \quad P_L = \frac{1}{3} (\mathcal{L}_0 + 2\mathcal{L}_1), \quad P_T = \frac{1}{3} \left( \mathcal{L}_0 - \mathcal{L}_1 - \frac{3}{2} \mathcal{M}_0 \right). \]
Conclusions

• Absence of early-time attractor in the normalized shear and bulk stress channel (in both non-conformal kinetic theory and ‘modern’ hydrodynamics).

• In KT, normalized longitudinal pressure $P_L/P$ continues to exhibit universal early-time attractive behavior driven by rapid longitudinal expansion.

• Rapid free-streaming expansion (and not interactions) is the reason for early-time attractor. Should be tested in other expansion geometries.

• 2nd order non-conformal hydrodynamics fails to capture this early-time attractor, but modified aHydro reproduces it very well.

Thank You!
Backup Slides
Second-order non-conformal hydrodynamics

- Hydrodynamic equations for gas undergoing Bjorken expansion:
  
  \[
  \frac{d\epsilon}{d\tau} = -\frac{1}{\tau} (\epsilon + P + \Pi - \pi) \quad \left| \quad \frac{d\Pi}{d\tau} + \frac{\Pi}{\tau_R} = -\frac{\beta\Pi}{\tau} - \delta_{\Pi\Pi} \frac{\Pi}{\tau} + \lambda_{\Pi\pi} \frac{\pi}{\tau}
  \right.
  
  \[
  \frac{d\pi}{d\tau} + \frac{\pi}{\tau_R} = \frac{4}{3} \frac{\beta\pi}{\tau} - \left( \frac{1}{3} \tau_{\pi\pi} + \delta_{\pi\pi} \right) \frac{\pi}{\tau} + \frac{2}{3} \lambda_{\pi\Pi} \frac{\Pi}{\tau}
  \]

  Free streaming fixed points

  - Set \( \tau_R \to \infty \), evaluate transport coefficients in \( m/T \to 0 \) limit:
    
    \[
    \{ P_L/P, P_T/P \} = \{-0.212, 1.61\}; \{3.64, -0.32\}; \{-1, -3.33\}.
    \]
    
    \[
    \{ \Pi/P, \pi/P \} = \{0, 1.214\}; \{0, -2.64\}; \{-3.56, -1.56\}.
    \]

  - **Blue** and **Red** fixed points are precisely the **stable** and **unstable** fixed points of second-order conformal hydrodynamics.

  - **Green** fixed point—absent in conformal hydrodynamics: a feature of non-conformal hydrodynamics.
Collisionless regime: comparing KT and hydro

Free-streaming evolution of KT and 2nd-order hydrodynamics:
- Pushing hydro in free streaming, $\tau_R \to \infty$

- All free-streaming KT trajectories evolves upwards after reaching $P_L/P = 0$ line. Not the same for hydro.
  - Reason: incorrect description of the saddle point; acts like attractive fixed point (according to hydro equations) for perturbations near this FP.
Hydrodynamics derived from KT: expansion of the distribution function around the thermal distribution.

- Does not capture the rapid shrinking at early times of the $p_z$ distribution caused by the rapid longitudinal Bjorken expansion.

**anisotropic hydrodynamics**: Standard derivation assumes the form for distribution function (for Bjorken flow):

$$f(\tau; p_T, w) \approx f_a \equiv \exp \left( -\frac{\sqrt{m^2 + p_T^2 / \alpha_T^2(\tau)} + (w/\tau)^2 / \alpha_L^2(\tau)}{\Lambda(\tau)} \right)$$

- However, unable to span the entire allowed region: fails to generate large bulk.

Modified ansatz for the anisotropic distribution:

$$f \approx \tilde{f}_a = \frac{1}{\alpha(\tau)} \exp \left( -\frac{\sqrt{m^2 + p_T^2 + (1 + \xi(\tau))(w/\tau)^2}}{\Lambda(\tau)} \right)$$

Spans entire solution space allowed by KT.
Excellent agreement with KT. Reproduces the far-from-equilibrium attractor in $P_L/P$. 