Causal and stable third-order hydrodynamics

A derivation from the Boltzmann equation

(work in progress)

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Motivation: Third-order hydrodynamics

- Better agreement with solutions of the BE (large viscosity, Bjorken flow)

[A. Jaiswal, PRC **87** 051901 (2013)]

- Formal derivation: Chapman-Enskog + Relaxation time approximation

  Neglecting bulk and diffusion, shear-stress only

[A. Jaiswal, PRC **88** 021903 (2013)]

- What about linear causality and stability? [CVB & G. S. Denicol, arXiv:2107.10319 [nucl.th]]
  - Acausal and unstable; Navier-Stokes-like behavior

- Is it possible to fix these problems *ad hoc*?
  - Propose a modification to the original theory:

$$\nabla^{\langle\alpha}\pi^{\mu\nu\rangle} \longrightarrow \rho^{\alpha\mu\nu}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\langle\mu\nu\rangle} = 2\eta\sigma^{\mu\nu} - \tau_\pi \nabla_\alpha \rho^{\alpha\mu\nu} + \cdots$$

$$\tau_\rho \dot{\rho}^{\langle\mu\nu\lambda\rangle} + \rho^{\mu\nu\lambda} = \frac{3}{7}\eta_\rho \nabla^{\langle\mu}\pi^{\nu\lambda\rangle} + \text{nonlinear terms}$$

- Causality-stability conditions:

  (Causality and stability now depend on the transport coefficients)

$$\left[3\tau_\pi\left(1-c_{\mathrm{s}}^2\right) - 4\frac{\eta}{\varepsilon_0+P_0}\right]\tau_\rho > \frac{27}{35}\eta_\rho\tau_\pi\left(1-c_{\mathrm{s}}^2\right),$$

$$3(1-\mathrm{c}_{\mathrm{s}}^2)\tau_\pi \geq \frac{4\eta}{\varepsilon_0+P_0}.$$

[S. Pu et al., PRD 81 114039 (2010)]

- Next step: formally derive the complete nonlinear third-order theory

Method of moments

Relativistic Boltzmann equation: $k^{\mu}\partial_{\mu}\underbrace{f_{\mathbf{k}}}_{\text{distribution function}} = \overbrace{C[f]}^{\text{collision term}}$

Factorize the distribution function: $f_{\mathbf{k}} = \underbrace{f_{0\mathbf{k}}}_{\text{equilibrium}} + \overbrace{\delta f_{\mathbf{k}}}^{\text{nonequilibrium}} = f_{0\mathbf{k}}(1 + \tilde{f}_{0\mathbf{k}}\phi_{\mathbf{k}})$, with $\overset{\text{near equilibrium (fluid-dynamical regime)}}{\delta f_{\mathbf{k}} \ll f_{0\mathbf{k}}}$

Expand $\phi_{\mathbf{k}}$:

- Israel and Stewart: $1, k_{\mu}, k_{\mu}k_{\nu}, \ldots$ [W. Israel & J. M. Stewart, Phys. Lett. **58A**, 213 (1976)]
- Denicol et al.: $1, k_{\langle\mu\rangle}, k_{\langle\mu}k_{\nu\rangle}, \ldots$ [G. S. Denicol et al., PRD **85** 114047 (2012)]

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} k_{\langle\mu_1\cdots\mu_\ell\rangle},$$

$$\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} = \sum_{n=0}^{N_\ell} \Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} P_{\mathbf{k}n}^{(\ell)},$$

$$\Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{\mathcal{N}^{(\ell)}}{\ell!} \sum_{r=0}^{n} \rho_n^{\mu_1\cdots\mu_\ell} a_{nr}^{\ell}$$

Irreducible moments of the nonequilibrium distribution function

$$\rho_r^{\mu_1\cdots\mu_\ell} \equiv \left\langle E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \right\rangle_\delta$$

$$\text{where} \begin{cases} \langle\cdots\rangle_0 \equiv \int dK (\cdots) f_{0\mathbf{k}}, \\ \langle\cdots\rangle_\delta \equiv \int dK (\cdots) \delta f_{\mathbf{k}}. \end{cases}$$

Method of moments

S. R. De Groot, W. A. van Leeuwen and C. G. van Weert
“Relativistic Kinetic Theory: Principles and Applications” (1980)

Net-charge current

$$N^{\mu} = \langle E_{\mathbf{k}} \rangle u^{\mu} + \left\langle k^{\langle \mu \rangle} \right\rangle$$

Energy-momentum tensor

$$T^{\mu\nu} = \langle E_{\mathbf{k}}^2 \rangle u^{\mu} u^{\nu} + \frac{1}{3} \Delta^{\mu\nu} \langle b_{\mathbf{k}} \rangle + \left\langle E_{\mathbf{k}} k^{\langle \mu \rangle} \right\rangle u^{\nu} + \left\langle E_{\mathbf{k}} k^{\langle \nu \rangle} \right\rangle u^{\mu} + \left\langle k^{\langle \mu} k^{\nu \rangle} \right\rangle$$

Connection between the irreducible moments and hydro currents

$$\Pi = -\frac{1}{3} m^2 \rho_0, \; V^{\mu} = \rho_0^{\mu}, \; \xcancel{W^{\mu} =} \rho_1^{\mu}, \; \pi^{\mu\nu} = \rho_0^{\mu\nu}.$$

Landau matching conditions

Time-evolution of the moments

$$\dot{\rho}_r^{\langle \mu_1 \cdots \mu_\ell \rangle} = \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} \frac{d}{d\tau} \int dK \, E_{\mathbf{k}}^r k^{\langle \nu_1} \cdots k^{\nu_\ell \rangle} \delta f_{\mathbf{k}}$$

Ambiguity: infinite number of equations for the irreducible moments;
Without loss of generality, we take **r=0**

Equations of motion for the **irreducible moments** → Equations of motion for the **dissipative currents**

Truncation scheme

How should we truncate the expansion of $\phi_{\mathbf{k}}$?

Israel and Stewart: $$\phi_{\mathbf{k}}^{\mathrm{IS}} = \lambda_{\mathbf{k}} + \lambda_{\mathbf{k}}^{\mu} k_{\mu} + \lambda_{\mathbf{k}}^{\mu\nu} k_{\mu} k_{\nu} + \mathcal{O}(k^3)$$

Denicol et al.: $$\phi_{\mathbf{k}}^{14\mathrm{M}} = \lambda_{\mathbf{k}} + \lambda_{\mathbf{k}}^{\langle\mu\rangle} k_{\langle\mu\rangle} + \lambda_{\mathbf{k}}^{\langle\mu\nu\rangle} k_{\langle\mu} k_{\nu\rangle} + \mathcal{O}(k^3)$$

In both approaches, the expansion was truncated in second order in momenta

30-moments: $$\phi_{\mathbf{k}}^{30\mathrm{M}} = \lambda_{\mathbf{k}} + \lambda_{\mathbf{k}}^{\langle\mu\rangle} k_{\langle\mu\rangle} + \lambda_{\mathbf{k}}^{\langle\mu\nu\rangle} k_{\langle\mu} k_{\nu\rangle} + \lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\rangle} k_{\langle\mu} k_{\nu} k_{\alpha\rangle} + \lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\beta\rangle} k_{\langle\mu} k_{\nu} k_{\alpha} k_{\beta\rangle} + \mathcal{O}(k^5)$$

Minimal truncation

$$\begin{aligned}
\lambda_{\mathbf{k}} &= \Phi_0 + P_{\mathbf{k}1}^{(0)} \Phi_1 + P_{\mathbf{k}2}^{(0)} \Phi_2, \\
\lambda_{\mathbf{k}}^{\langle\mu\rangle} &= \Phi_0^{\langle\mu\rangle} + P_{\mathbf{k}1}^{(1)} \Phi_1^{\langle\mu\rangle}, \\
\lambda_{\mathbf{k}}^{\langle\mu\nu\rangle} &= \Phi_0^{\langle\mu\nu\rangle}, \\
\lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\rangle} &= \Phi_0^{\langle\mu\nu\alpha\rangle}, \\
\lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\beta\rangle} &= \Phi_0^{\langle\mu\nu\alpha\beta\rangle}.
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
\rho_r &= \gamma_r^{\Pi} \Pi, \\
\rho_r^{\mu} &= \gamma_r^{V} V^{\mu} + \gamma_r^{W} W^{\mu}, \\
\rho_r^{\mu\nu} &= \gamma_r^{\pi} \pi^{\mu\nu}, \\
\rho_r^{\mu\nu\alpha} &= \gamma_r^{\Omega} \Omega^{\mu\nu\alpha} \\
\rho_r^{\mu\nu\alpha\beta} &= \gamma_r^{\Theta} \Theta^{\mu\nu\alpha\beta},
\end{aligned}$$

Equations of motion

Hydrodynamic currents

[G. S. Denicol et al., Eur. Phys. J. A **48** 170 (2012)]

$$\dot{\Pi} + \frac{1}{\tau_\Pi}\Pi = -\frac{\zeta}{\tau_\Pi}\theta + \tau_{\Pi V} V_\mu \dot{u}^\mu - \ell_{\Pi V}\partial_\mu V^\mu - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi V} V^\mu \nabla_\mu \alpha_0 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}.$$

$$\begin{aligned}\dot{V}^\mu + \frac{1}{\tau_V}V^\mu = {} & \frac{\kappa_q}{\tau_V \beta_0^2 h_0^2}\nabla^\mu\alpha_0 + V_\nu\omega^{\mu\nu} + \lambda_{VV}V_\nu\sigma^{\mu\nu} + \delta_{VV}V^\mu\theta - \tau_{q\Pi}\Pi\dot{u}^\mu - \tau_{q\pi}\pi^{\mu\nu}\dot{u}_\nu + \\ & + \ell_{q\Pi}\nabla^\mu\Pi - \ell_{q\pi}\Delta^\mu_\alpha\partial_\beta\pi^{\alpha\beta} + \lambda_{q\Pi}\Pi\nabla^\mu\alpha_0 + \lambda_{q\pi}\pi^{\mu\nu}\nabla_\nu\alpha_0 \boxed{- \gamma^\Omega_{-2}\Omega^{\mu\alpha\beta}\sigma_{\alpha\beta},}\end{aligned}$$

$$\begin{aligned}\dot{\pi}^{\mu\nu} + \frac{1}{\tau_\pi}\pi^{\mu\nu} = {} & 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} + 2\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} + 2\tau_{\pi V}V^{\langle\mu}\dot{u}^{\nu\rangle} + 2\ell_{\pi V}\nabla^{\langle\mu}V^{\nu\rangle} - 2\lambda_{\pi V}V^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 - 2\lambda_{\pi\pi}\pi_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \\ & + 2\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - 2\delta_{\pi\pi}\pi^{\mu\nu}\theta \boxed{- \gamma^\Omega_{-1}\Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\Omega^{\alpha\beta\lambda} + \lambda_{\pi\Omega}\Omega^{\mu\nu\lambda}\nabla_\lambda\alpha_0 + \tau_{\pi\Omega}\dot{u}_\alpha\Omega^{\mu\nu\alpha} - \gamma^\Theta_{-2}\Theta^{\mu\nu\alpha\beta}\sigma_{\alpha\beta},}\end{aligned}$$

Contributions of third order to the dissipative currents

Higher-order currents

$$\begin{aligned}\dot{\Omega}^{\mu\nu\alpha} + \frac{1}{\tau_\Omega}\Omega^{\mu\nu\alpha} = {} & \delta_{\Omega\Omega}\Omega^{\mu\nu\alpha}\theta + \ell_{\Omega V}\sigma^{\langle\mu\nu}V^{\alpha\rangle} - 3\Omega_\lambda^{\langle\mu\nu}\omega^{\alpha\rangle\lambda} + \ell_{\Omega\Omega}\sigma_\lambda^{\langle\mu}\Omega^{\nu\alpha\rangle\lambda} + \frac{3}{7}\eta_\Omega\nabla^{\langle\mu}\pi^{\nu\alpha\rangle} + \lambda_{\Omega\pi}\pi^{\langle\mu\nu}\nabla^{\alpha\rangle}\alpha_0 \\ & + \tau_{\Omega\pi}\dot{u}^{\langle\mu}\pi^{\nu\alpha\rangle} + \tau_{\Omega\Theta}\dot{u}_\beta\Theta^{\mu\nu\alpha\beta} + \lambda_{\Omega\Theta}\Theta^{\mu\nu\alpha\beta}\nabla_\beta\alpha_0 - \gamma^\Theta_{-1}\Delta^{\mu\nu\alpha}_{\lambda\sigma\rho}\nabla_\beta\Theta^{\lambda\sigma\rho\beta}.\end{aligned}$$

$$\begin{aligned}\dot{\Theta}^{\mu\nu\alpha\beta} + \frac{1}{\tau_\Theta}\Theta^{\mu\nu\alpha\beta} = {} & \delta^{(r)}_{\Theta\Theta}\Theta^{\mu\nu\alpha\beta}\theta - 4\omega_\lambda^{\langle\mu}\Theta^{\nu\alpha\beta\rangle\lambda} + \ell_{\Theta\Theta}\sigma_\lambda^{\langle\mu}\Theta^{\nu\alpha\beta\rangle\lambda} + \ell_{\Theta\pi}\sigma^{\langle\mu\nu}\pi^{\alpha\beta\rangle} + \\ & + \ell_{\Theta\Omega}\nabla^{\langle\mu}\Omega^{\nu\alpha\beta\rangle} + \tau_{\Theta\Omega}\dot{u}^{\langle\mu}\Omega^{\nu\alpha\beta\rangle} + \lambda_{\Theta\Omega}\Omega^{\langle\mu\nu\alpha}\nabla^{\beta\rangle}\alpha_0,\end{aligned}$$