Causal instabilities of Chern-Simons magnetohydrodynamics
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Maxwell-Chern-Simons theory

Add $F_{\mu\nu}^* F_{\mu\nu} = \partial_\mu (\epsilon_{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}) \equiv \partial_\mu J_{CS}^\mu$ with a non-constant factor $\Theta(x)$ to the Maxwell theory (e.g. [Kharzeev (2010)])

$$\mathcal{L}_{\text{MCS}} = -\frac{1}{4} F_{\mu\nu}^* F_{\mu\nu} + A_\mu J_\mu + \frac{C_A}{4} \left( P_\mu J_{CS}^\mu \right) \text{ integration by part }$$

$P_\mu = \partial_\mu \Theta = \left( \dot{\Theta}, \vec{P} \right)$

Maxwell equation $\rightarrow$ MCS equation

$$\partial_\mu * F_{\mu\nu} = 0 \quad \partial_\mu F_{\mu\nu} = J_\mu + C_A * F_{\mu\nu} \partial_\nu \Theta$$

Energy-momentum conservation [Sadooghi and Shokri (2018)]

$$\partial_\mu T_{\text{Fluid}}^{\mu\nu} = F^{\nu\lambda} J_\lambda - \frac{C_A}{4} \left( F_{\mu\nu}^* F_{\mu\nu} \right) P^\nu$$

$$\nabla \cdot \vec{E} = n_e + C_A \vec{P} \cdot \vec{B}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} + \begin{cases} C_A \dot{\Theta} \vec{B} & \text{Chiral Magnetic} \\ \mu_5 \end{cases} - C_A \vec{P} \times \vec{E} \text{ Anomalous Hall}$$
Second law of thermodynamics demands new thermodynamic relations [Ozonder (2010)]:

\[ d\epsilon = T \, ds + \mu_e \, dn_e + \mathcal{R}_\Theta \, d\Theta \]
\[ \mathcal{R}_\Theta = -\frac{CA}{4} F^{\mu\nu} F_{\mu\nu} \]
\[ dp = s \, dT + n_e \, d\mu_e - \mathcal{R}_\Theta \, d\Theta \quad \text{but still} \quad s = \frac{1}{T} \left( \epsilon + p - n_e \mu_e \right) \]

Assume a hydrostatic configuration for an uncharged fluid

\[ \epsilon = \epsilon_0 \quad u^\mu = (1, 0) \quad B^\mu = (0, B_0) \quad E^\mu = 0 \quad \Theta = \Theta_0(x) \quad n_e,0 = P_0 \cdot B_0 = 0 \]

Linear stability analysis: Perturb quantities using \( \delta X(\omega, k) \exp(-i\omega t + i\vec{k} \cdot \vec{x}) \) and linearize the EOM to find \( M\delta X = 0 \). Finally find the dispersion relations \( \omega = \omega(\vec{k}) \) from \( \det(M) = 0 \)

Non-hydro modes: \( \omega(0) \neq 0 \)

Stable: \( \text{Im}(\omega) < 0 \)

Causal: \( \frac{1}{|k|} \lim_{|k| \to \infty} \text{Re}(\omega) \leq 1 \)
The collective modes

The nonchiral channel

Resistive non-dissipative MHD is linearly stable and causal: 2 slow and 2 fast magneto-sonic modes [Gedalin (1993)], plus a genuine nonhydro mode.

\[ \omega_{\text{slow,} \pm} = \pm v_s k \sqrt{A - B} - \frac{iv_s^2 (1 - v_s^2) (1 - v_s^2 (A - B)) (1 - A + B)}{4 \sigma_e B} k^2 + O(k^3) \]

The chiral channel

CSMHD is unstable but causal: 2 Alfven modes [Gedalin (1993)], plus a genuine nonhydro mode \((0 \leq \Delta < 2\pi \text{ is the angle between } P_0 \text{ and } \hat{k}_\perp = \sin \theta)\).

\[ \omega_{\text{Alfven,} \pm} = \pm v_a k \cos \theta - \frac{i(1 - v_a^2)(1 - v_a^2 \cos^2(\theta))}{2 \sigma_{\text{eff.}}} k^2 + O(k^3) \]

\[ \sigma_{\text{eff.}} \equiv \sigma_e - C_A P \sin \Delta \tan \theta \]

Parity breakdown: The Alfven modes are damped in one direction, and amplified in the opposite one.
The only possible constraint for stability is \( \nabla \Theta \big|_{\text{equilibrium}} = 0 \).
The instability illustrated

The cone is depicted for $\theta = \theta_c = \arctan\left(\frac{\sigma_{e}}{C_A P \sin \Delta}\right)$ (At $\theta_c$, $\sigma_{\text{eff.}} = 0$): unstable inside, stable outside.

