The hydodynamic expansion through regularized moments

QUARK MATTER
KRAKÓW
2022

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arXiv:1808.06436
arXiv:2003.09268

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Relativistic hydrodynamics

$$\left.\begin{aligned}\partial_\mu \hat{T}^{\mu\nu} &= 0\\ T^{\mu\nu} &= tr(\hat{\rho}\,\hat{T}^{\mu\nu})\end{aligned}\right\} \rightarrow \partial_\mu T^{\mu\nu} = 0$$

Hydro

$$T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu - \mathcal{P}\Delta^{\mu\nu} + \delta T^{\mu\nu}$$

From quantum field theory, but at least ten degrees of freedom and only four equations

Gradient expansion

- Requires small gradients
- Unstable (even in the non-relativistic limit)
- Not converging

A Buchel, M P Heller, J Noronha, arXiv:1603.05344

G Denicol, J Noronha, arXiv:1608.07869

$$\delta T^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots$$

↓

transport coefficients times gradients

From kinetic theory?...

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta\Delta_{\mu\nu}, \qquad T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}^{\mu\nu} = \mathcal{E}u^\mu u^\nu - (\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

Relativistic Boltzmann equation

$$p\cdot\partial f = -\mathcal{C}[f]$$

$$\begin{cases} u_\nu \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu T^{\mu\langle\nu\rangle} = 0 \end{cases} \Rightarrow \begin{cases} \dot{\mathcal{E}} = -\theta(\mathcal{E}+\mathcal{P}+\Pi) + \pi^{\mu\nu}\sigma_{\mu\nu} \\ (\mathcal{E}+\mathcal{P}+\Pi)\,\dot{u}^\nu = \nabla^\nu(\mathcal{P}+\Pi) - \nabla_\mu \pi^{\mu\langle\nu\rangle} + \pi^{\nu\alpha}\dot{u}_\alpha \end{cases}$$

$$u\cdot\partial f = \dot{f} = -\frac{p\cdot\nabla f}{(p\cdot u)} - \frac{C[f]}{(p\cdot u)}$$

convenient (non unique) basis

$$\mathfrak{f}_r^{\mu_1\cdots\mu_l} = \int_{\boldsymbol{p}} (p\cdot u)^r\, p^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} f$$

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C_{-1}^{\langle\mu\rangle\langle\nu\rangle} = 2(\mathcal{P}+\Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P}+\Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi_\alpha^{(\mu}\sigma^{\nu)\alpha} + 2\pi_\alpha^{(\mu}\omega^{\nu)\alpha}$$
$$-\nabla_\alpha \mathfrak{f}_{-1}^{\alpha\langle\mu\rangle\langle\nu\rangle} - \left(\sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta}\right)\mathfrak{f}_{-2}^{\alpha\beta\mu\nu}$$

Generalization?

multiple particle species

$$\Theta(p_0)\delta(p^2 - m^2)f \to \sum_i \Theta(p_0)\delta(p^2 - m_i^2)f_i$$

$$C[f] \to \sum_i C_i[f_1, \cdots, f_n]$$

long range interactions (not-immediate)

$$p \cdot \partial f \to p \cdot \partial f + F \cdot \partial_{(p)} f$$

Divergencies at higher orders

$\mathfrak{f}_{-1}^{\alpha\mu\nu}$ $\mathfrak{f}_{-2}^{\alpha\beta\mu\nu}$

L T, G Vujnovich, J Noronha, U Heinz **arXiv:1808.06436**

Wigner distribution (quantum)

L T, **arXiv:2003.09268zz**

$$\Theta(p_0)\delta(p^2 - m^2)f \to W$$

$$p \cdot \partial f \to k \cdot \partial W$$

Needs regularization from the start

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\alpha k^\mu k^\nu}{(k \cdot u)} W = \int \frac{d^4k}{(2\pi)^4} \frac{k^{\langle\alpha\rangle} k^{\langle\mu\rangle} k^{\langle\nu\rangle}}{(k \cdot u)} W + 3u^{(\alpha} T^{\mu\nu)} - 2\mathcal{E} u^\alpha u^\mu u^\nu$$

Resumed moments

$$\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int \frac{d^4k}{(2\pi)^4}\,(k\cdot u)^n\, e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\ldots k^{\langle\mu_s\rangle}W(x,k)$$

+exactly solvable case, *Bjorken symmetry* RTA

$$k\cdot\partial W = -\frac{k\cdot u}{\tau_R}\left(W - W_{eq}\right) = -\frac{k\cdot u}{\tau_R}\left(W - \frac{2\delta(k^2)}{(2\pi)^3}e^{-\frac{1}{T(\tau)}\sqrt{k_T^2+\frac{w^2}{\tau^2}}}\right) \Rightarrow \partial_\tau W + 2\frac{v^2-w^2}{\tau}\partial v^2 W = \frac{1}{\tau_R}\delta W$$

$$L_n = \phi_2^{\mu_1\cdots\mu_{2n}} z_{\mu_1}\cdots z_{\mu_{2n}}, \qquad T_n = \phi_2^{\mu_1\cdots\mu_{2n}\alpha\beta} z_{\mu_1}\cdots z_{\mu_{2n}} x_\alpha x_\beta$$

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta')$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathrm{P}_L}{\tau}$$

$$\dot{\mathrm{P}}_L + \frac{1}{\tau_R}\left(\mathrm{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathrm{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathrm{P}}_T + \frac{1}{\tau_R}\left(\mathrm{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathrm{P}_L + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

Higher orders

$$\mathcal{R}_T^{(n)} = \int_0^\infty d\zeta\,\left(\hat{\mathcal{L}}\right)^n T_n\,, \qquad \mathcal{R}_L^{(n)} = \int_0^\infty d\zeta\,\left(\hat{\mathcal{L}}\right)^n L_{n+1}$$

$$\dot{\mathcal{R}}_T^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+1}{\tau}\mathcal{R}_T^{(n)} + \frac{1}{\tau}\mathcal{R}_T^{(n+1)}$$

$$\dot{\mathcal{R}}_L^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+3}{\tau}\mathcal{R}_L^{(n)} + \frac{1}{\tau}\mathcal{R}_L^{(n+1)}$$

Exact solutions for the wigner distribution

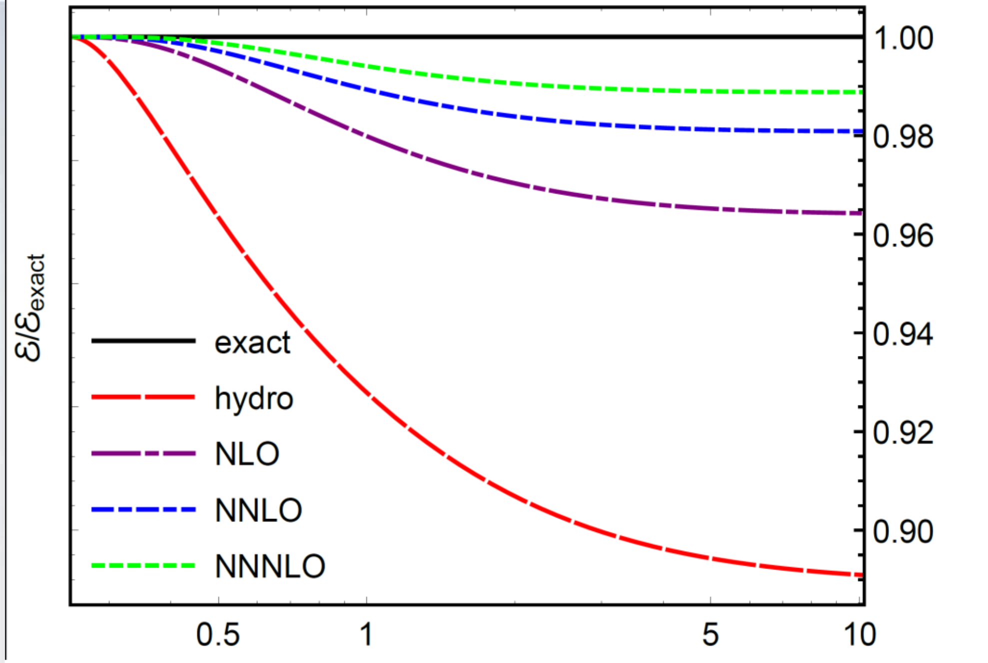
- Constant shear-viscosity over entropy ratio: $\tau_R = 5\bar{\eta}/T$
- $\bar{\eta} = 3/(4\pi)$
- $\tau_0 = 1/4$ fm/c, $T_0 = 0.6$ GeV, two possible initial conditions:

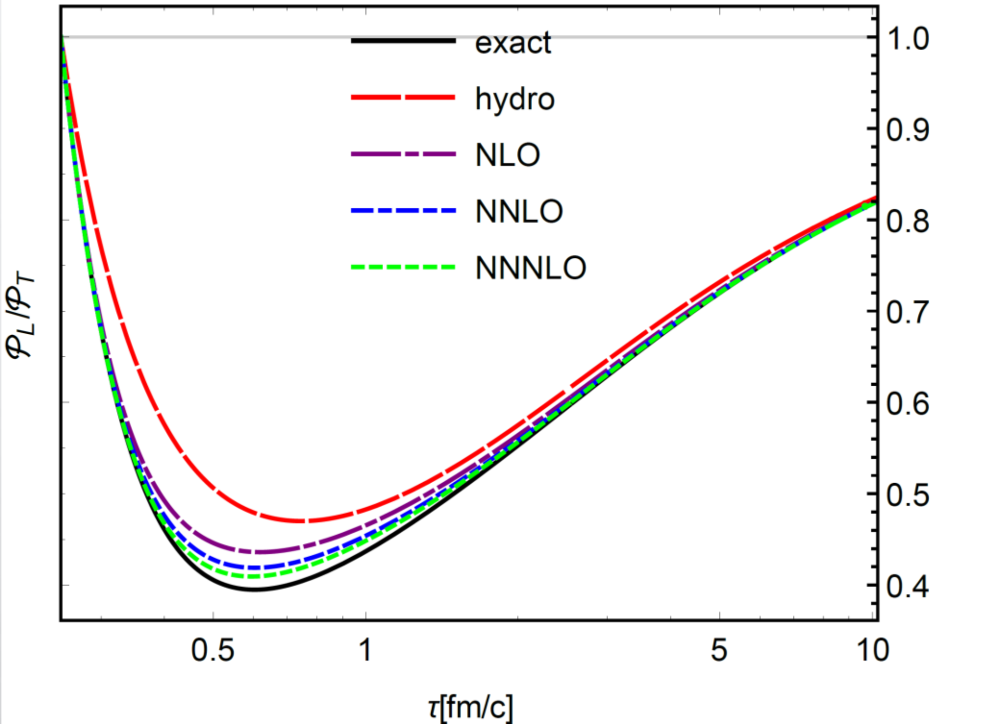
$$v = \left(tk^0 - zk^z\right) = \tau\, k_\mu u^\mu$$

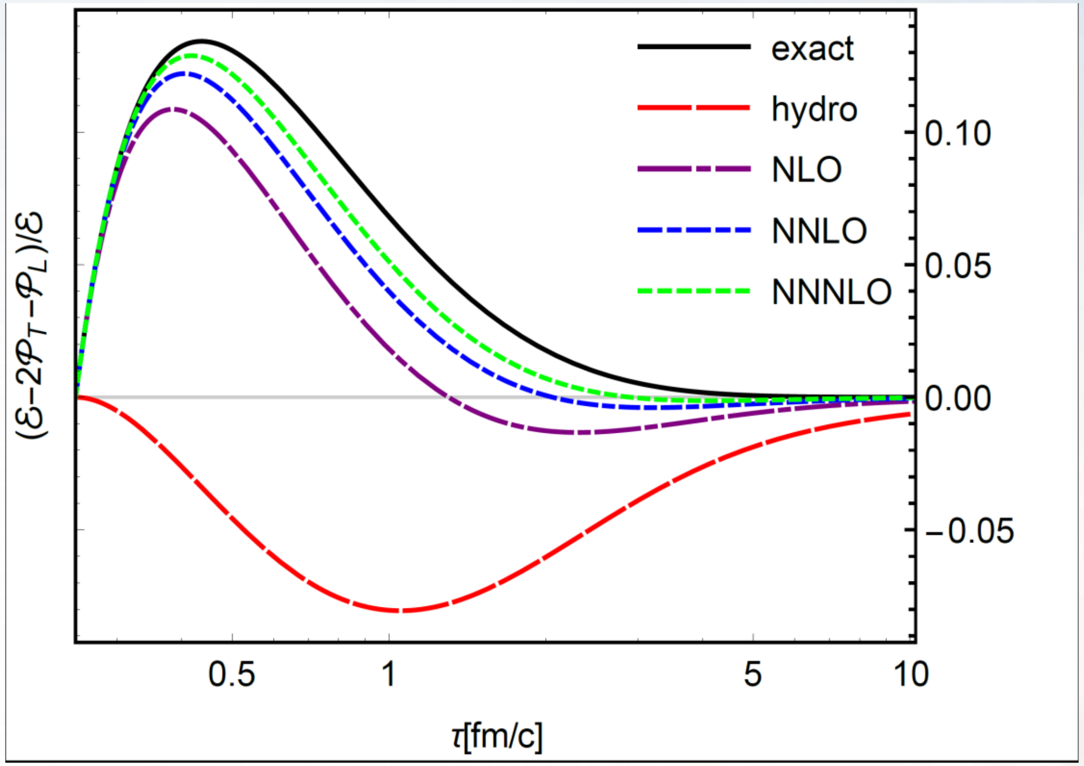
$$w = \left(z\, k^0 - t\, k^z\right) = \tau\, k_\mu z^\mu$$

$$W_0^{iso} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}}\, \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}}\, e^{-\frac{1}{T_0}\sqrt{\sigma = k_T^2 + \frac{w^2}{\tau_0^2}}} \longrightarrow \mathcal{P}_0 = \mathcal{P}_{\mathrm{eq.}} = \frac{1}{3}\,\mathcal{E}$$

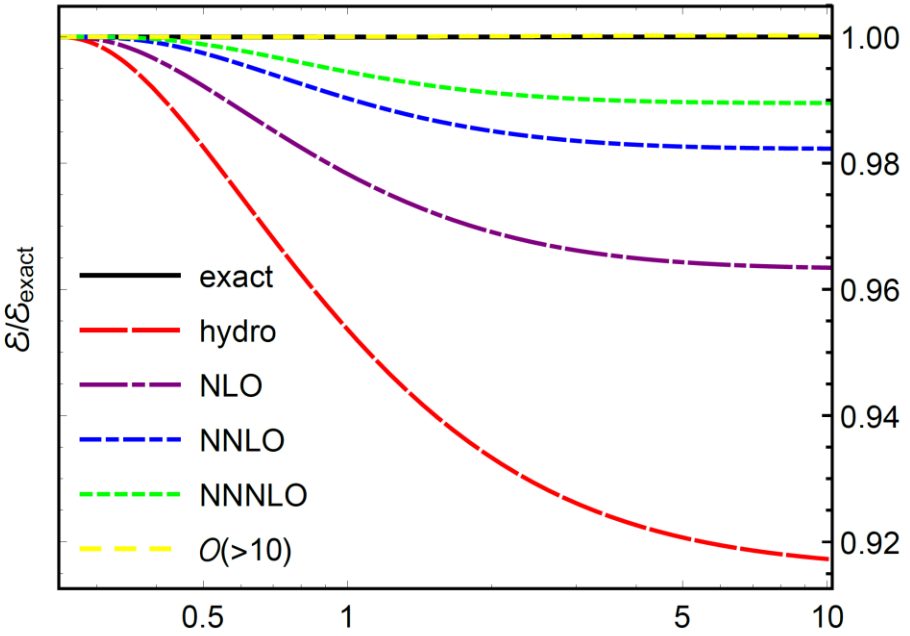
$$W_0^{a} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}}\, \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}}\, e^{-\frac{1}{T_0}\sqrt{\sigma = k_T^2 + \frac{w^2}{\tau_0^2}}}\left[1 - 3P_2\left(\frac{w}{\tau_0\sqrt{\sigma}}\right)\right] \longrightarrow \mathcal{P}_T^0 = \frac{8}{5}\mathcal{P}_{\mathrm{eq.}},\quad \mathcal{P}_L^0 = -\frac{1}{5}\mathcal{P}_{\mathrm{eq.}}$$

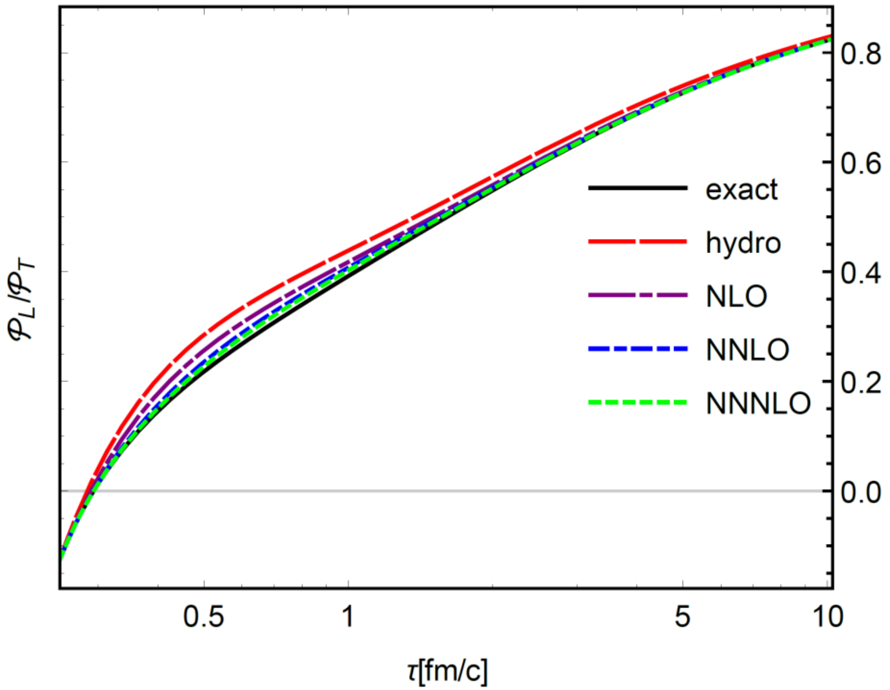


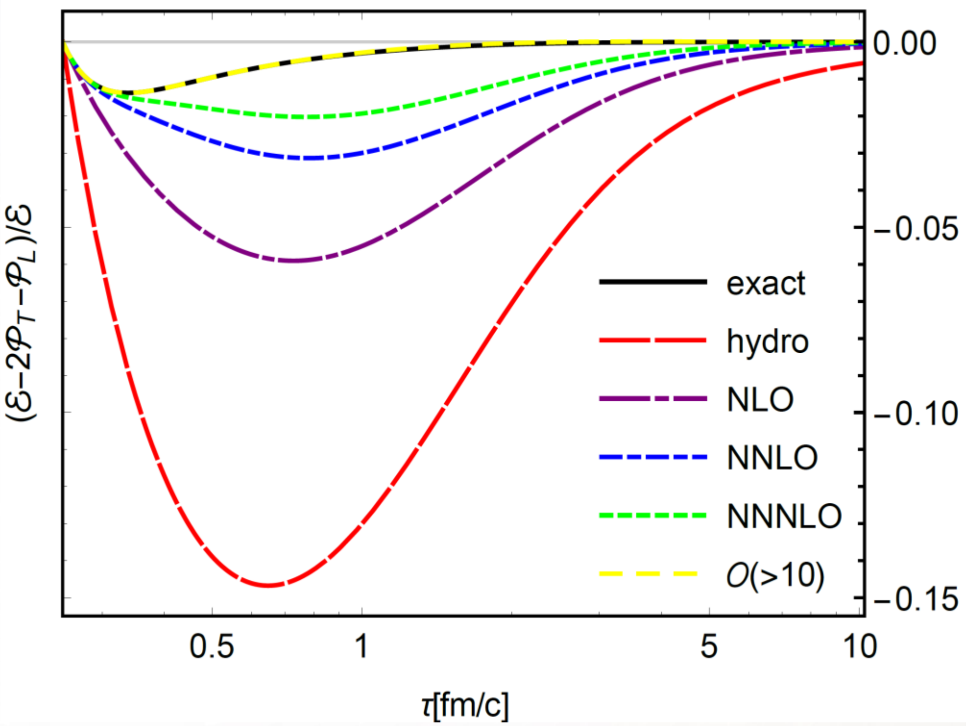




$$(\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L)/\mathcal{E} = -\frac{3\Pi}{\mathcal{E}} = -\frac{\Pi}{\mathcal{P}}$$







similar conclusions

Back up slides

$$\int [g(x) + h(x)]\, dx \neq \int g(x)dx + \int h(x)dx$$

$$\int \lim_{\varepsilon\to 0} f(\varepsilon, x)dx \neq \lim_{\varepsilon\to 0} \int f(\varepsilon, x)dx$$

$$\frac{1}{\beta} = \int_0^\infty \left[-\partial_\beta \left(\frac{e^{-\beta x}}{x}\right)\right] dx \neq -\partial_\beta \left(\int_0^\infty \frac{e^{-\beta x}}{x} dx \equiv \infty\right)$$

$$\frac{1}{x} = \int_0^\infty e^{-\alpha x} d\alpha$$

$$\frac{1}{(\alpha+\beta)^2} = \int_0^\infty dx \left[-\partial_\beta\left(e^{-(\alpha+\beta)x}\right)\right] = -\partial_\beta \left(\int_0^\infty dx\, e^{-(\alpha+\beta)x} = \frac{1}{\alpha+\beta}\right),$$

$$\int_0^\infty d\alpha \left[\frac{1}{(\alpha+\beta)^2} = \partial_\alpha\left(-\frac{1}{\alpha+\beta}\right)\right] = \frac{1}{\beta}$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}\Big|_{eq}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}\Big|_{eq}$$

What can we say for the isotropic case

$$\mathrm{R}_L^{eq.} = \frac{1}{5}\mathcal{E} \qquad \mathrm{R}_L^{0} = -\frac{1}{5}\mathcal{E}$$

$$\mathrm{R}_T^{eq.} = \frac{1}{15}\mathcal{E} \qquad \mathrm{R}_T^{0} = -\frac{1}{15}\mathcal{E}$$

$$\frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}\Big|_0 = -\frac{1}{3}$$

$$\frac{\delta\dot{\mathcal{P}}_T}{\dot{\mathcal{P}}_T}\Big|_0 = -\frac{1}{3}$$

$$\delta\mathcal{P}_L = \int_{\tau_0}^{\tau} ds\, \delta\dot{\mathcal{P}}_L \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\int \delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \text{Maximum if } 0 = \partial_\tau\left(\frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\right) = \frac{\delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} - \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\frac{\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}$$

$$\frac{\delta\mathcal{E}}{\mathcal{E}} = \frac{\delta\dot{\mathcal{E}}}{\dot{\mathcal{E}}} = \frac{\delta\mathcal{E}+\delta\mathcal{P}_L}{\mathcal{E}+\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{E}}{\mathcal{E}} \simeq \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}$$

…but forthe trace anomaly $\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L = -3\Pi$ $\qquad \frac{\delta\dot{\Pi}}{\dot{\Pi}} = -1$

Resummed moments

Approach introduced for the Boltzmann-vlasov equation helps

$$\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int \frac{d^4k}{(2\pi)^4}\,(k\cdot u)^n\, e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\dots k^{\langle\mu_s\rangle} W(x,k)$$

$$\partial_\zeta \phi_n^{\mu_1\cdots\mu_s} = -\phi_{n+2}^{\mu_1\cdots\mu_s}$$

$$\int_\zeta^\infty dv\, \phi_{n+2}^{\mu_1\cdots\mu_s} = \phi_n^{\mu_1\cdots\mu_s}$$

$$\phi_n^{\mu_1\cdots\mu_s}(x,0) = \Delta^{\boldsymbol{\mu_1}}_{\boldsymbol{\alpha_1}}\cdots\Delta^{\boldsymbol{\mu_s}}_{\boldsymbol{\alpha_s}}\mathcal{F}_{\mathrm{n}}^{\alpha_1\cdots\alpha_s} = \mathfrak{f}_n^{\alpha_1\cdots\alpha_s}$$

All (well-defined) previous moments recovered from the resumed ones, including $T^{\mu\nu}$

2 generations of dynamical moments needed

$$\dot\phi_2^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde C_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_2^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot u_\alpha\left[2\phi_1^{\alpha\mu_1\cdots\mu_s} + 2\zeta\partial_\zeta\phi_1^{\alpha\mu_1\cdots\mu_s}\right]$$
$$-s\dot u^{(\mu_1}\partial_\zeta\phi_1^{\mu_2\cdots\mu_s)} + \nabla_\alpha u_\beta\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$

$$\dot\phi_1^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde C_0^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_1^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\int_\zeta^\infty dv\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot u_\alpha\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$
$$+s\dot u^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)} - 2\zeta\nabla_\alpha u_\beta\phi_1^{\alpha\beta\mu_1\cdots\mu_s}$$

Exactly solvable case

Bjorken symmetry

$$\tau = \sqrt{t^2 - z^2}, \qquad v = k^0 t - z\, k^z, \qquad u = (\cosh\eta, 0, 0, \sinh\eta)$$

$$\eta = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right), \qquad w = z k^0 - t\, k^z, \qquad z = (\sinh\eta, 0, 0, \cosh\eta)$$

(as a consequence)

$$T^{\mu\nu} = \mathcal{E}(\tau) u^\mu u^\nu + \mathcal{P}_T(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \mathcal{P}_L(\tau) z^\mu z^\nu$$

$$\pi^{\mu\nu} = -\frac{1}{2}\pi(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \pi(\tau) z^\mu z^\nu$$

$$\mathcal{P}_T = \mathcal{P} + \Pi - \frac{1}{2}\pi, \qquad \mathcal{P}_L = \mathcal{P} + \Pi + \pi$$

RTA

$$k\cdot\partial W = -\frac{k\cdot u}{\tau_R}\left(W - W_{eq}\right) = -\frac{k\cdot u}{\tau_R}\left(W - \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2 + \frac{w^2}{\tau^2}}}\right) \Rightarrow \partial_\tau W + 2\frac{v^2 - w^2}{\tau}\partial v^2 W = \frac{1}{\tau_R}\delta W$$

in addition $W(\tau, v^2, k_T, w^2)$

Particles interacting with external fields

Boltzmann-Vlasov equation

$$p\cdot\partial f + m\partial_\alpha m\,\partial^\alpha_{(p)} f + qF_{\alpha\beta}p^\beta\,\partial^\alpha_{(p)} f = -\mathcal{C}[f]$$

Immediate (but problematic) generalization

$$\begin{aligned}\dot{\mathcal{F}}_r^{\mu_1\cdots\mu_s} + C_{r-1}^{\mu_1\cdots\mu_s} \;=\;& r\,\dot{u}_\alpha\,\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} + (r-1)\,\nabla_\alpha u_\beta\,\mathcal{F}_{r-2}^{\alpha\beta\mu_1\cdots\mu_s}\\ &+m\dot{m}\,(r-1)\,\mathcal{F}_{r-2}^{\mu_1\cdots\mu_s} + s\,m\partial^{(\mu_1}m\,\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)}\\ &-q(r-1)\,E_\alpha\,\mathcal{F}_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\,s\,g_{\alpha\beta}F^{\alpha(\mu_1}\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)\beta}\end{aligned}$$

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \varepsilon_{\mu\nu\rho\sigma}u^\rho B^\sigma$$

Moments with large negative r needed, infrared catastrophe!

Unphysical moments as a source

$$\mathcal{F}_r^{\mu_1\cdots\mu_s} = \int_{\mathbf{p}} (p\cdot u)^r \, p^{\mu_1}\cdots p^{\mu_s} f$$

$r < -2 - s$, diverging integral in the massless case

Numerical problems for small non-vanishing masses

$$\frac{\mathcal{F}_r^{\mu_1\cdots\mu_s}}{T^{r+s+2}} \propto \left(\frac{m}{T}\right)^{r+s+2}$$

Any non-trivial coupling to an electromagnetic field introduces numerical problems at higher orders

Moments with large negative r needed, infrared catastrophe!

Resummed expansion

- Resummed moments

$$\Phi^{\mu_1\cdots\mu_s} = \int_{\mathbf{p}} (p\cdot u) p^{\mu_1}\cdots p^{\mu_s} e^{-\xi^2 (p\cdot u)^2} f$$

- All reducible moments recovered

$$\mathcal{F}_n^{\mu_1\cdots\mu_l} = \frac{2}{\sqrt{\pi}} \int_0^\infty d\xi\, \Phi^{\mu_1\cdots\mu_l\nu_1\cdots\nu_n}\, u_{\nu_1}\cdots u_{\nu_n}$$

- Well defined equations

$$\begin{aligned}\dot{\Phi}^{\mu_1\cdots\mu_s} + \delta\Phi^{\mu_1\cdots\mu_s}_{\text{coll.}} = \; & \frac{2}{\sqrt{\pi}} \int_\xi^\infty d\zeta \frac{\zeta}{\sqrt{\zeta^2-\xi^2}} \Big\{ \dot{u}_\alpha\, \Phi^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha \Phi^{\alpha\mu_1\cdots\mu_s} \\ & + s\left[m\partial^{(\mu_1} m\, \Phi^{\mu_2\cdots\mu_s)} - q\, g_{\alpha\beta} F^{\alpha(\mu_1} \Phi^{\mu_2\cdots\mu_s)\beta}\right]\Big\} \\ & -2\xi^2 \left[\partial_\alpha u_\beta\, \Phi^{\alpha\beta\mu_1\cdots\mu_s} + m\dot{m}\, \Phi^{\mu_1\cdots\mu_s} - qE_\alpha\, \Phi^{\alpha\mu_1\cdots\mu_s}\right]\end{aligned}$$

Contribution from the collisional kernel

$$\delta\Phi^{\mu_1\cdots\mu_s}_{\text{coll.}} = \int_{\mathbf{p}} p^{\mu_1}\cdots p^{\mu_s} e^{-\xi^2 (p\cdot u)^2} \mathcal{C}[f]$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$f_r^{\mu_1\cdots\mu_s} = \mathcal{F}_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$
$$\phi_r^{\mu_1\cdots\mu_s} = \Phi_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$

$$\begin{aligned}
\dot{f}_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\mathcal{F}_{\text{coll.}})_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\,s\,\varepsilon^{\rho\sigma\alpha(\mu_1} f_{r-1}^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta} u_\rho B_\sigma - q(r-1)\,E_\alpha\, f_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\,s\,E^{(\mu_1} f_r^{\mu_2\cdots\mu_s)} \\
& + m\dot{m}\,(r-1)\, f_{r-2}^{\mu_1\cdots\mu_s} + s\, m\nabla^{(\mu_1} m\, f_{r-1}^{\mu_2\cdots\mu_s)} \\
& + r\,\dot{u}_\alpha f_{r-1}^{\alpha\mu_1\cdots\mu_s} - s\dot{u}^{(\mu_1} f_{r+1}^{\mu_2\cdots\mu_s)} \\
& - \nabla_\alpha f_{r-1}^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} - \theta\, f_r^{\mu_1\cdots\mu_s} - s\,\nabla_\alpha u^{(\mu_1} f_r^{\mu_2\cdots\mu_s)\alpha} \\
& + (r-1)\nabla_\alpha u_\beta\, f_{r-2}^{\alpha\beta\mu_1\cdots\mu_s},
\end{aligned}$$

$$\begin{aligned}
\dot{\phi}_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\Phi_{\text{coll.}})_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\left[s\,E^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)} - 2\xi^2\left(E_\alpha\,\phi_1^{\alpha\mu_1\cdots\mu_s} + m\dot{m}\,\phi_1^{\mu_1\cdots\mu_s}\right)\right] \\
& + s\frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[m\nabla^{(\mu_1} m\,\phi_1^{\mu_2\cdots\mu_s)} - q\,\varepsilon^{\rho\sigma\alpha(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta} u_\rho B_\sigma\right] \\
& + \frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[\dot{u}_\alpha\phi_1^{\alpha\mu_1\cdots\mu_s} + s\,\dot{u}^{(\mu_1}\partial_\upsilon\phi_1^{\mu_2\cdots\mu_s)} + 2\xi^2\,\dot{u}_\alpha\,\partial_\upsilon\phi_1^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle}\right] \\
& - \theta\,\phi_1^{\mu_1\cdots\mu_s} - s\,\nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - 2\xi^2\nabla_\alpha u_\beta\,\phi_1^{\alpha\beta\mu_1\cdots\mu_s}.
\end{aligned}$$

Exact solutions of the Boltzmann-Vlasov equation

- Maxwell equations, particles as the source: $\partial_\mu F^{\mu\nu} = J^\nu$ $\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0$
- Longitudinally boost invariant expansion, and homogeneous in the transverse plane (no parity invariance), RTA

Because of symmetry

$$\partial\tau f(\tau, p_T, p_\eta) = -q\, E_\eta \frac{\partial f}{\partial p_\eta} - \frac{1}{\tau_R}(f - f_{eq.})$$

$$\partial\tau \bar{f}(\tau, p_T, p_\eta) = +q\, E_\eta \frac{\partial \bar{f}}{\partial p_\eta} - \frac{1}{\tau_R}(\bar{f} - f_{eq.})$$

$$\partial_\tau E_\eta(\tau) = \frac{1}{\tau} E_\eta - J_\eta$$

$$u \cdot J = 0$$

$$f_{eq.} = k\, e^{-\frac{1}{T}(p\cdot u)} \qquad E_\eta = -\tau\, E_L \qquad k = \frac{N_{dof}}{(2\pi)^3} \qquad J_\eta = -\tau\, J_L$$

- Massless particles, $4\pi\,\bar{\eta} = 1$
- Local equilibrium initial conditions, $\tau_0 = 1$ fm/c, $\mathrm{T}_0 = 0.3$ GeV, $\mathrm{E}_L^0/T_0 = 0.2\ \mathrm{fm}^{-1}$.

R Ryblewski, W Florkowski, arXiv:1307.0356

Set of independent moments

- Linearly independent moments: $\Phi_l^{\pm} = \Phi^{\mu_1 \ldots \mu_l} z_{\mu_1} \cdots z_{\mu_l} \pm \bar{\Phi}^{\mu_1 \ldots \mu_l} z_{\mu_1} \cdots z_{\mu_l}$

$$[z^\mu = (\sinh\eta, 0,0, \cosh\eta), \quad u^\mu = (\cosh\eta, 0,0, \sinh\eta)]$$

- Normalized (dimensionless) moments: $M_l^{\pm} = \dfrac{\Phi_l^{\pm}}{(8\pi k)(l+2)l!T_0^{l+3}}$

In particular

$$(48\,\pi\,k)T^4 = \mathcal{E} = \frac{2}{\sqrt{\pi}}\int_0^\infty d\,\xi\left(-\frac{\partial}{\partial\xi^2}\Phi_0^+\right) = 32\sqrt{\pi}\,k\,T_0^3\int_0^\infty d\,\xi\left(-\frac{\partial}{\partial\xi^2}M_0^+\right)$$

$$\mathcal{P}_L = \frac{2}{\sqrt{\pi}}\int_0^\infty d\,\xi\,\Phi_2^+ = 128\sqrt{\pi}\,k\,T_0^5\int_0^\infty d\,\xi\,M_2^+$$

$$J_L = -q\frac{2}{\sqrt{\pi}}\int_0^\infty d\,\xi\,\Phi_1^- = -\,q(48\sqrt{\pi}\,k\,T_0^4)\int_0^\infty d\,\xi\,M_1^-$$

Equations to test numerically

From the resummed moments and Maxwell equations

$$\tau_R \partial_\tau M_l^{\pm} + \left(M_l^{\pm} - M_l^{\pm}\Big|_{eq.} \right) = -\frac{\tau_R}{\tau}\Bigg[(l+1)M_l^{\pm} - 2(\xi T_0)^2 (l+4)(l+1) M_{l+2}^{\pm} + \frac{qE_\eta}{T_0}\left(\frac{l+1}{l+2} M_{l-1}^{\mp} - 2(\xi T_0)^2 \frac{(l+3)(l+1)}{l+2} M_{l+1}^{\mp} \right)\Bigg]$$

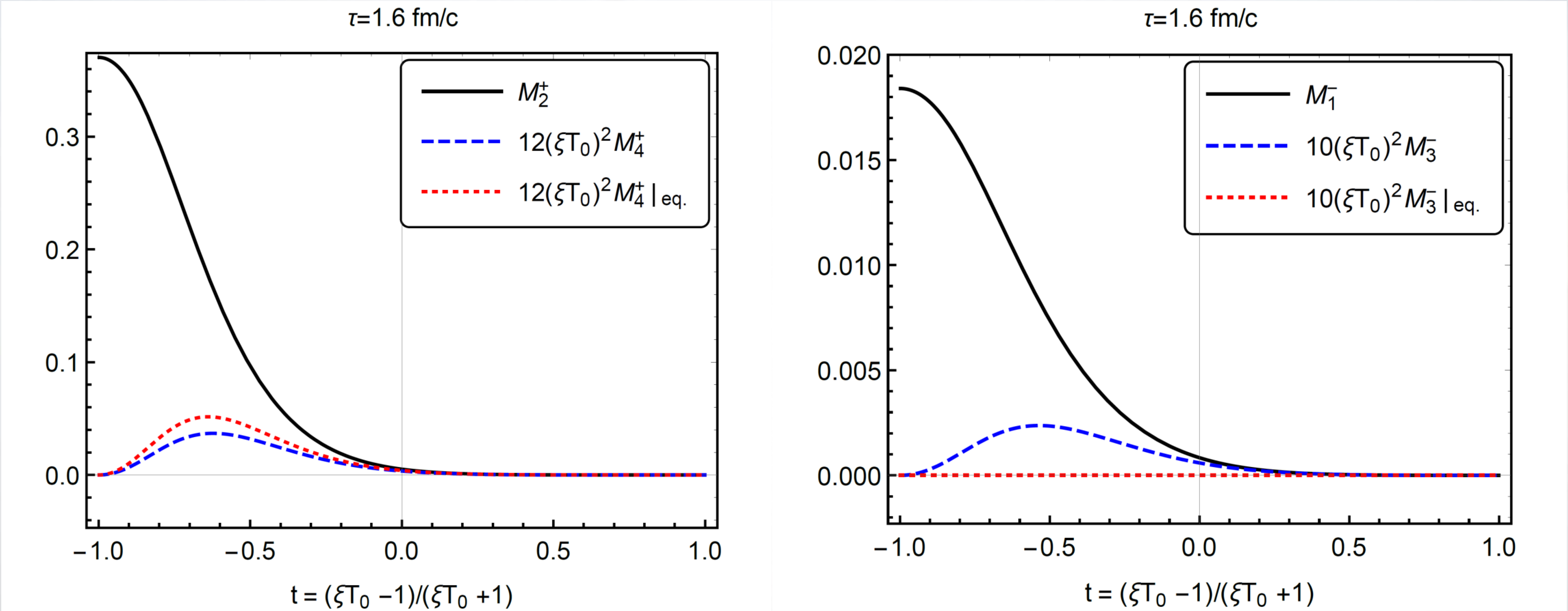
$$\frac{\partial_\tau T}{T} = -\frac{1}{4\tau}\left[1 + \frac{4T_0^5}{3\sqrt{\pi}T^4}\int_0^\infty d\xi\, M_2^+ - q\, E_\eta \frac{T_0^4}{\sqrt{\pi}T^4}\int_0^\infty d\xi\, M_1^- \right]$$

$$\partial_\tau E_\eta = \frac{1}{\tau} E_\eta - \tau q (48\sqrt{\pi}\, k\, T_0^4) \int_0^\infty d\,\xi\, M_1^-$$

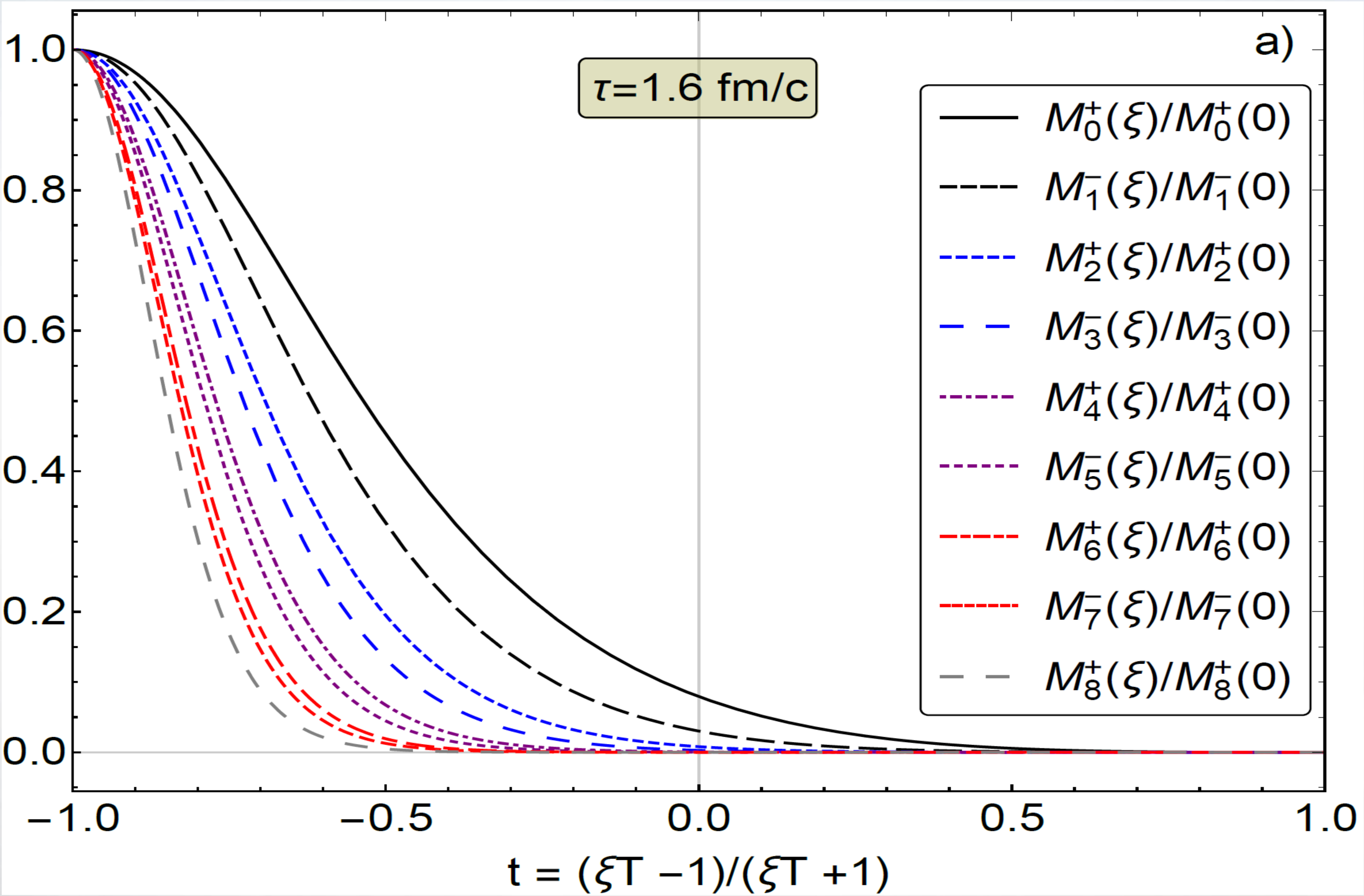
Suppression for small ξ

$$\xi T_0 = \frac{1+t}{1-t} \Rightarrow t = \frac{\xi T_0 - 1}{\xi T_0 + 1}$$

$\tau = 1.6$ fm
$E_L^0/T_0 = 0.2\ \text{fm}^{-1}$

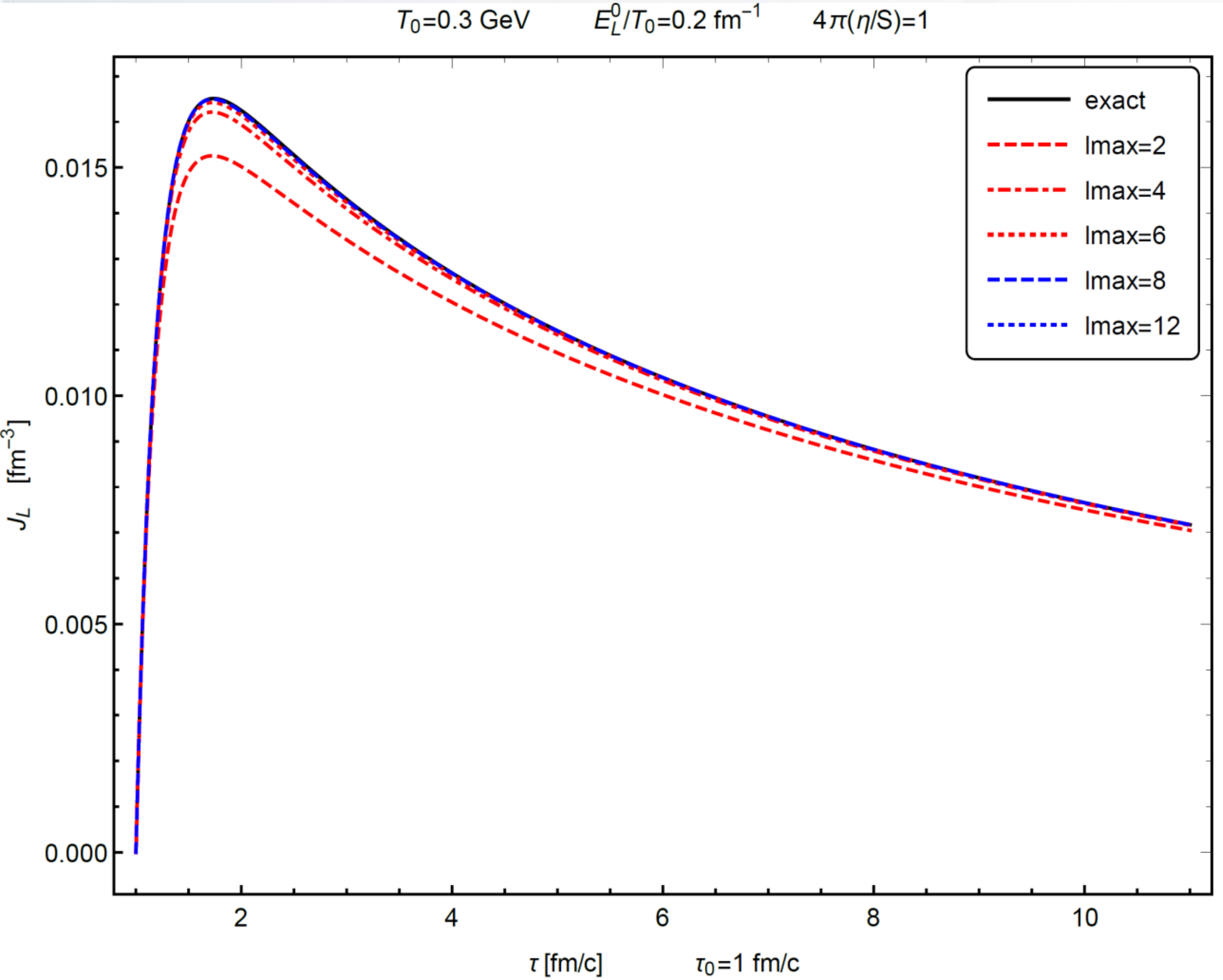


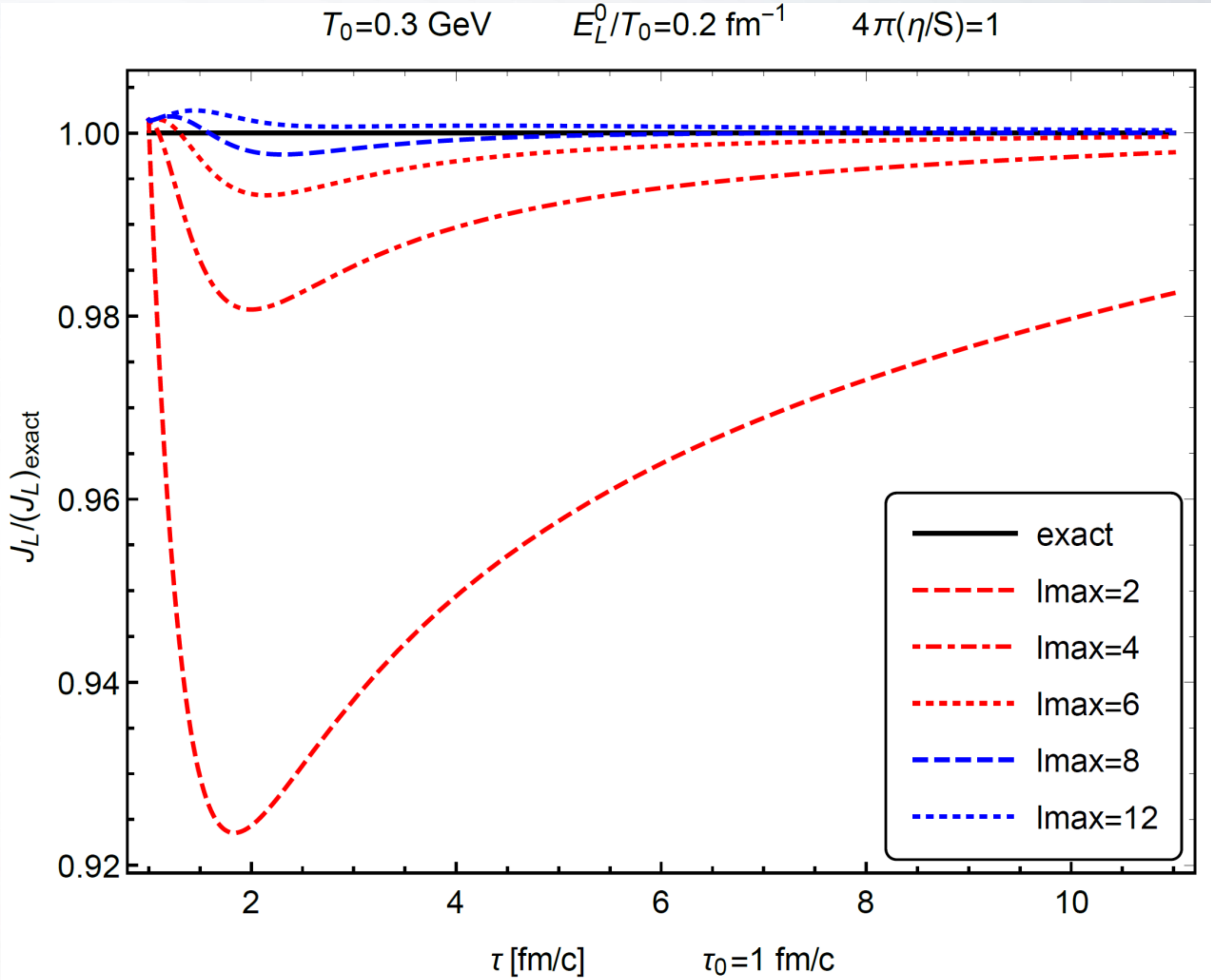
from the EOM: $\tau_R \partial_\tau M_l^\pm = -\frac{\tau_R}{\tau}\left[(l+1)M_l^\pm - 2(\xi T_0)^2(l+4)(l+1)M_{l+2}^\pm + \cdots\right.$



τ=1.6 fm/c
a)
$M_0^+(\xi)/M_0^+(0)$
$M_1^-(\xi)/M_1^-(0)$
$M_2^+(\xi)/M_2^+(0)$
$M_3^-(\xi)/M_3^-(0)$
$M_4^+(\xi)/M_4^+(0)$
$M_5^-(\xi)/M_5^-(0)$
$M_6^+(\xi)/M_6^+(0)$
$M_7^-(\xi)/M_7^-(0)$
$M_8^+(\xi)/M_8^+(0)$
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0
t = (ξT −1)/(ξT +1)

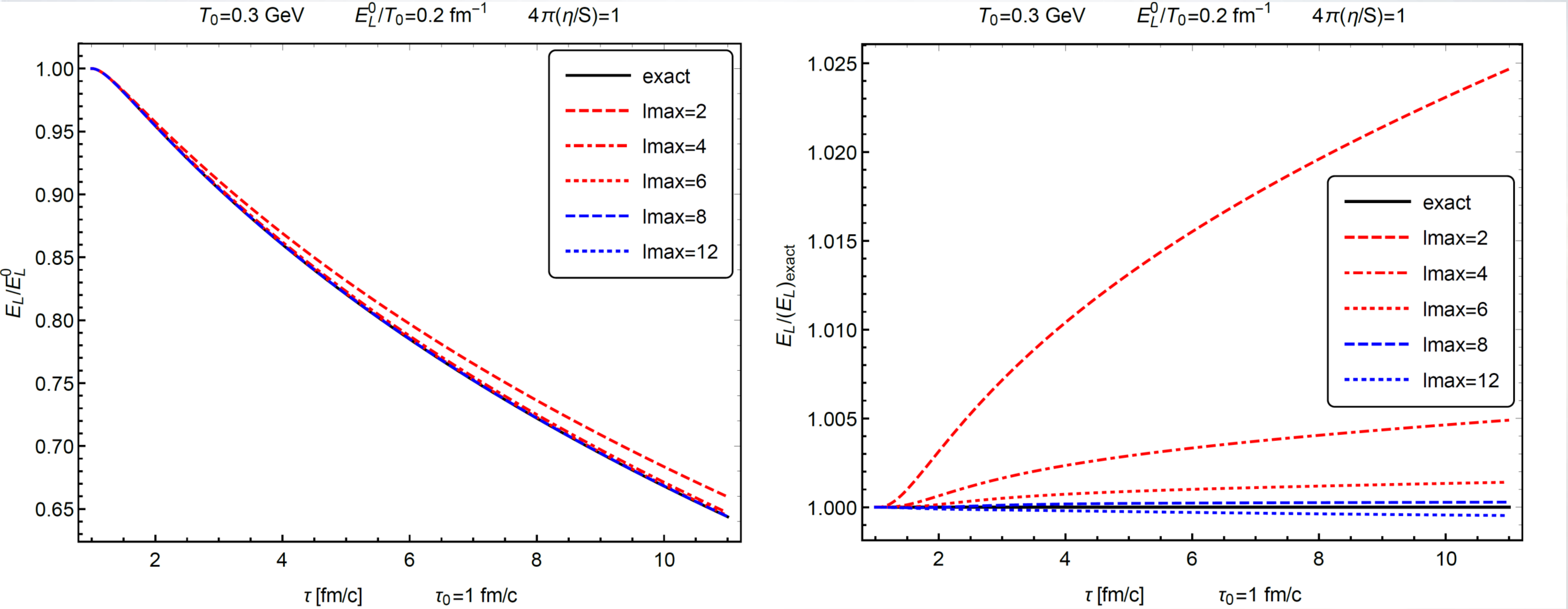
Comparisons:electric current





Higher orders ill-defined in the traditional expasion

Comparisons: electric field



Higher orders ill-defined in the traditional expasion