Inverse Reynolds-dominance approach to transient fluid dynamics

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2nd order hydro

- In the Landau frame ($T^{\mu}{}_{\nu} u^{\nu} = \varepsilon u^{\mu}$), N^{μ} and $T^{\mu\nu}$ can be decomposed as

$$N^{\mu} = n u^{\mu} + n^{\mu}, \qquad T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{1}$$

- The cons. eqs. $\partial_{\mu} N^{\mu} = \partial_{\nu} T^{\mu\nu} = 0$ give the evolution of n, ε and u^{μ}.
- Π, n^{μ} and $\pi^{\mu\nu}$ are regarded to be small and of $O(\mathrm{Re}^{-1})$, while gradients are taken as $O(\mathrm{Kn})$.
- 2nd order hydro gives the evolution eqs. for Π, n^{μ} and $\pi^{\mu\nu}$ up to second order w.r.t. Kn and Re^{-1}:

$$\begin{aligned} \tau_{\Pi}\dot{\Pi} + \Pi &= -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R}\,, \\ \tau_{n}\dot{n}^{\langle\mu\rangle} + n^{\mu} &= \kappa I^{\mu} + \mathcal{J}^{\mu} + \mathcal{K}^{\mu} + \mathcal{R}^{\mu}\,, \\ \tau_{\pi}\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu}\,, \end{aligned} \tag{2}$$

containig the 1st order (NS) limit as well as 2nd order terms of orders $O(\mathrm{Re}^{-1}\mathrm{Kn})$, $O(\mathrm{Kn}^{2})$ and $O(\mathrm{Re}^{-2})$.

- Problem: The $O(\mathrm{Kn}^{2})$ are parabolic and must be discarded.

IReD: From parabolic to hyperbolic

- Let us look at the $O(\mathrm{Kn}^2)$ and $O(\mathrm{Re}^{-1}\mathrm{Kn})$ terms (just $\mathcal{K}^{\mu\nu}$ and $\mathcal{J}^{\mu\nu}$ for brevity):

$$\mathcal{K}^{\mu\nu} = \tilde{\eta}_1 \omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda + \tilde{\eta}_2 \theta\sigma^{\mu\nu} + \tilde{\eta}_3 \sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \tilde{\eta}_4 \sigma^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu} I^{\nu\rangle} + \tilde{\eta}_6 F^{\langle\mu} F^{\nu\rangle} + \tilde{\eta}_7 I^{\langle\mu} F^{\nu\rangle} + \tilde{\eta}_8 \nabla^{\langle\mu} I^{\nu\rangle} + \tilde{\eta}_9 \nabla^{\langle\mu} F^{\nu\rangle},$$

$$\mathcal{J}^{\mu\nu} = 2\tau_\pi \pi^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu}F^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu} I^{\nu\rangle},$$

- The main idea of the IReD $=$ *Inverse Reynolds Dominance* approach is to render $\mathcal{K}^{\mu\nu}$ of $O(\mathrm{Re}^{-1}\mathrm{Kn})$ by trading one power of Kn for one of Re^{-1} via

$$\theta \simeq -\Pi/\zeta, \qquad I^\mu \simeq n^\mu/\kappa, \qquad \sigma^{\mu\nu} \simeq \pi^{\mu\nu}/2\eta. \tag{3}$$

- The terms in red can be related to

$$\dot{\sigma}^{\langle\mu\nu\rangle} = -\omega^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda - \frac{2}{3}\theta\sigma^{\mu\nu} - \sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}{}_\lambda - \frac{\tilde{\eta}_6}{\tilde{\eta}_1}F^{\langle\mu}F^{\nu\rangle} - \frac{D_{20}\mathcal{H}}{(\varepsilon+P)^3} I^{\langle\mu}F^{\nu\rangle} - \frac{\tilde{\eta}_9}{\tilde{\eta}_1}\nabla^{\langle\mu}F^{\nu\rangle}.$$

- Since $\dot{\sigma}^{\langle\mu\nu\rangle} = \frac{1}{2\eta}\dot{\pi}^{\langle\mu\nu\rangle} - \frac{1}{2\eta^2}\pi^{\mu\nu}\dot{\eta}$, $\tilde{\eta}_1$ leads to a modification of τ_π:

$$\tau_\pi^{\mathrm{IReD}} = \tau_\pi^{\mathrm{DNMR}} + \frac{\tilde{\eta}_1}{2\eta}. \tag{4}$$

Connection to DNMR: Ultrarelativistic hard spheres

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URHS	IReD	DNMR	URHS
$1.66\lambda_{\rm mfp}$	τ_π	$\tilde{\tau}_\pi + \dfrac{\tilde{\eta}_1}{2\eta}$	$2\lambda_{\rm mfp}$
$4\tau_\pi/3$	$\delta_{\pi\pi}$	$\tilde{\delta}_{\pi\pi} + \dfrac{\tilde{\eta}_1}{3\eta} - \dfrac{\tilde{\eta}_2}{2\eta} - \dfrac{\tilde{\eta}_1}{2\eta^2}\overline{\mathcal{H}}\dfrac{\partial\eta}{\partial\beta}$	$4\tilde{\tau}_\pi/3$
$1.69\tau_\pi$	$\tau_{\pi\pi}$	$\tilde{\tau}_{\pi\pi} + \dfrac{\tilde{\eta}_1 - \tilde{\eta}_3}{2\eta}$	$1.69\tilde{\tau}_\pi$
$-0.57\tau_\pi/\beta P$	$\tau_{\pi n}$	$\tilde{\tau}_{\pi n} - \dfrac{\tilde{\eta}_7}{\kappa} - \dfrac{\tilde{\eta}_8}{\kappa^2(\varepsilon+P)}\dfrac{\partial\kappa}{\partial\ln\beta}$	$-0.69\tau_\pi/\beta P$
$-0.57\tau_\pi/\beta$	$\ell_{\pi n}$	$\tilde{\ell}_{\pi n} + \dfrac{\tilde{\eta}_8}{\kappa}$	$-0.69\tilde{\tau}_\pi/\beta$
$0.21\tau_\pi/\beta$	$\lambda_{\pi n}$	$\tilde{\lambda}_{\pi n} + \dfrac{\tilde{\eta}_5}{\kappa} - \dfrac{\tilde{\eta}_8}{\kappa^2}\left(\dfrac{\partial\kappa}{\partial\alpha} + \dfrac{1}{h}\dfrac{\partial\kappa}{\partial\beta}\right)$	$0.24\tilde{\tau}_\pi/\beta$

- In the original DNMR approach, $\mathcal{K}^{\mu\nu} \neq 0$ is simply ignored.
- Properly accounting for $\mathcal{K}^{\mu\nu}$ within IReD gives a 17% difference in τ_π, together with substantial differences in $\tau_{\pi n}/\tau_\pi$, $\ell_{\pi n}/\tau_\pi$ and $\lambda_{\pi n}/\tau_\pi$.

Microscopic implementation: ~~Separation of scales~~

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r	$\tau_{\pi;r}^{\rm IReD}[\lambda_{\rm mfp}]$	$\tau_{\pi;r}^{\rm DNMR}[\lambda_{\rm mfp}]$
0	1.66	2
1	1.79	1.65
2	1.90	1.48
3	2.00	1.39

- The hydro eqs. can be derived from the Boltzmann eq., $k^\mu \partial_\mu f_{\mathbf{k}} = C[f_{\mathbf{k}}]$.
- The collision term $C[f_{\mathbf{k}}]$ drives $f_{\mathbf{k}}$ towards $f_{0\mathbf{k}}$, so $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}}$ is small.
- The moments $\rho_r^{\mu\nu} = \int dK \delta f_{\mathbf{k}} E_{\mathbf{k}}^r k^{\langle\mu} k^{\nu\rangle}$ are linked to the dissipative quantities ($\rho_0^{\mu\nu} = \pi^{\mu\nu}$) and obey

$$\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\mu\nu} = 2\alpha_r^{(2)}\sigma^{\mu\nu} + \text{h.o.t.}, \qquad C_{r-1}^{\mu\nu} = -\sum_{n=0}^{N_2} \mathcal{A}_{rn}\rho_n^{\mu\nu}. \tag{5}$$

- Multiplying Eq. (5) by $\tau_{0r} = [\mathcal{A}^{-1}]_{0r}$ and summing over r gives

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta_0\sigma^{\mu\nu} + \text{h.o.t.}, \qquad \eta_n = \sum_{r=0}^{N_2} \tau_{nr}\alpha_r^{(2)}. \tag{6}$$

- The key to deriving the above Eq. is matching $\rho_r^{\mu\nu}$ to $\pi^{\mu\nu}$.
- DNMR: $\mathcal{A} = \Omega\, \text{diag}(\chi_0, \chi_1, \dots)\Omega^{-1}$ with $\chi_0 \le \chi_1 \le \dots$ and $X_0^{\mu\nu} = \sum_{r=0}^{N_2} \Omega_{0r}^{-1}\rho_r^{\mu\nu}$ is kept in the transient regime (*separation of scales*), leading to

$$\text{DNMR}: \qquad \rho_r^{\mu\nu} = \Omega_{r0}\pi^{\mu\nu} + 2\eta(\mathcal{C}_r - \Omega_{r0})\sigma^{\mu\nu}, \qquad \mathcal{C}_r = \frac{\eta_r}{\eta}, \qquad \Rightarrow \tau_\pi = \chi_0^{-1}, \tag{7}$$

  where the blue term gives rise to $\mathcal{K}^{\mu\nu}$.
- In IReD, $\rho_r^{\mu\nu} = \mathcal{C}_r\pi^{\mu\nu}$, such that $\mathcal{K}^{\mu\nu} = 0$ and $\tau_\pi = \sum_{r=0}^{N_2} \tau_{0r}\mathcal{C}_r$.
- Looking now at $\tau_{\pi;r}\dot{\rho}_r^{\langle\mu\nu\rangle} + \rho_r^{\mu\nu} = 2\eta_r\sigma^{\mu\nu} + \dots$, we find

$$\tau_{\pi;r}^{\rm DNMR} = \chi_r^{-1}, \qquad \tau_{\pi;r}^{\rm IReD} = \sum_{n=0}^{N_2} \tau_{rn}\mathcal{C}_{n;r}, \qquad \mathcal{C}_{n;r} = \frac{\eta_n}{\eta_r}. \tag{8}$$

- As the table shows, $\tau_{\pi;0}^{\rm IReD} < \tau_{\pi;1}^{\rm IReD} < \dots$, thus contradicting the *separation of scales* paradigm.

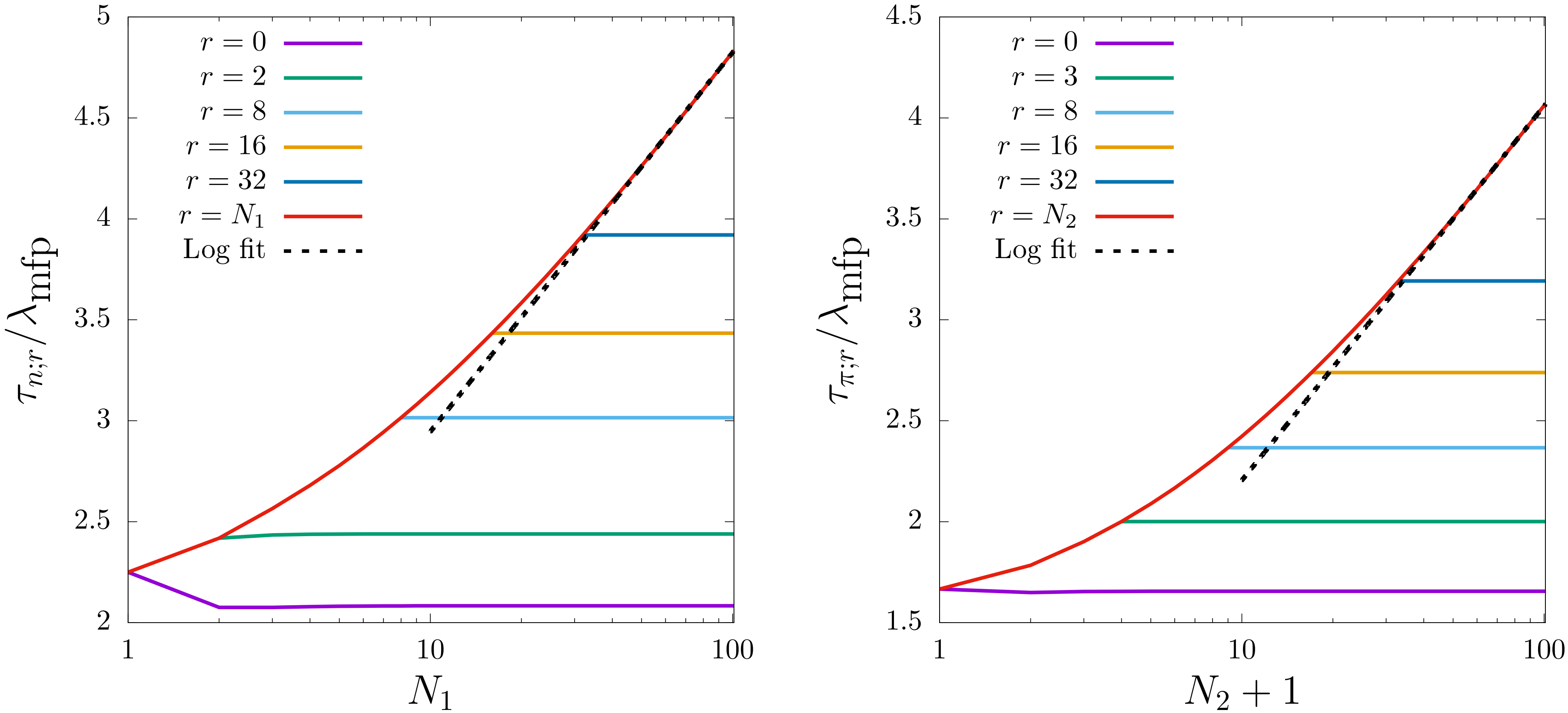
Conclusions and references

- The goal of the IReD [1] approach is to eliminate the parabolic $O(\mathrm{Kn}^2)$ terms appearing in DNMR [2] in favor of the hyperbolic $O(\mathrm{Re}^{-1}\mathrm{Kn})$ ones.
- By this procedure, the transport coefficients and the relaxation times are modified compared to the usual DNMR ones.
- The IReD approach is second order accurate w.r.t. Kn and Re^{-1} and the formal equivalence to DNMR is analytically established.
- The IReD relaxation times no longer satisfy the *separation of scales*.
- The IReD approach is crucial for solving the hydro equations, while in the DNMR approach, the $O(\mathrm{Kn}^2)$ terms are omitted [3].

[1] D. Wagner, A. Palermo, V. E. Ambruș, arXiv:2203.12608.
[2] G. S. Denicol, H. Niemi, E. Molnar, D. H. Rischke, Phys. Rev. D **85** (2012) 114047.
[3] E. Molnár, H. Niemi, G. S. Denicol, D. H. Rischke, Phys. Rev. D **89** (2014) 074010.

Appendix

Higher order relaxation times



- Fitting indicates a logarithmic increase of $\tau_{n;r}$ and $\tau_{\pi;r}$ with r.
- Convergence with respect to $N_1 = N_2 + 1$ very fast.

Hard spheres collision matrix

- The collision matrix is linked with the expansion of $\delta f_{\mathbf{k}}$ with respect to a complete basis,

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_n^{\mu_1\cdots\mu_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \mathcal{H}_{\mathbf{k}n}^{(\ell)},$$

where $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ is defined such that $\rho_n^{\mu_1\cdots\mu_\ell} \equiv \int dK E_{\mathbf{k}}^n k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f_{\mathbf{k}}$.

- The linearized collision integrals are given by

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{\mathbf{kk}'\to\mathbf{pp}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^{r-1} k^{\langle\nu_1} \cdots k^{\nu_\ell\rangle}$$
$$\times \left(\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle\nu_1} \cdots k_{\nu_\ell\rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle\nu_1} \cdots k'_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle\nu_1} \cdots p_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle\nu_1} \cdots p'_{\nu_\ell\rangle}\right),$$

- In the case of the UR ideal HS gas, $W_{\mathbf{kk}'\to\mathbf{pp}'} = s(2\pi)^6 \delta^{(4)}(k+k'-p-p') \frac{\sigma_T \nu}{4\pi}$ and

$$\mathcal{A}_{r=0,n}^{(1)} = \frac{16(-\beta)^n g^2}{\lambda_{\rm mfp}(n+3)!} \left[S_n^{(1)}(N_1) - \frac{\delta_{n0}}{2}\right], \qquad \mathcal{A}_{r=0,n}^{(2)} = \frac{432 g^2 (-\beta)^n}{\lambda_{\rm mfp}(n+5)!} S_n^{(2)}(N_2),$$

$$\mathcal{A}_{r>0,n\le r}^{(1)} = \frac{g^2 \beta^{n-r} (r+2)! [n(r+4)-r]}{\lambda_{\rm mfp}(n+2)! r(n+3)} \times \left(\delta_{nr} + \delta_{n0} - \frac{2}{r+1}\right),$$

$$\mathcal{A}_{r>0,n\le r}^{(2)} = \frac{g^2 \beta^{n-r} (r+4)! (n+1)}{\lambda_{\rm mfp}(n+5)! r(r+1)} \times (9n + nr - 4r) \left(\delta_{nr} - \frac{2}{r+2}\right),$$

while $\mathcal{A}_{r>0,n>r}^{(1)} = \mathcal{A}_{r>0,n>r}^{(2)} = 0$ and $S_n^{(\ell)}(N_\ell) = \sum_{m=n}^{N_\ell} \binom{m}{n} \frac{1}{(m+\ell)(m+\ell+1)}$.