Inverse Reynolds-dominance approach to transient fluid dynamics

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In the Landau frame \( (T^\mu_\nu u^\nu = \varepsilon u^\mu) \), \( N^\mu \) and \( T^{\mu\nu} \) can be decomposed as

\[
N^\mu = n u^\mu + n^\mu, \quad T^{\mu\nu} = \varepsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{1}
\]

The cons. eqs. \( \partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0 \) give the evolution of \( n, \varepsilon \) and \( u^\mu \).

\( \Pi, n^\mu \) and \( \pi^{\mu\nu} \) are regarded to be small and of \( O(Re^{-1}) \), while gradients are taken as \( O(Kn) \).

2nd order hydro gives the evolution eqs. for \( \Pi, n^\mu \) and \( \pi^{\mu\nu} \) up to second order w.r.t. \( Kn \) and \( Re^{-1} \):

\[
\begin{align*}
\tau_\Pi \dot{\Pi} + \Pi &= -\zeta \theta + \mathcal{J} + \mathcal{K} + \mathcal{R}, \\
\tau_n \dot{n}^{\langle \mu \rangle} + n^\mu &= \kappa I^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu, \\
\tau_\pi \dot{\pi}^{\langle \mu\nu \rangle} + \pi^{\mu\nu} &= 2\eta \sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu},
\end{align*}
\tag{2}
\]

containing the 1st order (NS) limit as well as 2nd order terms of orders \( O(Re^{-1}Kn), O(Kn^2) \) and \( O(Re^{-2}) \).

Problem: The \( O(Kn^2) \) are parabolic and must be discarded.
Let us look at the $O(Kn^2)$ and $O(Re^{-1}Kn)$ terms (just $K^{\mu\nu}$ and $J^{\mu\nu}$ for brevity):

\[
K^{\mu\nu} = \tilde{\eta}_1 \omega^{\lambda \langle \mu \omega \nu \rangle} + \tilde{\eta}_2 \theta \sigma^{\mu\nu} + \tilde{\eta}_3 \sigma^{\lambda \langle \mu \sigma \nu \rangle} + \tilde{\eta}_4 \sigma^{\langle \mu \omega \nu \rangle} + \tilde{\eta}_5 I^{(\mu I \nu)} + \tilde{\eta}_6 F^{\langle \mu F \nu \rangle} + \tilde{\eta}_7 I^{(\mu F \nu)} + \tilde{\eta}_8 \nabla^{\langle \mu I \nu \rangle} + \tilde{\eta}_9 \nabla^{\langle \mu F \nu \rangle},
\]

\[
J^{\mu\nu} = 2\tau_{\pi \pi} \pi^{\langle \mu \omega \nu \rangle} - \delta_{\pi \pi} \pi^{\mu\nu} \theta - \tau_{\pi \pi} \pi^{\lambda \langle \mu \sigma \nu \rangle} + \lambda_{\pi \Pi} \Pi \sigma^{\mu\nu} - \tau_{\pi \mu} n^{\langle \mu F \nu \rangle} + \ell_{\pi \mu} \nabla^{\langle \mu n \nu \rangle} + \lambda_{\pi \mu} n^{\langle \mu I \nu \rangle},
\]

The main idea of the IReD = *Inverse Reynolds Dominance* approach is to render $K^{\mu\nu}$ of $O(Re^{-1}Kn)$ by trading one power of $Kn$ for one of $Re^{-1}$ via

\[
\theta \simeq -\Pi/\zeta, \quad I^{\mu} \simeq n^{\mu}/\kappa, \quad \sigma^{\mu\nu} \simeq \pi^{\mu\nu}/2\eta.
\]  

The terms in red can be related to

\[
\dot{\sigma}^{\langle \mu\nu \rangle} = -\omega^{\lambda \langle \mu \omega \nu \rangle} - \frac{2}{3} \theta \sigma^{\mu\nu} - \sigma^{\lambda \langle \mu \sigma \nu \rangle} - \tilde{\eta}_6 F^{\langle \mu F \nu \rangle} - \frac{D_{20}A}{(\varepsilon + P)^3} I^{(\mu F \nu)} - \tilde{\eta}_9 \nabla^{\langle \mu F \nu \rangle}.
\]

Since $\dot{\sigma}^{\langle \mu\nu \rangle} = \frac{1}{2\eta} \pi^{\langle \mu\nu \rangle} - \frac{1}{2\eta^2} \pi^{\mu\nu} \tilde{\eta}_1$, $\tilde{\eta}_1$ leads to a modification of $\tau_{\pi}$:

\[
\tau^{\text{IReD}}_{\pi} = \tau_{\pi}^{\text{DNMR}} + \frac{\tilde{\eta}_1}{2\eta}.
\]
Connection to DNMR: Ultrarelativistic hard spheres

<table>
<thead>
<tr>
<th>URHS</th>
<th>IReD</th>
<th>DNMR</th>
<th>URHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.66\lambda_{\text{mfp}}$</td>
<td>$\tau_\pi$</td>
<td>$\tilde{\tau}_\pi + \frac{\tilde{\eta}_1}{2\eta}$</td>
<td>$2\lambda_{\text{mfp}}$</td>
</tr>
<tr>
<td>$4\tau_\pi/3$</td>
<td>$\delta_{\pi\pi}$</td>
<td>$\tilde{\delta}_{\pi\pi} + \frac{\tilde{\eta}_1}{3\eta} - \frac{\tilde{\eta}_2}{2\eta} - \frac{\tilde{\eta}_1}{2\eta^2}\mathcal{H}\frac{\partial\eta}{\partial\beta}$</td>
<td>$4\tilde{\tau}_\pi/3$</td>
</tr>
<tr>
<td>$1.69\tau_\pi$</td>
<td>$\tau_{\pi\pi}$</td>
<td>$\tilde{\tau}_{\pi\pi} + \frac{\tilde{\eta}_1 - \tilde{\eta}_3}{2\eta}$</td>
<td>$1.69\tilde{\tau}_\pi$</td>
</tr>
<tr>
<td>$-0.57\tau_\pi/\beta P$</td>
<td>$\tau_{\pi n}$</td>
<td>$\tilde{\tau}_{\pi n} - \frac{\tilde{\eta}_7}{\kappa} - \frac{\tilde{\eta}_8}{\kappa^2(\varepsilon + P)}\frac{\partial\kappa}{\partial\ln\beta}$</td>
<td>$-0.69\tau_\pi/\beta P$</td>
</tr>
<tr>
<td>$-0.57\tau_\pi/\beta$</td>
<td>$\ell_{\pi n}$</td>
<td>$\tilde{\ell}_{\pi n} + \frac{\tilde{\eta}_8}{\kappa}$</td>
<td>$-0.69\tilde{\tau}_\pi/\beta$</td>
</tr>
<tr>
<td>$0.21\tau_\pi/\beta$</td>
<td>$\lambda_{\pi n}$</td>
<td>$\tilde{\lambda}_{\pi n} + \frac{\tilde{\eta}_5}{\kappa} - \frac{\tilde{\eta}_8}{\kappa^2}\left(\frac{\partial\kappa}{\partial\alpha} + \frac{1}{\hbar}\frac{\partial\kappa}{\partial\beta}\right)$</td>
<td>$0.24\tilde{\tau}_\pi/\beta$</td>
</tr>
</tbody>
</table>

▶ In the original DNMR approach, $\mathcal{K}^{\mu\nu} \neq 0$ is simply ignored.

▶ Properly accounting for $\mathcal{K}^{\mu\nu}$ within IReD gives a 17% difference in $\tau_\pi$, together with substantial differences in $\tau_{\pi n}/\tau_\pi$, $\ell_{\pi n}/\tau_\pi$ and $\lambda_{\pi n}/\tau_\pi$. 

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The hydro eqs. can be derived from the Boltzmann eq., $k^\mu \partial_\mu f_k = C[f_k]$.
The collision term $C[f_k]$ drives $f_k$ towards $f_{0k}$, so $\delta f_k = f_k - f_{0k}$ is small.
The moments $\rho^{\mu\nu}_r = \int dK \delta f_k E^r_k k^{(\mu k \nu)}$ are linked to the dissipative quantities ($\rho^{\mu\nu}_0 = \pi^{\mu\nu}$) and obey

$$\dot{\rho}^{(\mu\nu)}_r - C^{\mu\nu}_r = 2\alpha^{(2)}_r \sigma^{\mu\nu} + \text{h.o.t.}, \quad C^{\mu\nu}_r = -\sum_{n=0}^{N_2} A_{rn} \rho^{\mu\nu}_n.$$  \hfill (5)

Multiplying Eq. (5) by $r_0 = [A^{-1}]_0r$ and summing over $r$ gives

$$\tau \dot{\pi}^{(\mu\nu)} + \pi^{\mu\nu} = 2\eta_0 \sigma^{\mu\nu} + \text{h.o.t.}, \quad \eta_n = \sum_{r=0}^{N_2} \tau_{nr} \alpha^{(2)}_r.$$  \hfill (6)

The key to deriving the above Eq. is matching $\rho^{\mu\nu}_r$ to $\pi^{\mu\nu}$.

DNMR: $A = \Omega \text{ diag}(\chi_0, \chi_1, \ldots) \Omega^{-1}$ with $\chi_0 \leq \chi_1 \leq \ldots$ and $X^{\mu\nu}_0 = \sum_{r=0}^{N_2} \Omega_{0r}^{-1} \rho^{\mu\nu}_r$ is kept in the transient regime (separation of scales), leading to

$$\text{DNMR} : \quad \rho^{\mu\nu}_r = \Omega_{r0} \pi^{\mu\nu} + 2\eta(C_r - \Omega_{r0}) \sigma^{\mu\nu}, \quad C_r = \frac{\eta_r}{\eta}, \quad \Rightarrow \tau_\pi = \chi_0^{-1},$$  \hfill (7)

where the blue term gives rise to $K^{\mu\nu}$.

In IReD, $\rho^{\mu\nu}_r = C_r \pi^{\mu\nu}$, such that $K^{\mu\nu} = 0$ and $\tau_\pi = \sum_{r=0}^{N_2} \tau_{0r} C_r$.

Looking now at $\tau^{\mu\nu}_r \dot{\rho}^{(\mu\nu)}_r + \rho^{\mu\nu}_r = 2\eta_r \sigma^{\mu\nu} + \ldots$, we find

$$\tau^{\text{DNMR}}_\pi = \chi_r^{-1}, \quad \tau^{\text{IReD}}_\pi = \sum_{n=0}^{N_2} \tau_{rn} C_{n;r}, \quad C_{n;r} = \frac{\eta_n}{\eta_r}.$$  \hfill (8)

As the table shows, $\tau^{\text{IReD}}_{\pi;0} < \tau^{\text{IReD}}_{\pi;1} < \ldots$, thus contradicting the separation of scales paradigm.
The goal of the IReD [1] approach is to eliminate the parabolic $O(Kn^2)$ terms appearing in DNMR [2] in favor of the hyperbolic $O(Re^{-1}Kn)$ ones.

By this procedure, the transport coefficients and the relaxation times are modified compared to the usual DNMR ones.

The IReD approach is second order accurate w.r.t. $Kn$ and $Re^{-1}$ and the formal equivalence to DNMR is analytically established.

The IReD relaxation times no longer satisfy the separation of scales.

The IReD approach is crucial for solving the hydro equations, while in the DNMR approach, the $O(Kn^2)$ terms are omitted [3].

Appendix
Higher order relaxation times

▶ Fitting indicates a logarithmic increase of $\tau_{n;r}$ and $\tau_{\pi;r}$ with $r$.
▶ Convergence with respect to $N_1 = N_2 + 1$ very fast.
The collision matrix is linked with the expansion of $\delta f_k$ with respect to a complete basis,

$$\delta f_k = f_{0k} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_{n1\cdots\mu\ell}^k \mathcal{H}_{kn}^{(\ell)},$$

where $\mathcal{H}_{kn}^{(\ell)}$ is defined such that $\rho_{n1\cdots\mu\ell}^k \equiv \int dK E_k^n k^{(\mu} \cdots k^{\ell)} \delta f_k$.

The linearized collision integrals are given by

$$A_{r_n}^{(\ell)} = \frac{1}{\nu(2\ell + 1)} \int dKdK' dPdP' W_{kk' \rightarrow pp'} f_{0k} f_{0k'} E_k^{r-1} k^{(\nu_1} \cdots k^{\nu_\ell)}$$

$$\times \left( \mathcal{H}_{kn}^{(\ell)} k^{(\nu_1} \cdots k^{\nu_\ell)} + \mathcal{H}_{k'n}^{(\ell)} k'^{(\nu_1} \cdots k'^{(\nu_\ell)} - \mathcal{H}_{p'n}^{(\ell)} p'^{(\nu_1} \cdots p'^{(\nu_\ell)} - \mathcal{H}_{p'n}^{(\ell)} p'^{(\nu_1} \cdots p'^{(\nu_\ell)} \right),$$

In the case of the UR ideal HS gas, $W_{kk' \rightarrow pp'} = s(2\pi)^6 \delta^{(4)}(k + k' - p - p') \frac{\sigma T}{4\pi}$ and

$$A_{r=0,n}^{(1)} = \frac{16(-\beta)^n g^2}{\lambda_{mfp}(n + 3)!} \left[ S_n^{(1)}(N_1) - \delta_{n0} \right], \quad A_{r=0,n}^{(2)} = \frac{432g^2(-\beta)^n}{\lambda_{mfp}(n + 5)!} S_n^{(2)}(N_2),$$

$$A_{r>0,n \leq r}^{(1)} = \frac{g^2 \beta^{n-r}(r + 2)!(n(r + 4) - r)}{\lambda_{mfp}(n + 2)!r(n + 3)} \left( \delta_{nr} + \delta_{n0} - \frac{2}{r + 1} \right), \quad A_{r>0,n \leq r}^{(2)} = \frac{g^2 \beta^{n-r}(r + 4)!(n + 1)}{\lambda_{mfp}(n + 5)!r(r + 1)} \left( \delta_{nr} - \frac{2}{r + 2} \right),$$

while $A_{r>0,n > r}^{(1)} = A_{r>0,n > r}^{(2)} = 0$ and

$$S_n^{(\ell)}(N_\ell) = \sum_{m=n}^{N_\ell} \binom{m}{n} \frac{1}{(m + \ell)(m + \ell + 1)}.$$

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