Scattering Amplitudes: QCD & Gravity applications

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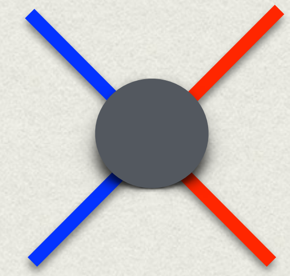
Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Scattering Amplitudes

* Particle interactions

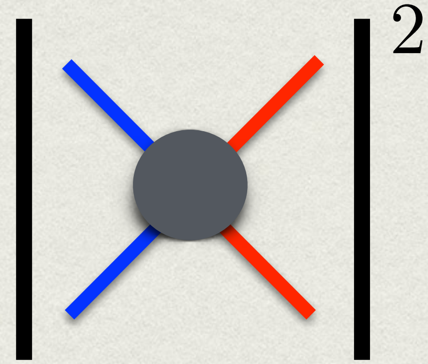
$$1 + 2 \to 3 + 4$$

2->2 scattering



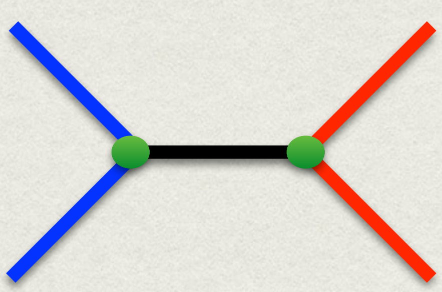
The simplest process

* Quantum probability $\sim$

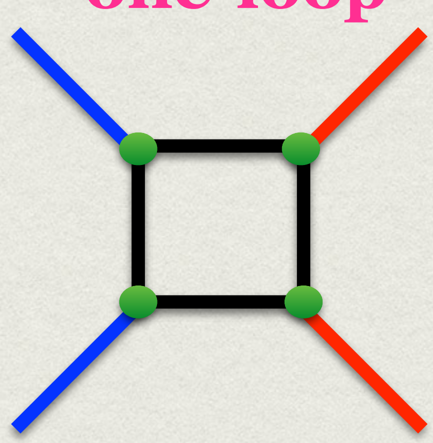


* Amplitudes ~ Feynman diagrams

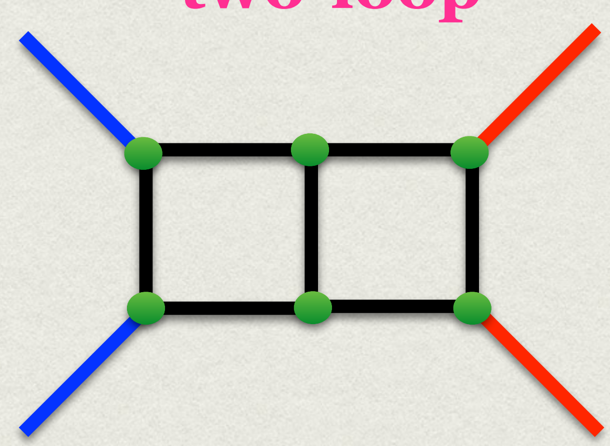
Tree-level



one-loop



two-loop



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Perturbation expansion

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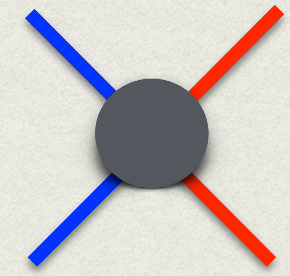
Scattering Amplitudes

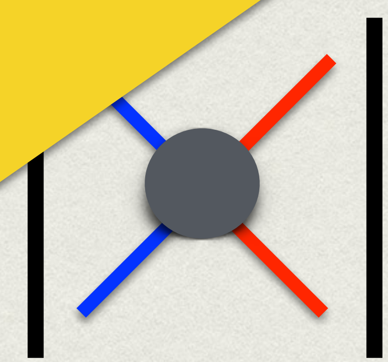
* Particle interactions



The simplest process

$1 +$

* Quantum p

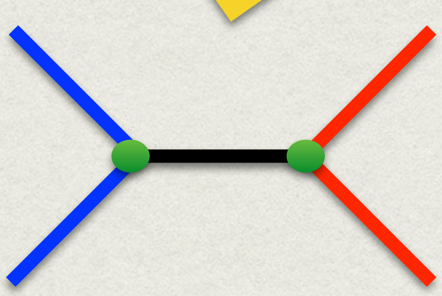


* A … Feynman diagrams

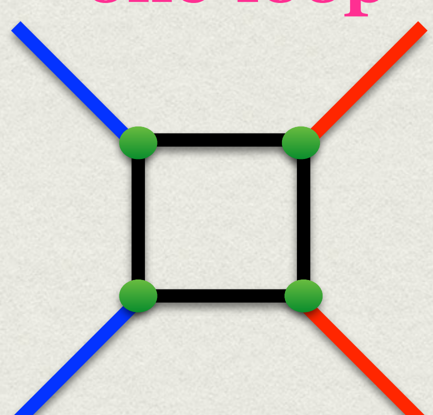
$$e^{ix} = \left(1 + \frac{-x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} + \dots\right) + i\left(x + \frac{-x^3}{3!} + \frac{x^5}{5!} + \frac{-x^7}{7!} + \dots\right)$$
$$= \cos x + i \sin x$$

precision depends on the number of couplings =

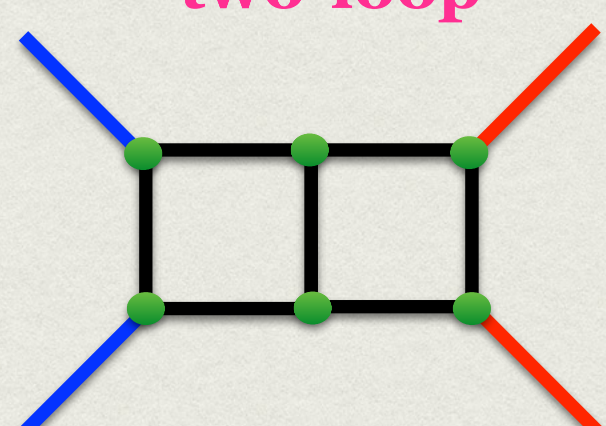
Tr…vel



one-loop



two-loop



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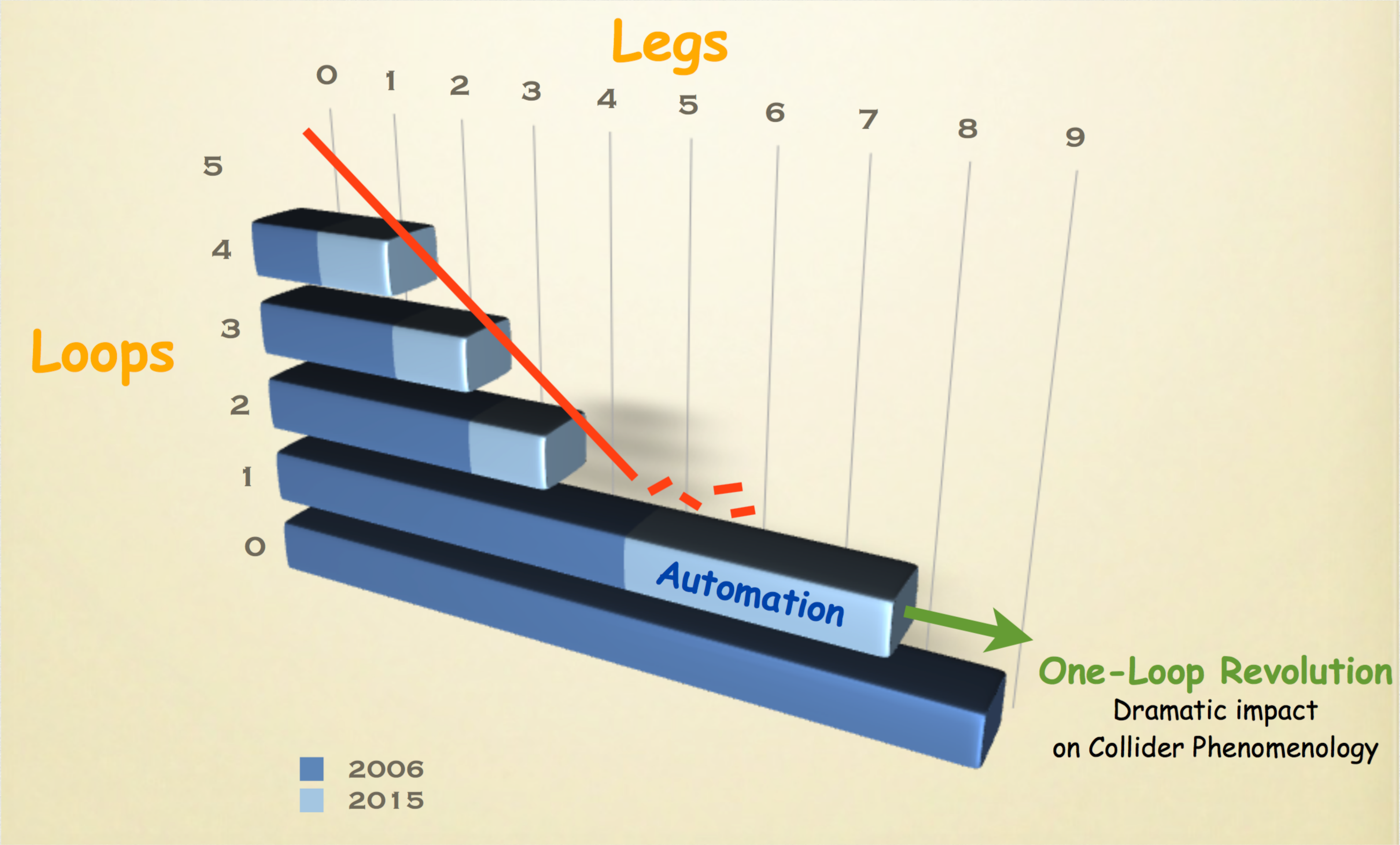
Dyson series

Perturbation expansion

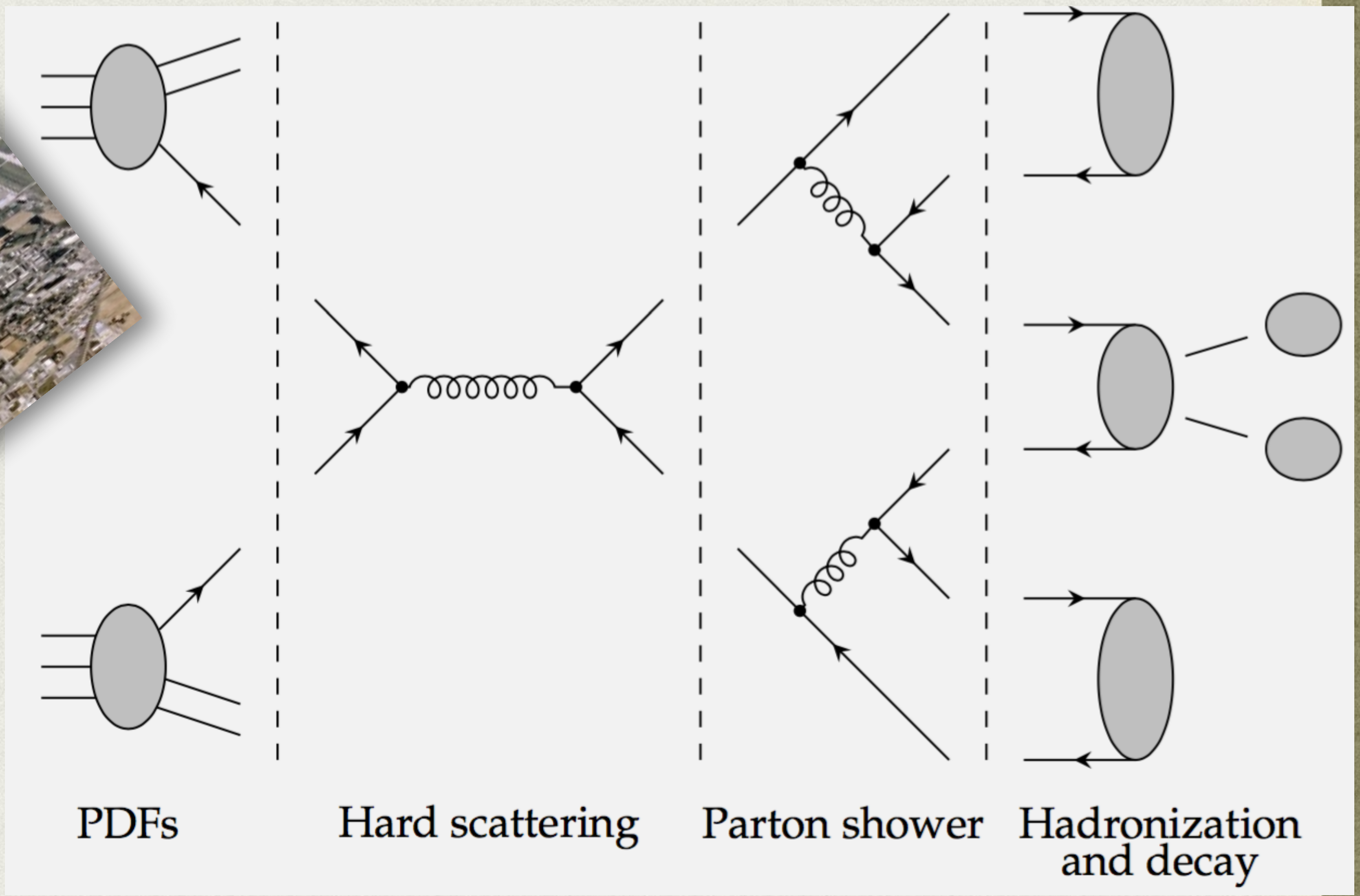

Legs
0
1
2
3
4
5
6
7
8
9
Loops
5
4
3
2
1
0
Automation
One-Loop Revolution
Dramatic impact
on Collider Phenomenology
2006
2015

Outline

- Scattering amplitudes in a nutshell
- Tree-level
- Multi-loop
- Applications
- Conclusions

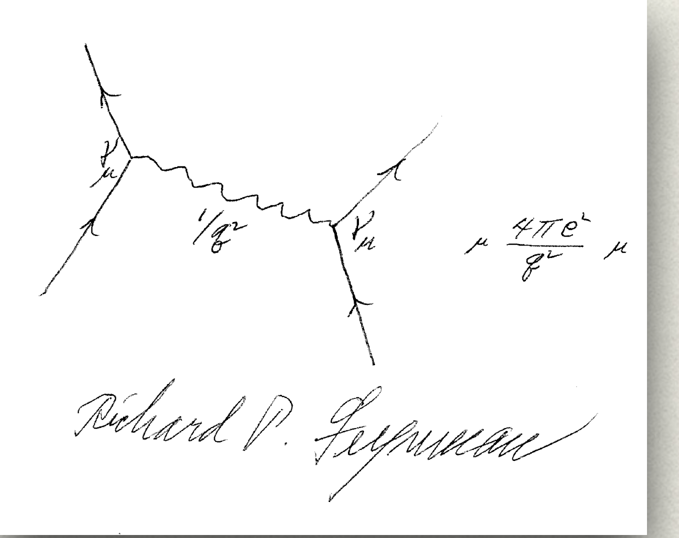
Introduction

- LHC results demand a refinement of our understanding of the SM physics
  High precision predictions in background processes —> New physics at the TeV scale



- Relevant observables
  —> computation of Quantum Chromodynamics (QCD) **Scattering Amplitudes**

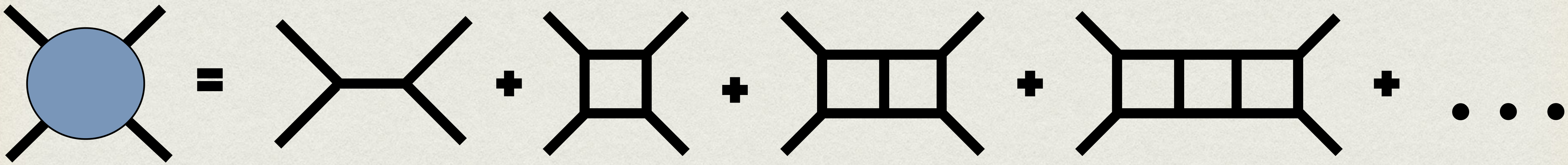
Introduction



- **Scattering Amplitudes**

  — Practical applications in particle physics
  — Mathematical elegance
  — Gauge invariant objects

- Perturbative expansion



Motivation

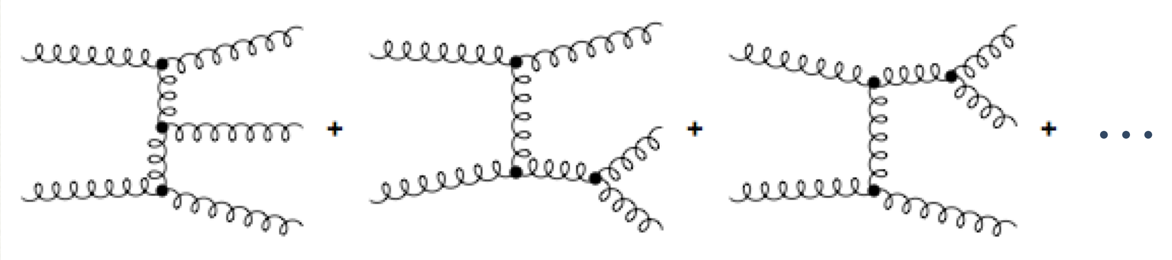
- Compute the uncomputable
- Simplify the calculations in High-Energy Physics.
- Discover hidden properties of Quantum Field Theories
- Towards NNLO is the **Next Frontier.**

Tree-level scattering amplitudes

Tree-level amplitudes

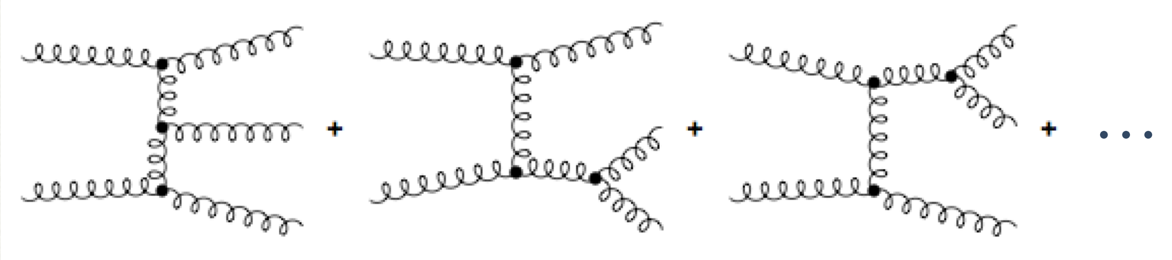
gg —> ggg @ tree-level

— Feynman diagram —> gauge dependent quantities
— A factorial growth in the number of terms



Tree-level amplitudes

gg —> ggg @ tree-level



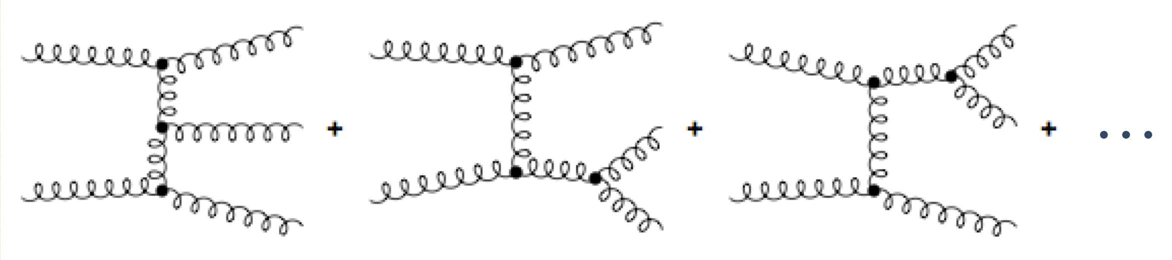
— Feynman diagram —> gauge dependent quantities
— A factorial growth in the number of terms

Result of a brute force calculation (1 of 25 diags)

$$f^{\text{Glu1 Glu2 Glu6}}\Bigg(\frac{1}{(-p(3)-p(4)-p(5))^2}$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +$$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) -$$
$$(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(4)))\,f^{\text{Glu3 Glu6 \$AL\$33004}}\,f^{\text{Glu4 Glu5 \$AL\$33004}} - \frac{1}{(-p(3)-p(4)-p(5))^2}$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +$$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) -$$
$$(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu6 \$AL\$33006}}\,f^{\text{Glu4 Glu5 \$AL\$33006}} -$$
$$\frac{1}{(-p(3)-p(4)-p(5))^2}\,g_s^3\,((p(1)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) -$$
$$(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) +$$
$$(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) -$$
$$(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))))$$
$$(\varepsilon^*(p(4))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu5 \$AL\$33002}}\,f^{\text{Glu4 Glu6 \$AL\$33002}} + \frac{1}{(-p(3)-p(4)-p(5))^2}$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +$$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) -$$
$$(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(4)))\,f^{\text{Glu3 Glu5 \$AL\$33003}}\,f^{\text{Glu4 Glu6 \$AL\$33003}} -$$
$$\frac{1}{(-p(3)-p(4)-p(5))^2}\,g_s^3\,((p(1)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) -$$
$$(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) +$$
$$(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) -$$
$$(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))))$$
$$(\varepsilon^*(p(4))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu4 \$AL\$33001}}\,f^{\text{Glu5 Glu6 \$AL\$33001}} + \frac{1}{(-p(3)-p(4)-p(5))^2}$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +$$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) -$$
$$(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu4 \$AL\$33005}}\,f^{\text{Glu5 Glu6 \$AL\$33005}}\Bigg)$$

Tree-level amplitudes

gg —> ggg @ tree-level



— Feynman diagram —> gauge dependent quantities
— A factorial growth in the number of terms

Result of a brute force calculation (1 of 25 diags)

$$
f^{\text{Glu1 Glu2 Glu6}}\Bigg(\frac{1}{(-p(3)-p(4)-p(5))^2}
g_s^3\,\big((p(1)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +
(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +
(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) -
(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5)))\big)\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(4)))\, f^{\text{Glu3 Glu6 \$AL\$33004}}\, f^{\text{Glu4 Glu5 \$AL\$33004}} - \frac{1}{(-p(3)-p(4)-p(5))^2}
$$

$$
g_s^3\,\big((p(1)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +
(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +
(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) -
(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))\big)\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(5)))\, f^{\text{Glu3 Glu6 \$AL\$33006}}\, f^{\text{Glu4 Glu5 \$AL\$33006}} -
$$

$$
\frac{1}{(-p(3)-p(4)-p(5))^2} g_s^3\,\big((p(1)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) -
(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) +
(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) -
(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3)))\big)
(\varepsilon^*(p(4))\cdot\varepsilon^*(p(5)))\, f^{\text{Glu3 Glu5 \$AL\$33002}}\, f^{\text{Glu4 Glu6 \$AL\$33002}} + \frac{1}{(-p(3)-p(4)-p(5))^2}
$$

$$
g_s^3\,\big((p(1)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +
(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5))) +
(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))) -
(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5)))\big)\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(4)))\, f^{\text{Glu3 Glu5 \$AL\$33003}}\, f^{\text{Glu4 Glu6 \$AL\$33003}} -
$$

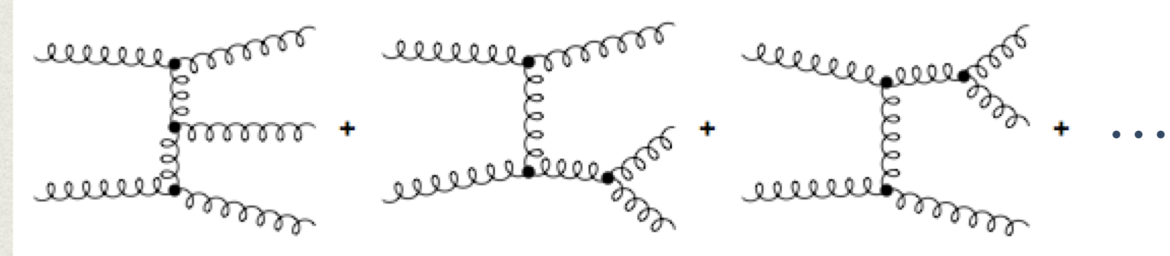
$$
\frac{1}{(-p(3)-p(4)-p(5))^2} g_s^3\,\big((p(1)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) -
(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) +
(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3))) + (p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) -
(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3))) - (p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3)))\big)
(\varepsilon^*(p(4))\cdot\varepsilon^*(p(5)))\, f^{\text{Glu3 Glu4 \$AL\$33001}}\, f^{\text{Glu5 Glu6 \$AL\$33001}} + \frac{1}{(-p(3)-p(4)-p(5))^2}
$$

$$
g_s^3\,\big((p(1)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(2)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2))) - (p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +
(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) + (p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4))) +
(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) - (p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))) -
(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))\big)\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(5)))\, f^{\text{Glu3 Glu4 \$AL\$33005}}\, f^{\text{Glu5 Glu6 \$AL\$33005}}\Bigg)
$$

Presence of colour structures
&
Kinematics

Tree-level amplitudes

gg —> ggg @ tree-level

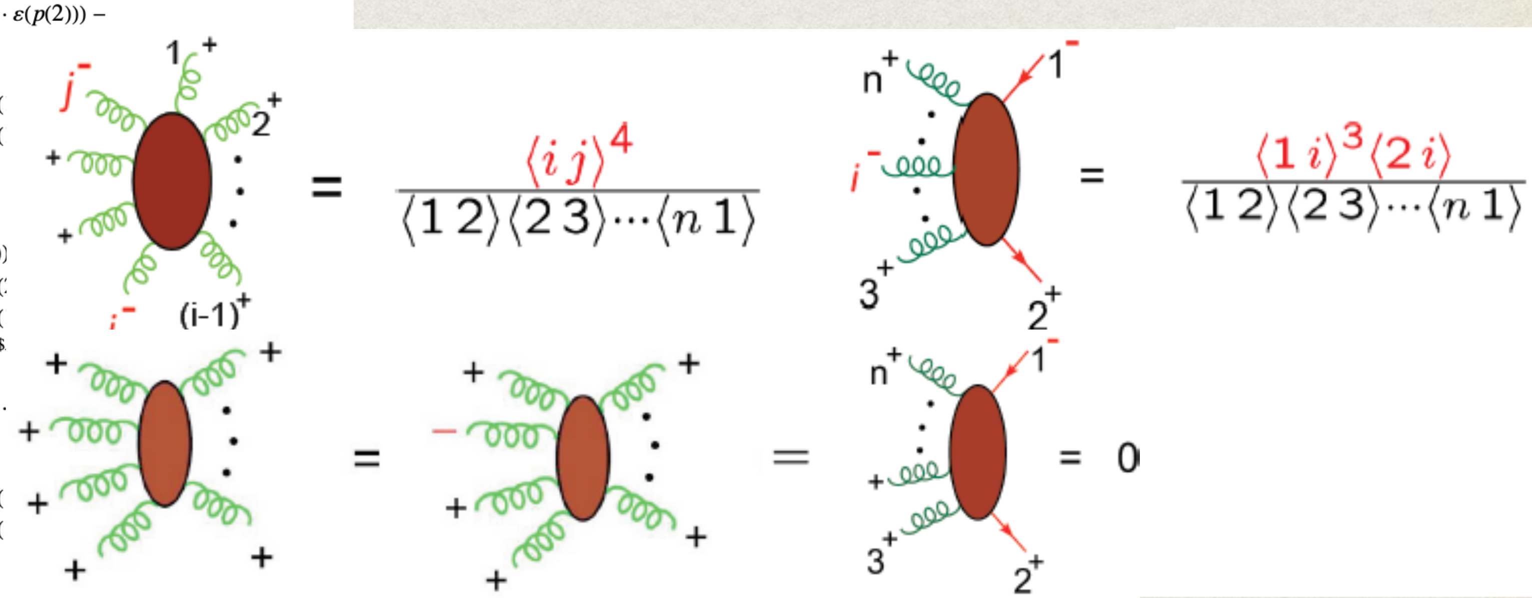


— Feynman diagram —> gauge dependent quantities
— A factorial growth in the number of terms

Result of a brute force calculation (1 of 25 diags)

$$f^{\text{Glu1 Glu2 Glu6}}\left(\frac{1}{(-p(3)-p(4)-p(5))^2}\right.$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(2)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5)))+$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5)))+(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5)))+(p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5)))+$$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5)))-(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5)))-(p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5)))-$$
$$(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(4)))\,f^{\text{Glu3 Glu6 \$AL\$33004}}\,f^{\text{Glu4 Glu5 \$AL\$33004}}-\frac{1}{(-p(3)-p(4)-p(5))^2}$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(2)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+(p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+$$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))-(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))-(p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))-$$
$$(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu6 \$AL\$33006}}\,f^{\text{Glu4 Glu5 \$AL\$33006}}-$$
$$\frac{1}{(-p(3)-p(4)-p(5))^2}\,g_s^3\,((p(1)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(2)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-$$
$$(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+$$
$$(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+(p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+(p(2)\cdot\varepsilon(p($$
$$(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3)))-(p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3)))-(p(5)\cdot\varepsilon(p($$
$$(\varepsilon^*(p(4))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu5 \$AL\$33002}}\,f^{\text{Glu4 Glu6 \$AL\$33002}}+\frac{1}{(-p(3)-p(4)-p(5))^2}$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(2)\cdot\varepsilon^*(p(5)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(1)\cdot\varepsilon(p(2))$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5)))+(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(5)))+(p(5)\cdot\varepsilon(p($$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5)))-(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5)))-(p(4)\cdot\varepsilon(p($$
$$(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(5))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(4)))\,f^{\text{Glu3 Glu5 \$AL\$33003}}\,f^{\text{Glu4 Glu6 \$}}$$
$$\frac{1}{(-p(3)-p(4)-p(5))^2}\,g_s^3\,((p(1)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(2)\cdot\varepsilon^*(p(3)))\,(\varepsilon(p(1))\cdot$$
$$(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+$$
$$(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+(p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(3)))+(p(2)\cdot\varepsilon(p($$
$$(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3)))-(p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(3)))-(p(5)\cdot\varepsilon(p($$
$$(\varepsilon^*(p(4))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu4 \$AL\$33001}}\,f^{\text{Glu5 Glu6 \$AL\$33001}}+\frac{1}{(-p(3)-p(4)-p(5))^2}$$
$$g_s^3\,((p(1)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(2)\cdot\varepsilon^*(p(4)))\,(\varepsilon(p(1))\cdot\varepsilon(p(2)))-(p(1)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+$$
$$(p(3)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+(p(4)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+(p(5)\cdot\varepsilon(p(2)))\,(\varepsilon(p(1))\cdot\varepsilon^*(p(4)))+$$
$$(p(2)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))-(p(3)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))-(p(4)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4)))-$$
$$\left.(p(5)\cdot\varepsilon(p(1)))\,(\varepsilon(p(2))\cdot\varepsilon^*(p(4))))\,(\varepsilon^*(p(3))\cdot\varepsilon^*(p(5)))\,f^{\text{Glu3 Glu4 \$AL\$33005}}\,f^{\text{Glu5 Glu6 \$AL\$33005}}\right)$$

Presence of colour structures
&
Kinematics



Maximally Helicity Violating (**MHV**) amplitudes

[Parke and Taylor (1986)]

Tree-level amplitudes

In QCD any amplitude can be decomposed as

$$\mathcal{A}_n\left(\{k_i, a_i, h_i\}\right) = A_n\left(1^{h_1}, 2^{h_2}, \cdots, n^{h_n}\right) \times \text{Colour Structure} + \text{all non-cyclic permutations}$$

Momenta

Color

Helicities

Primitive amplitudes
depends on Lorentz variables only

Can contain traces or
products of generators T

At tree-level

For the **n**-gluon tree-level amplitude, the **colour decomposition** is

$$\mathcal{A}_{\rm n}^{\rm tree}\left(\{k_i, a_i, h_i\}\right) = g^{n-2} \mathrm{Tr}\left(T^{a_1} T^{a_2} \cdots T^{a_n}\right) A_{\rm n}^{\rm tree}\left(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}\right)$$ + all non-cyclic permutations

Properties between amplitudes

- Reflection invariance
- Cyclic invariance

→ **(n-1)!** Independent amplitudes

$$\mathcal{A}_n^{\rm tree}(\{p_i, h_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A_n^{\rm tree}(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n}))$$

Tree-level amplitudes

In QCD any amplitude can be decomposed as

$$\mathcal{A}_n\left(\{k_i, a_i, h_i\}\right) = A_n\left(1^{h_1}, 2^{h_2}, \cdots, n^{h_n}\right) \times \text{Colour Structure}$$
$$+ \text{ all non-cyclic permutations}$$

Momenta — Color — Helicities

Primitive amplitudes
depends on Lorentz variables only

Can contain traces or
products of generators T

At tree-level

An alternative representation

[Del Duca, Frizzo and Maltoni (1999)]
[Del Duca, Dixon and Maltoni (1999)]

$$\mathcal{A}_n^{\text{tree}}(\{p_i, h_i, a_i\}) = (ig)^{n-2}\ f^{a_1 a_2 x_1} f^{x_1 a_3 x_2} \cdots f^{x_{n-3} a_{\sigma_{n-1}} a_b} A_n^{\text{tree}}(1^{h_1}, \sigma(2^{h_2}), \ldots, n^h)$$ + all non-cyclic permutations

Properties between amplitudes

- Kleiss-Kuijf relations

$$A_n^{\text{tree}}(1, \alpha_1, ..., \alpha_j, n, \beta_1, ..., \beta_{n-2-j}) = (-1)^{n-2-j} \sum_{\sigma \in \vec{\alpha} \sqcup \vec{\beta}^T} A_n^{\text{tree}}(1, \sigma_1, ..., \sigma_{n-2-j}, n)$$

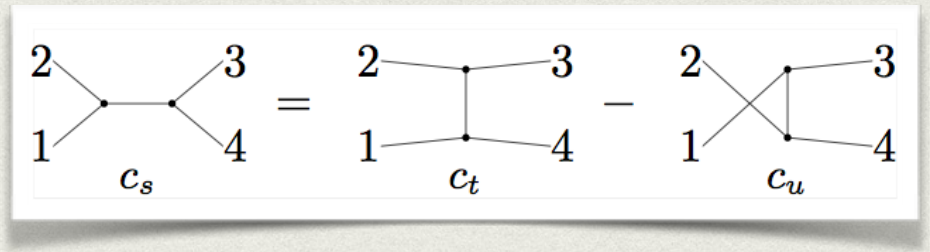
→ **(n-2)!** Independent amplitudes

[Kleiss and Kuijf (1989)]

$$\mathcal{A}_n^{\text{tree}}(\{p_i, h_i, a_i\}) = (ig)^{n-2} \sum_{\sigma \in S_{n-2}} f^{a_1 a_2 x_1} f^{x_1 a_3 x_2} \cdots f^{x_{n-3} a_{\sigma_{n-1}} a_b} A_n^{\text{tree}}(1^{h_1}, \sigma(2^{h_2}), \ldots, n^h)$$

Tree-level amplitudes

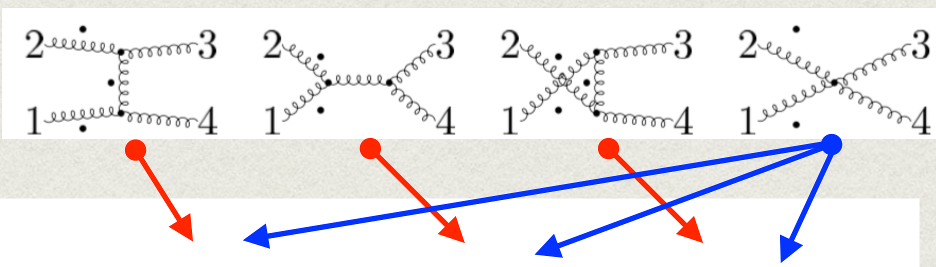
Jacobi Relation (colour) **colour-kinematics duality**



$$c_s = c_t - c_u$$
$$f^{a_1 a_2 b} f^{a_3 a_4 b} = f^{a_4 a_1 b} f^{a_2 a_3 b} - f^{a_1 a_3 b} f^{a_2 a_4 b}$$
$$f^{a_1 a_2 b} T^b = T^{a_1} T^{a_2} - T^{a_2} T^{a_1}$$

Write QCD amplitudes in terms of cubic graphs

$$\mathcal{A}_n = g^{n-2} \sum \frac{n_i c_i}{D_i}$$



$$\mathcal{A}(p_1, p_2, p_3, p_4) = c_1 \frac{n_1}{(p_1+p_2)^2} + c_2 \frac{n_2}{(p_2+p_3)^2} + c_3 \frac{n_3}{(p_2+p_4)^2}$$

- Satisfy automatically for 4-point tree amplitudes $n_s = n_t - n_u$
- For high multiplicity, is not trivially satisfied

[Zhu (1980)]
[Bern, Carrasco, Johansson (2008),(2010)]

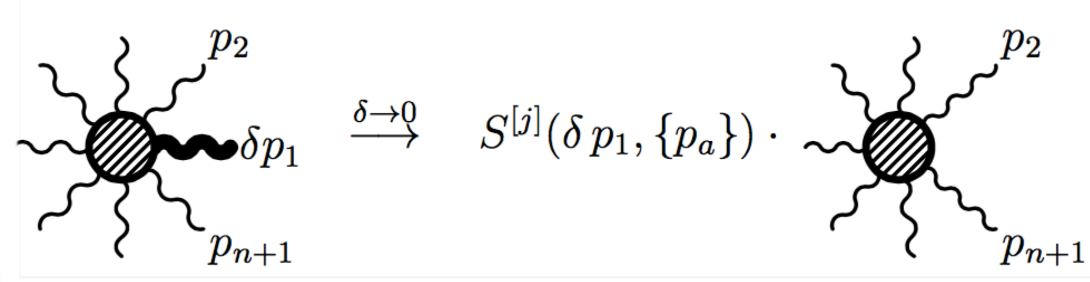
[Bern, Dennen, Huang, Kiermaier (2010)], [Boels, Isermann (2012)]
...
[Mastrolia, Primo, Schubert, W.J.T. (2015)]
[Primo, W.J.T. (2016)]
[Jurado, Rodrigo, W.J.T. (2017)]

- Bern-Carrasco-Johansson relations

$$\sum_{i=3}^{n} \left(\sum_{j=3}^{i} s_{2j} \right) A_n^{\text{tree}}(1, 3, \ldots, i, 2, i+1, \ldots, n) = 0$$

→ **(n-3)!** Independent amplitudes

Tree-level amplitudes



Soft theorems

- Scattering amplitudes display universal factorisation when a single photon (gluon) or graviton becomes soft: Parametrise soft momentum as δq and take $\delta \to 0$

[Low (1958)]
[Weinberg (1964)]

Gravity

$$M_N = \left(S_{\rm g}^{(0)} + S_{\rm g}^{(1)} + S_{\rm g}^{(2)}\right) M_{N-1} + \mathcal{O}(q)$$

[Cachazo and Strominger (2014)]

$$S_{\rm g}^{(0)} = \sum_{i=1}^{N-1} \frac{\epsilon_{\mu\nu} p_i^\mu p_i^\nu}{q.p_i}, \qquad S_{\rm g}^{(1)} = \sum_{i=1}^{N-1} \frac{\epsilon_{\mu\nu} p_i^\mu (q_\rho J_i^{\rho\nu})}{q.p_i}, \qquad S_{\rm g}^{(2)} = \sum_{i=1}^{N-1} \frac{\epsilon_{\mu\nu} (q_\lambda J_i^{\lambda\mu})(q_\rho J_i^{\rho\nu})}{q.p_i}$$

Yang Mills

$$A_N = \left(S_{\rm YM}^{(0)} + S_{\rm YM}^{(1)}\right) A_{N-1} + \mathcal{O}(q)$$

[Casali (2014)]
[Bern, Davies, Di Vecchia and Nohle (2014)]
[White (2014)]
[Luo, Mastrolia, W.J.T. (2014)]

$$S_{\rm YM}^{(0)} = \frac{\epsilon_\mu p_1^\mu}{q.p_1} - \frac{\epsilon_\mu p_n^\mu}{q.p_n}, \qquad S_{\rm YM}^{(1)} = \frac{\epsilon_\mu q_\nu J_1^{\mu\nu}}{q.p_1} - \frac{\epsilon_\mu q_\nu J_n^{\mu\nu}}{q.p_n}.$$

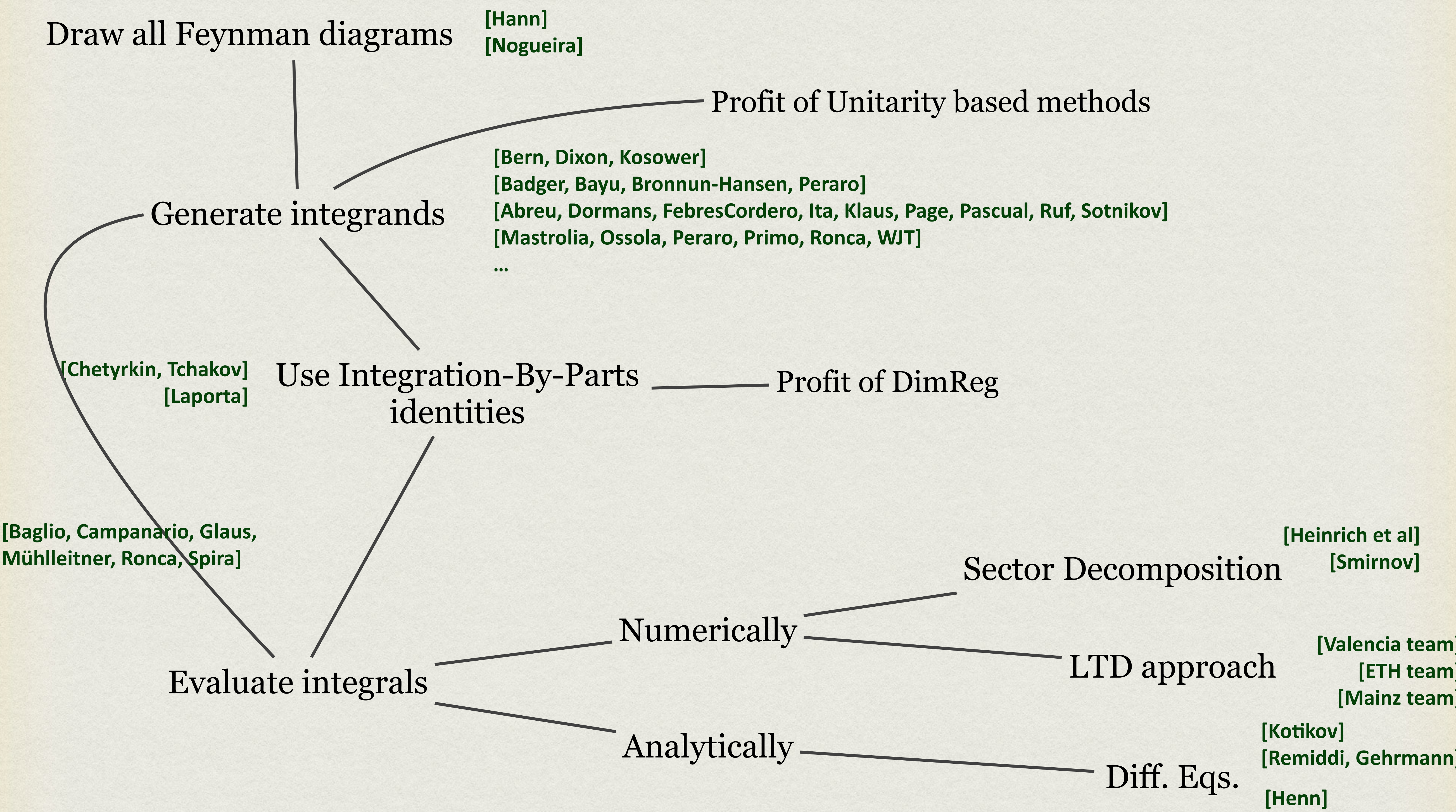
J is the total angular momentum of the emitter

$$J_i^{\mu\nu} = L_i^{\mu\nu} + \Sigma_i^{\mu\nu}$$

gives arise to the subleading-soft behaviour of the amplitude

Multi-loop scattering amplitudes

Standard approach @multi-loop level



Dimensional regularisation schemes

Before computing multi-loop amplitudes…

Consider

$$I_0 = \int_0^\infty \frac{dx}{x}$$

Dimensional regularisation schemes

Before computing multi-loop amplitudes…

Consider

$$I_0 = \int_0^\infty \frac{dx}{x} \qquad \text{does not exist}$$

Dimensional regularisation schemes

Before computing multi-loop amplitudes…

Consider

$$I_0 = \int_0^\infty \frac{dx}{x} \qquad \text{does not exist}$$

Tweak the integrand

$$I_\epsilon = \int_0^\infty \frac{dx}{x^{1+\epsilon}} = \int_0^1 \frac{dx}{x^{1+\epsilon}} + \int_1^\infty \frac{dx}{x^{1+\epsilon}} \qquad (\text{with } \epsilon \in \mathbb{C})$$

well defined

Dimensional regularisation schemes

Before computing multi-loop amplitudes...

Consider

$$I_0 = \int_0^\infty \frac{dx}{x}$$

does not exist ☹

Tweak the integrand

$$I_\epsilon = \int_0^\infty \frac{dx}{x^{1+\epsilon}} = \int_0^1 \frac{dx}{x^{1+\epsilon}} + \int_1^\infty \frac{dx}{x^{1+\epsilon}}$$

(with $\epsilon \in \mathbb{C}$)

well defined ☺

$$-\frac{1}{\epsilon} \qquad \Re(\epsilon) < 0$$

$$+\frac{1}{\epsilon} \qquad \Re(\epsilon) > 0$$

Dimensional regularisation schemes

Before computing multi-loop amplitudes…

Consider

$$I_0 = \int_0^\infty \frac{dx}{x}$$ does not exist 😵

Tweak the integrand

$$I_\epsilon = \int_0^\infty \frac{dx}{x^{1+\epsilon}} = \int_0^1 \frac{dx}{x^{1+\epsilon}} + \int_1^\infty \frac{dx}{x^{1+\epsilon}}$$ (with $\epsilon \in \mathbb{C}$)

well defined ☺

$$-\frac{1}{\epsilon}$$ ~~$\Re(\epsilon) < 0$~~

$$+\frac{1}{\epsilon}$$ ~~$\Re(\epsilon) > 0$~~

we are physicists 😜

$I_\epsilon = 0 \,, \forall \epsilon \in \mathbb{C} \longrightarrow I_0 = 0$ (analytical continuation)

Dimensional regularisation schemes

Before computing multi-loop amplitudes…

— All computations made in d=4-2ε
— Singularities manifest as poles in ε
— physical observables don't depend on ε

Consider

$$I_0 = \int_0^\infty \frac{dx}{x}$$ does not exist

Tweak the integrand

$$I_\epsilon = \int_0^\infty \frac{dx}{x^{1+\epsilon}} = \int_0^1 \frac{dx}{x^{1+\epsilon}} + \int_1^\infty \frac{dx}{x^{1+\epsilon}}$$ (with $\epsilon \in \mathbb{C}$)

well defined

$$-\frac{1}{\epsilon} \qquad\qquad +\frac{1}{\epsilon}$$

~~$\Re(\epsilon) < 0$~~ ~~$\Re(\epsilon) > 0$~~

we are physicists

$$I_\epsilon = 0\,, \forall \epsilon \in \mathbb{C} \longrightarrow I_0 = 0$$ (analytical continuation)

This was **dimensional regularisation**

Dimensional regularisation schemes

https://indico.ific.uv.es/event/3737/

WorkStop/ThinkStart 3.0: paving the way to alternative NNLO strategies

IR methods to boot NNLO calculations

- FDH/DRED
- FDR
- FDU
- IREG
- Local Analytic Sector Subtraction
- qT-subtraction
- Antenna subtraction

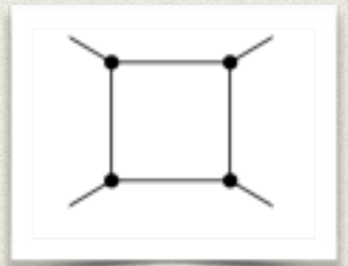
[W.J.T. et al (to appear)]

What about an actual calculation?

One-loop scattering amplitudes

Deal with with integrals of the form

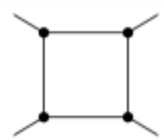
$\bar{l}^2,\ \bar{l}\cdot p_i,\ \bar{l}\cdot \varepsilon_i$



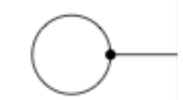
$$I_{i_1\cdots i_k}\left[\mathcal{N}(\bar{l},p_i)\right] = \int d^d\bar{l}\,\frac{\mathcal{N}_{i_1\cdots i_k}(\bar{l},p_i)}{D_{i_1}\cdots D_{i_k}}$$

Numerator and denominators are polynomials in the integration variable

Tensor reduction

$$A_n^{(1),D=4}(\{p_i\}) = \sum_{K_4} C^{[0]}_{4;K4}$$

$$+ \sum_{K_3} C^{[0]}_{3;K3}$$

$$+ \sum_{K_2} C^{[0]}_{2;K2}$$

$$+ \sum_{K_1} C^{[0]}_{1;K1}$$


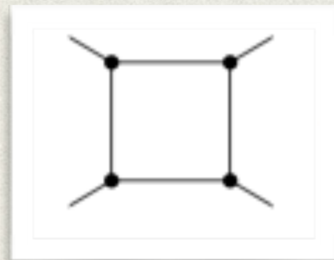
[Passarino - Veltman (1979)]

- ☑ Cut-constructible amplitude -> determined by its branch cuts
- ☑ All one-loop amplitudes are cut-constructible in dimensional regularisation.
- ☑ Master integrals are known

One-loop scattering amplitudes

Deal with with integrals of the form

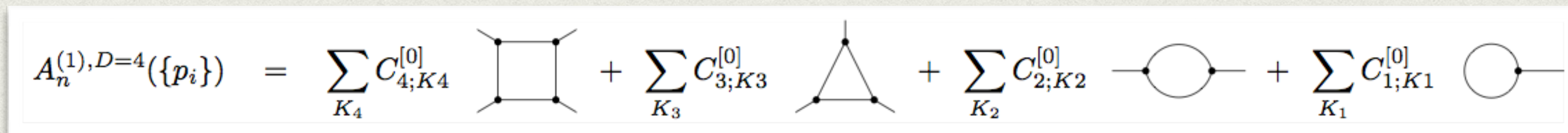
$\bar{l}^2,\, \bar{l}\cdot p_i,\, \bar{l}\cdot \varepsilon_i$



$$I_{i_1\cdots i_k}\left[\mathcal{N}(\bar{l},p_i)\right] = \int d^d\bar{l}\, \frac{\mathcal{N}_{i_1\cdots i_k}(\bar{l},p_i)}{D_{i_1}\cdots D_{i_k}}$$

Numerator and denominators are polynomials in the integration variable

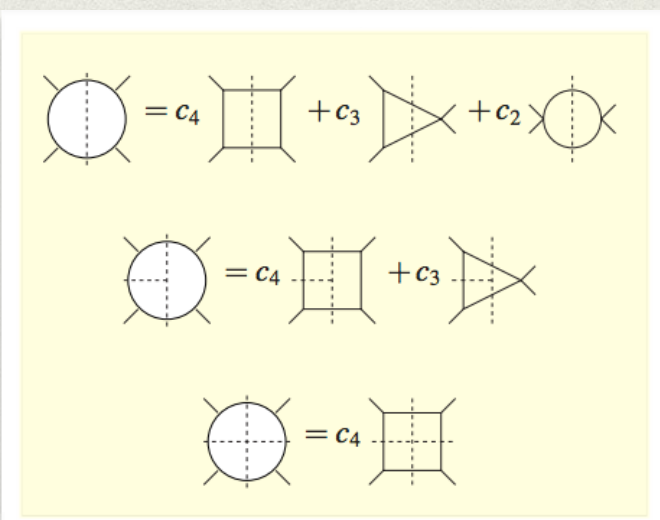
Tensor reduction



$$A_n^{(1),D=4}(\{p_i\}) = \sum_{K_4} C^{[0]}_{4;K4} \;\square\; + \sum_{K_3} C^{[0]}_{3;K3} \;\triangle\; + \sum_{K_2} C^{[0]}_{2;K2} \;\bigcirc\; + \sum_{K_1} C^{[0]}_{1;K1} \;\bigcirc$$

[Passarino - Veltman (1979)]

Unitarity based methods $\quad \dfrac{i}{q_i^2 - m^2 - i\epsilon} \to 2\pi\,\delta^{(+)}\left(q_i^2 - m_i^2\right)$



cut-4 :: Britto Cachazo Feng

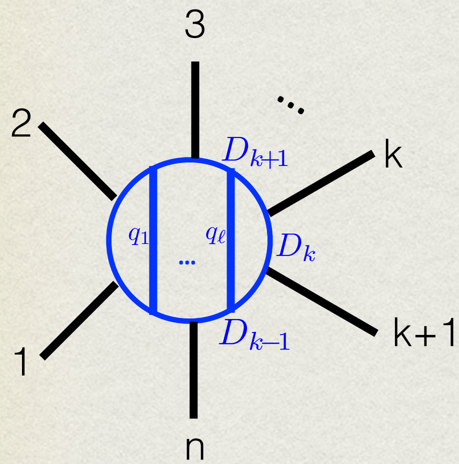
cut-3 :: Forde
Bjerrum-Bohr, Dunbar, Ita, Perkins
Mastrolia

cut-2 :: Bern, Dixon, Dunbar, Kosower.
Britto, Buchbinder, Cachazo, Feng.
Britto, Feng, Mastrolia.

Isolate the leading discontinuity!

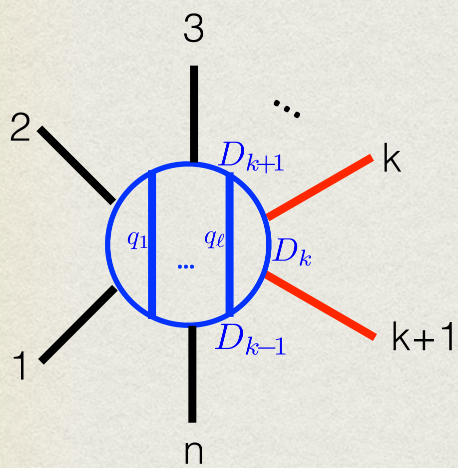
Integrand decomposition method

[Ossola, Papadopoulos, Pittau (2006)]
[Ellis, Giele, Kunszt, Melnikov (2007)]
[Mastrolia, Ossola, Papadopoulos, Pittau (2008)]



$$= \frac{\mathcal{N}_{i_1 \cdots i_m}(q_i)}{D_1 \cdots D_k \cdots D_m}$$

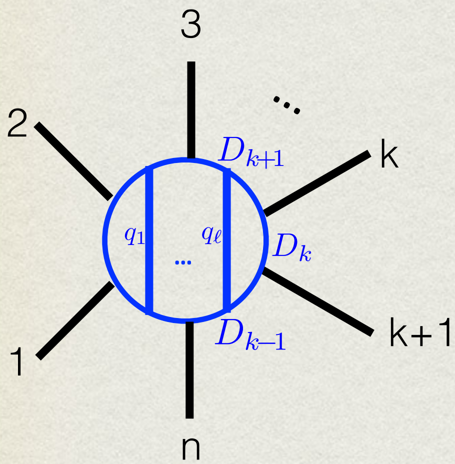
- ☑ Applicable to any theory
- ☑ Ideal for helicity amplitudes
- ☑ Work for any number of external legs
- ☑ Straightforwardly automated



$$= \sum_{k=1}^{m} \frac{\mathcal{N}_{i_1 \cdots i_{k+1} i_{k-1} \cdots i_m}(q_i)\, D_k}{D_1 \cdots D_k \cdots D_m} + \frac{\Delta_{i_1 \cdots i_k \cdots i_m}(q_i)}{D_1 \cdots D_k \cdots D_m}$$

Integrand decomposition method

[Ossola, Papadopoulos, Pittau (2006)]
[Ellis, Giele, Kunszt, Melnikov (2007)]
[Mastrolia, Ossola, Papadopoulos, Pittau (2008)]



$$= \frac{\mathcal{N}_{i_1 \cdots i_m}(q_i)}{D_1 \cdots D_k \cdots D_m}$$

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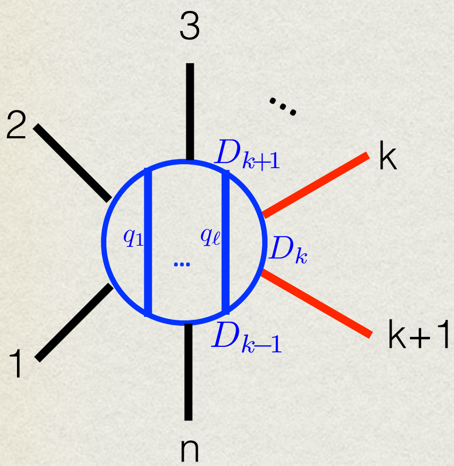
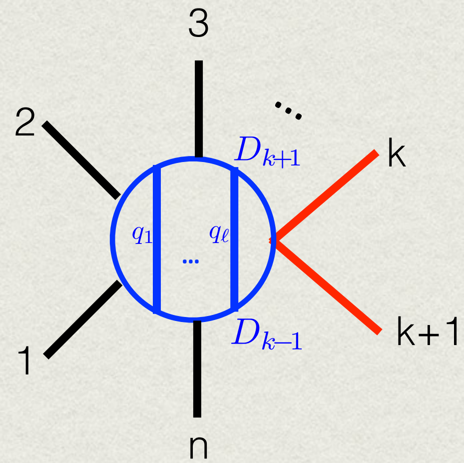


$$= \sum_{k=1}^{m}$$



$$+ \frac{\Delta_{i_1 \cdots i_k \cdots i_m}(q_i)}{D_1 \cdots D_k \cdots D_m}$$

Integrand decomposition method

[Ossola, Papadopoulos, Pittau (2006)]
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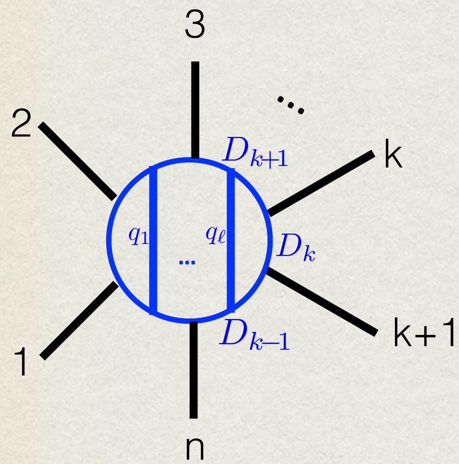

$$= \frac{\mathcal{N}_{i_1\cdots i_m}(q_i)}{D_1\cdots D_k\cdots D_m}$$

- ☑ Applicable to any theory
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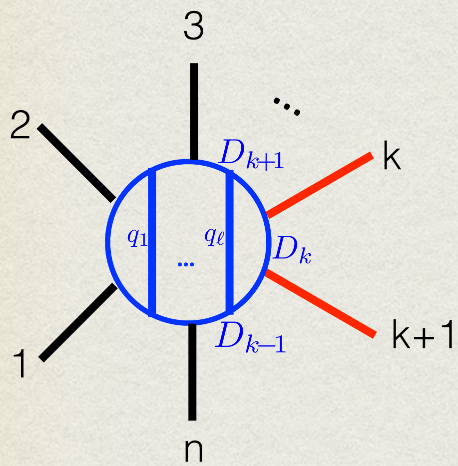
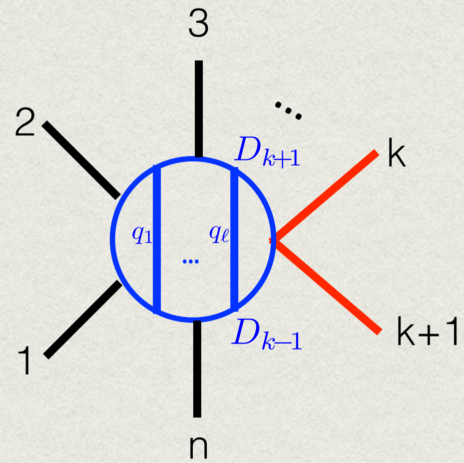
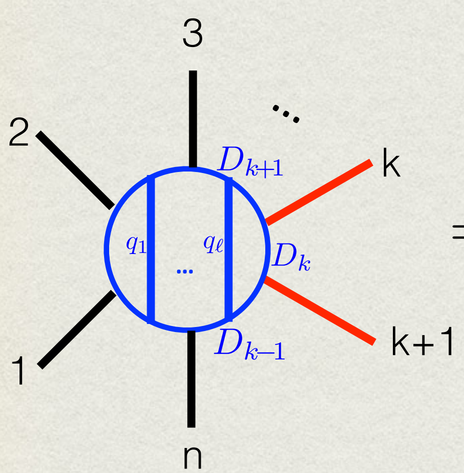


$$= \sum_{k=1}^{m}$$



$$+ \frac{\Delta_{i_1\cdots i_k\cdots i_m}(q_i)}{D_1\cdots D_k\cdots D_m}$$

Amplitude decomposed into all possible multi-particle cuts



$$= \sum_{k=0}^{m} \sum_{\{1,\cdots,m\}} \frac{\Delta_{i_1\cdots i_k}(q_i)}{D_1\cdots D_k}$$

Numerator in terms of
Irreducible polynomials

Polynomial division module Groebner basis

[Mastrolia, Ossola 11]
[Zhang 2012-2016]
[Badger, Frellesvig, Zhang 2012-2013]
[Mastrolia, Mirabella, Ossola, Peraro 2012]
…
[Mastrolia, Peraro, Primo, W.J.T. 2016]

Loop-Tree duality representation

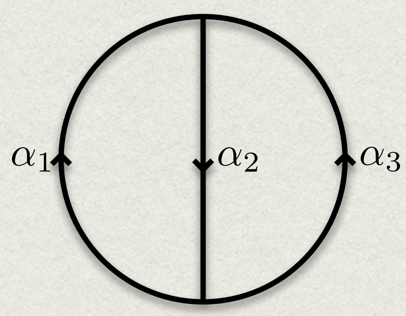
- Any multi-loop Feynman integral contains S sets of internal propagators

$$q_{i_S} = \ell_S + k_{i_S}\,, \quad i_S \in S$$

loop momenta

linear combination of external momenta

e.g. @ 2L



PHYSICAL REVIEW LETTERS **124,** 211602 (2020)

Open Loop Amplitudes and Causality to All Orders and Powers from the Loop-Tree Duality

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- Feynman propagators

$$G_F(q_{i_S}) = \frac{1}{q_{i_S}^2 - m_{i_S}^2 + \imath 0} = \frac{1}{q_{i_S,0}^2 - (q_{i_S,0}^{(+)})^2}$$

In terms of spatial components

$$q_{i_S,0}^{(+)} = +\sqrt{\mathbf{q}_{i_S}^2 + m_{i_S}^2 - \imath 0}$$

Pull out full dependence of the energy components of loop momenta

usual Feynman ı0 prescription!

- *Let's now apply the Cauchy residue thm for each "energy" integration*

Multi-loop LTD representation

[Aguilera-Verdugo et al (2020)]

- LTD representation is written in terms of nested residues

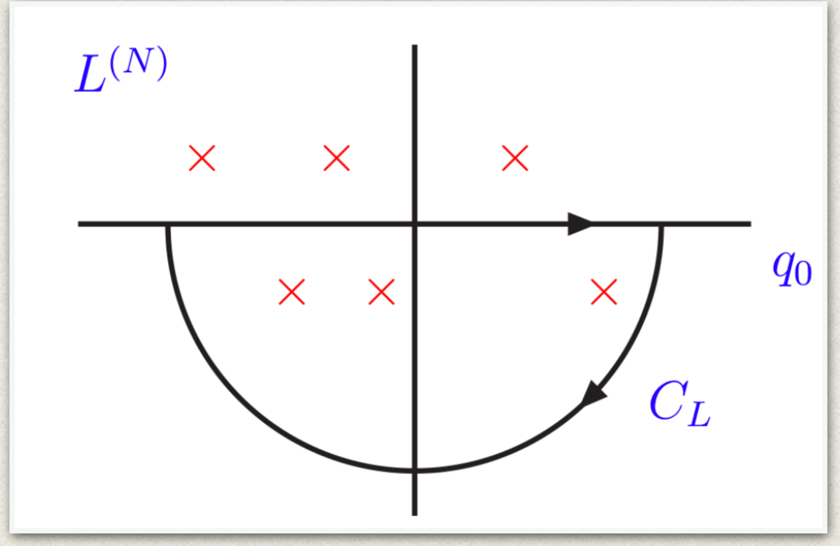
$$\mathcal{A}_D^{(L)}\left(1,\ldots,r;r+1,\ldots,n\right) \equiv -2\pi\imath \sum_{i_r\in r} \mathrm{Res}\left(\mathcal{A}_D^{(L)}\left(1,\ldots,r-1;r,\ldots,n\right), \mathrm{Im}\left(q_{i_r,0}\right)<0\right) ,$$

in terms of on-shell and off-shell propagators and

$$\mathcal{A}_D^{(L)}\left(1;2,\ldots,n\right) \equiv -2\pi \sum_{i_r\in r} \mathrm{Res}\left(d\mathcal{A}_F^{(L)}\left(1,\ldots,n\right), \mathrm{Im}\left(q_{i_1,0}\right)<0\right) ,$$

$$\mathcal{A}_F^{(L)}\left(1,\ldots,n\right) = \int_{\ell_1\cdots\ell_L} N\times G_F\left(1,\ldots,n\right)$$

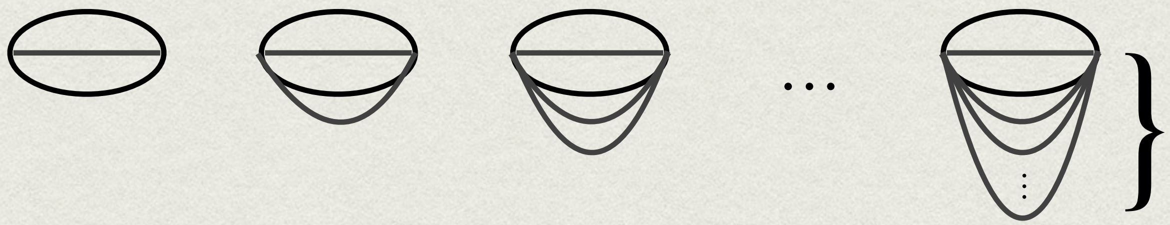
- Cauchy contour is always closed from below the real axis



Multi-loop LTD representation

[Aguilera-Verdugo et al (2020)]

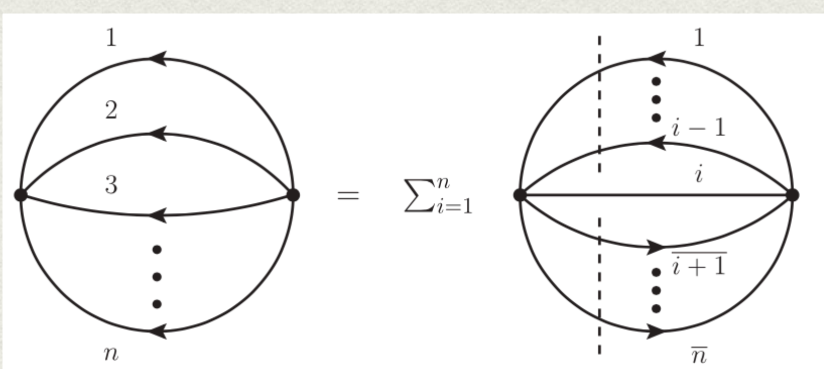
- Interesting decomposition of topologies built from



Characterised by

$$q_{i,S} = \begin{cases} \ell_S + k_{i_S}\,, & S \in \{1,\dots,L\} \\ -\sum_{S=1}^{L} \ell_S + k_{i_{L+1}} & S = L+1 \end{cases}$$

- We propose an Ansatz and prove it by induction

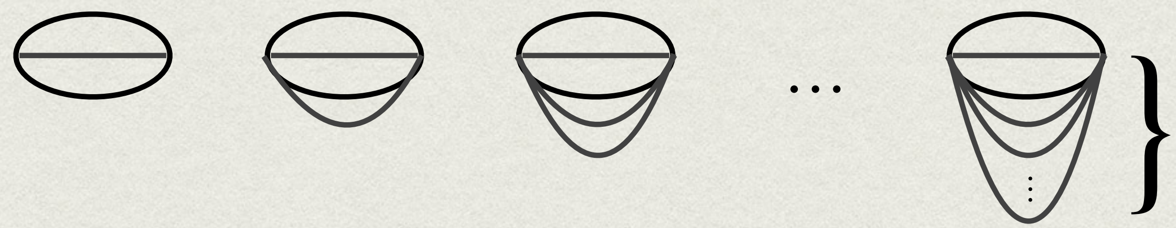


$$= \int_{\ell_1 \dots \ell_L} \sum_{i=1}^{L+1} \mathcal{A}_D^{(L)}\left(1,\dots,i-1,\overline{i+1},\dots,\overline{L+1}; i\right)$$

Multi-loop LTD representation

[Aguilera-Verdugo et al (2020)]

- Interesting decomposition of topologies built from



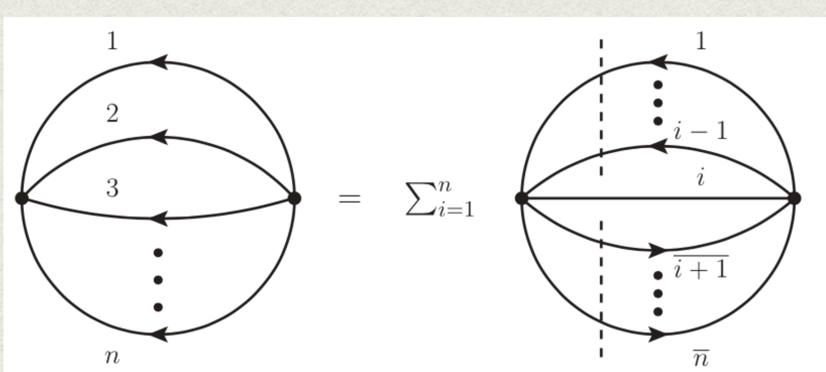
(L-2) loops

Characterised by

$$q_{i,S} = \begin{cases} \ell_S + k_{i_S}\,, & S \in \{1,\ldots,L\} \\ -\sum_{S=1}^{L} \ell_S + k_{i_{L+1}} & S = L+1 \end{cases}$$

- We propose an Ansatz and prove it by induction

MAXIMAL-LOOP TOPOLOGY (MLT)



$$= \int_{\ell_1 \ldots \ell_L} \sum_{i=1}^{L+1} \mathcal{A}_D^{(L)} \left(1, \ldots, i-1, \overline{i+1}, \ldots, \overline{L+1}; i\right)$$

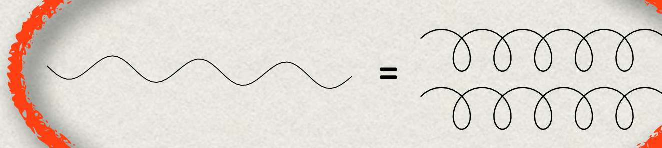
Applications

Einstein-Yang-Mills Amplitudes

$$\mathbf{R}_{\mu\nu} = \partial_\mu \Gamma^\rho_{\rho\nu} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu} - \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu},$$
$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\boldsymbol{g}^{\rho\sigma}\left(\partial_\mu \boldsymbol{g}_{\nu\sigma} + \partial_\nu \boldsymbol{g}_{\sigma\mu} - \partial_\sigma \boldsymbol{g}_{\mu\nu}\right),$$
$$\boldsymbol{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}.$$

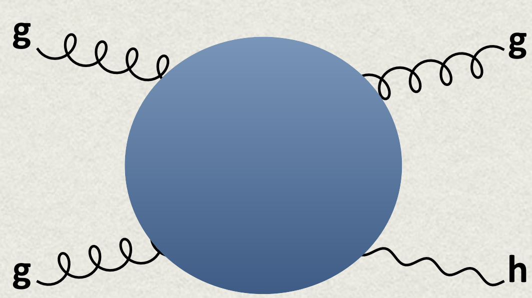
$$\mathcal{L}_{\rm EYM} = \frac{2}{\kappa^2}\sqrt{-\boldsymbol{g}}\mathbf{R} - \frac{1}{4}\sqrt{-\boldsymbol{g}}\boldsymbol{g}^{\mu\rho}\boldsymbol{g}^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} + \mathcal{L}_{\rm gf}$$

- 4-point process depending on 2 scales + d



$$g(p_1) + g(p_2) \to g(-p_3) + h(-p_4)$$

$$h^{\mu\nu}(p_i) \to \varepsilon^\mu_{\lambda_i}(p_i)\varepsilon^\nu_{\lambda_i}(p_i)$$



$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2$$

Warming up exercise

More gravitons →

$$\{I_4[\mu_{11}], I_4[\mu_{11}^2], I_4[\mu_{11}^3], I_4[\mu_{11}^4], I_3[\mu_{11}], I_3[\mu_{11}^2], I_2[\mu_{11}], I_2[\mu_{11}^2]\}$$

Einstein-Yang-Mills Amplitudes

Initialisation

- Identify parent topologies from Feynman graphs

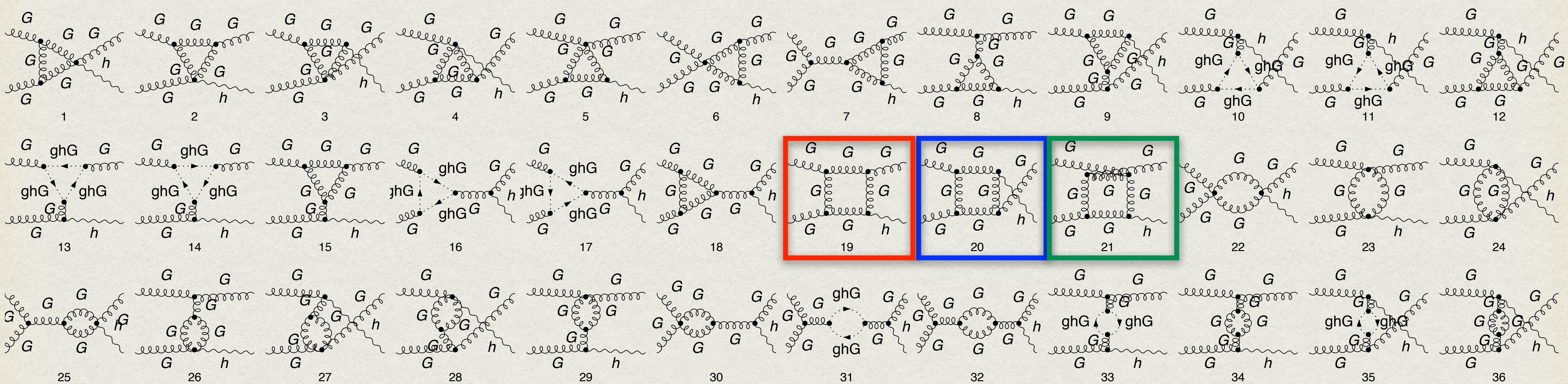
e.g. 1-Loop

Einstein-Yang-Mills Amplitudes

Initialisation

- Identify parent topologies from Feynman graphs

e.g. 1-Loop

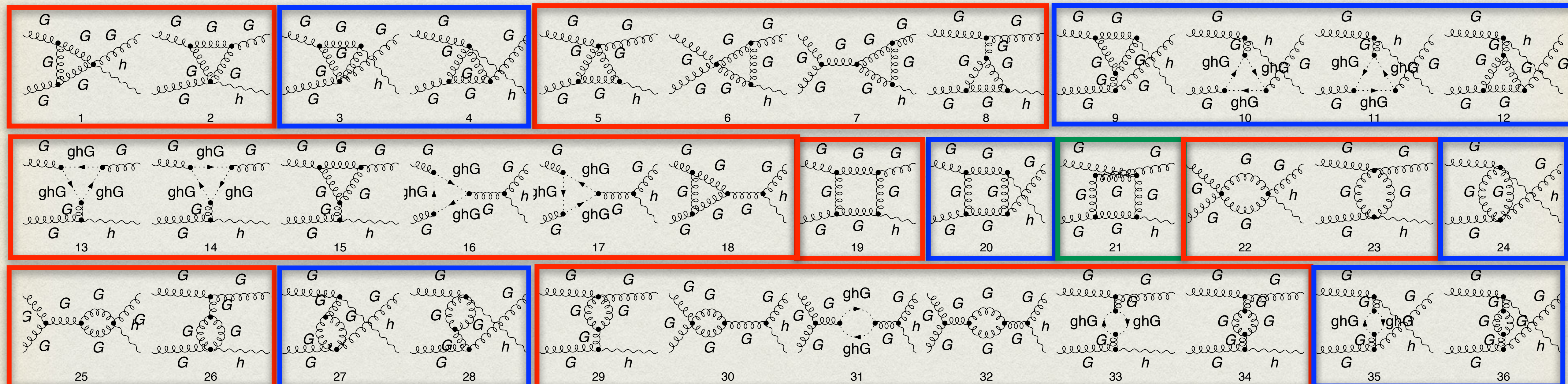


Einstein-Yang-Mills Amplitudes

Initialisation

- Identify parent topologies from Feynman graphs

e.g. 1-Loop

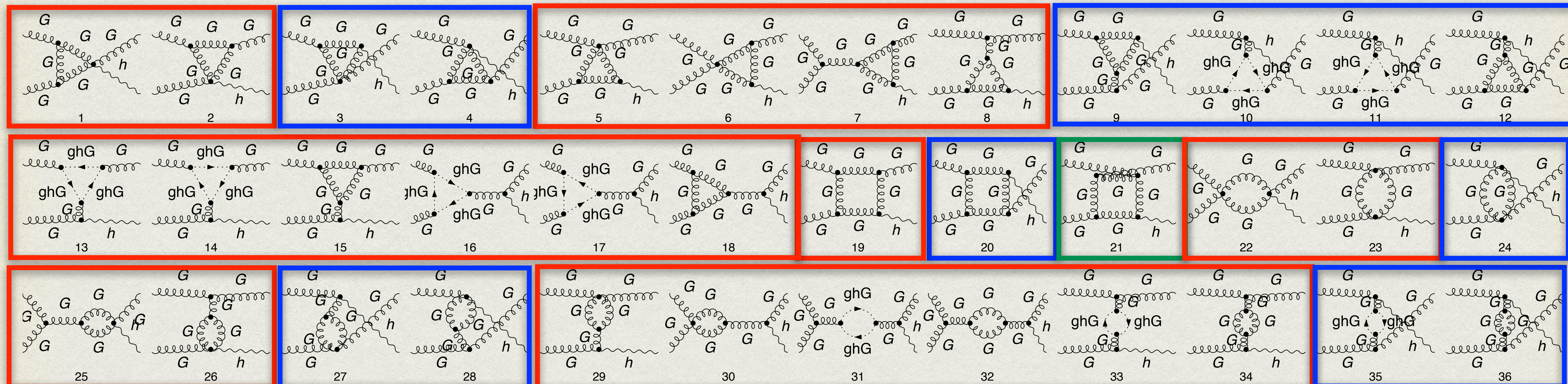


Einstein-Yang-Mills Amplitudes

Initialisation

- Identify parent topologies from Feynman graphs

e.g. 1-Loop



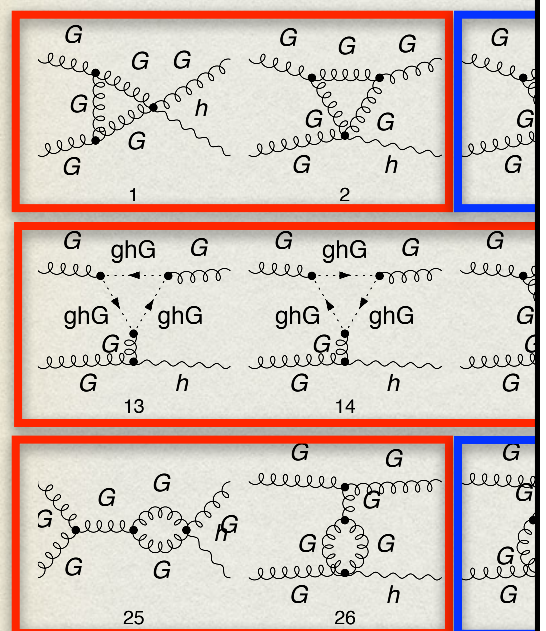
- Group diagrams

- Extract the leading colour contribution

$$\mathcal{A}\left(\{p_i,h_i\}_{i=1,3}\right)\Big|_{\text{leading colour}} = \sum_{\sigma\in S_3/Z_3} \text{Tr}\left(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}\right)\, g_0^3 \left(A_4^{(0)} + \frac{\alpha_0 N_C}{4\pi}A_4^{(1)} + \left(\frac{\alpha_0 N_C}{4\pi}\right)^2 A_4^{(2)} + \mathcal{O}\left(\alpha_0^3\right)\right)$$

Einstein-Yang-Mills Amplitudes

Input numerators



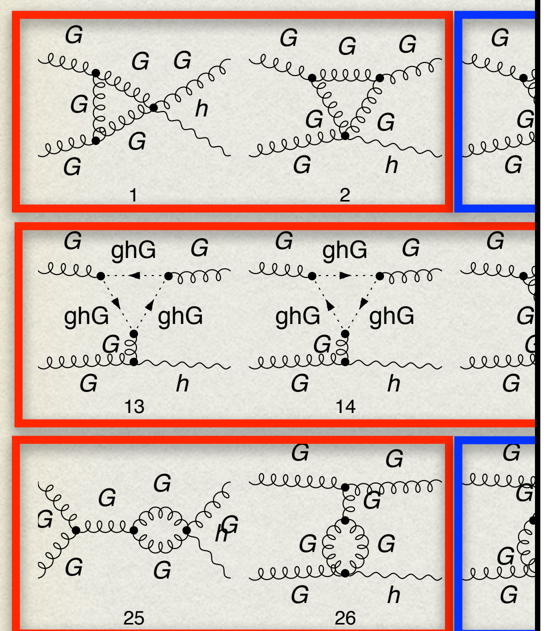
$$
\begin{aligned}
&\frac{1}{2}\Big(2\,(\mathrm{sp}(q,\varepsilon_4)+\mathrm{sp}(\varepsilon_4,p_1)+\mathrm{sp}(\varepsilon_4,p_4))\\
&\quad\Big(3\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)^2+\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_2)\,\mathrm{sp}(q,p_1)-2\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(q,p_1)+\\
&\qquad 7\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(q,p_1)-\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(q,p_1)+\\
&\qquad 5\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(q,p_1)-6\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_2)+\mathrm{sp}(\varepsilon_2,p_1))\,\mathrm{sp}(q,p_1)-\\
&\qquad \mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(q,p_1)+3\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(q,p_1)-\\
&\qquad \mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(q,p_1)+3\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(q,p_1)-\\
&\qquad 2\,\mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(q,p_1)-\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,p_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(q,p_1)+\\
&\qquad 4\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)-\\
&\qquad 6\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_2)+\mathrm{sp}(\varepsilon_2,p_1))\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+\\
&\qquad 2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_2)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-\\
&\qquad 2\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-\\
&\qquad 6\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-2\,\mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-\\
&\qquad \mathrm{sp}(\varepsilon_1,p_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)+\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_2)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)+\\
&\qquad \mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)-\\
&\qquad 2\,\mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)-\mathrm{sp}(\varepsilon_1,p_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)+\\
&\qquad 4\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)-\\
&\qquad 3\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_2)+\mathrm{sp}(\varepsilon_2,p_1))\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)+\\
&\qquad 2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)+8\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_4)\,\mathrm{sp}(q,p_1)+\\
&\qquad 4\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(\varepsilon_3,p_4)\,\mathrm{sp}(q,p_1)-6\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_2)+\mathrm{sp}(\varepsilon_2,p_1))\,\mathrm{sp}(\varepsilon_3,p_4)\,\mathrm{sp}(q,p_1)+\\
&\qquad 4\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(\varepsilon_3,p_4)\,\mathrm{sp}(q,p_1)+4\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(\varepsilon_3,p_4)\,\mathrm{sp}(q,p_1)-\\
&\qquad \mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)-\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,p_2)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)+\\
&\qquad \mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,p_3)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)+\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,p_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)-2\,s\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)+\\
&\qquad (\mathrm{sp}(q,q)-\mu 11)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)+3\,(-\mu 11+\mathrm{sp}(q,q)+\mathrm{sp}(q,p_1))\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)-\\
&\qquad \mathrm{sp}(q,p_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)+\mathrm{sp}(q,p_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)+
\end{aligned}
$$

Input: rank 5 numerator

Reduction time ~ 30 seconds

Einstein-Yang-Mills Amplitudes

Input numerators



$$\frac{1}{2}\Big(2\,(\mathrm{sp}(q,\varepsilon_4)+\mathrm{sp}(\varepsilon_4,p_1)+\mathrm{sp}(\varepsilon_4,p_4))$$

$$\Big(3\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,\varepsilon_2)\,\mathrm{sp}(q,p_1)^2+\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_2)\,\mathrm{sp}(q,p_1)-2\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(q,p_1)+$$

$$7\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(q,p_1)-\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(q,p_1)+$$
$$5\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(q,p_1)-6\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_2)+\mathrm{sp}(\varepsilon_2,p_1))\,\mathrm{sp}(q,p_1)-$$
$$\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(q,p_1)+3\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(q,p_1)-$$
$$\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(q,p_1)+3\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(q,p_1)-$$
$$2\,\mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(q,p_1)-\mathrm{sp}(q,\varepsilon_3)\,\mathrm{sp}(\varepsilon_1,p_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(q,p_1)+$$
$$4\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)-$$
$$6\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_2)+\mathrm{sp}(\varepsilon_2,p_1))\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+$$
$$2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}(\varepsilon_3,p_1)\,\mathrm{sp}(q,p_1)+\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_2)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-$$
$$2\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-$$
$$6\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-2\,\mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)-$$
$$\mathrm{sp}(\varepsilon_1,p_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_3)+\mathrm{sp}(\varepsilon_3,p_1))\,\mathrm{sp}(q,p_1)+\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,\varepsilon_2)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)+$$
$$\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)-$$
$$2\,\mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)-\mathrm{sp}(\varepsilon_1,p_1)\,\mathrm{sp}(\varepsilon_2,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_2)\,\mathrm{sp}(q,p_1)+$$
$$4\,\mathrm{sp}(q,\varepsilon_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)-$$
$$3\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,(\mathrm{sp}(q,\varepsilon_2)+\mathrm{sp}(\varepsilon_2,p_1))\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)+2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{sp}(\varepsilon_3,p_3)\,\mathrm{sp}(q,p_1)+$$
$$2\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_4)\,\mathrm{sp}\ldots$$
$$4\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_1)\,\mathrm{s}\ldots$$
$$4\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon_2,p_3)\,\mathrm{s}\ldots$$
$$\mathrm{sp}(q,\varepsilon_1)\,\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_3\ldots$$
$$\mathrm{sp}(q,\varepsilon_4)\,\mathrm{sp}(\varepsilon_1,p_3)\,\mathrm{sp}(\varepsilon\ldots$$
$$(\mathrm{sp}(q,q)-\mu 11)\,\mathrm{sp}(\varepsilon_1,\varepsilon\ldots$$
$$\mathrm{sp}(q,p_2)\,\mathrm{sp}(\varepsilon_1,\varepsilon_4)\,\mathrm{sp}(\varepsilon\ldots$$

$$A_4^{(1)}\left(1^-,2^-,3^+;4^{++}\right)=A_4^{(0)}c_\Gamma\left(-\frac{\mu^2}{s}\right)^\epsilon\Bigg[-\frac{3}{\varepsilon^2}-\frac{11}{3\varepsilon}-\frac{1}{\varepsilon}\left(\log\left(\frac{-s}{-t}\right)+\log\left(\frac{-s}{s+t}\right)\right)$$
$$-\frac{11}{3}\log\left(\frac{-s}{-t}\right)+\frac{t\left(14s^2+9st+6t^2\right)}{3s^3}\log\left(\frac{-t}{s+t}\right)$$
$$+\left(\frac{t(s+t)\left(2s^2+st+t^2\right)}{s^4}+\frac{1}{2}\right)\left(\log^2\left(\frac{-t}{s+t}\right)+\pi^2\right)+\pi^2$$
$$+\frac{t(s+t)}{s^2}-\frac{64}{9}+\frac{\delta}{6}\Bigg].$$

$\delta=-2$ or $\delta=0$.
tHV and FDH

Input: rank 5 numerator

Reduction time ~ 30 seconds

William J. Torres Bobadilla [W.J.T. (2018)]

Gravitational potential — Post-Newtonian corrections

- Start w/ the action $\mathcal{S} = \mathcal{S}_{\text{pp}} + \mathcal{S}_{\text{bulk}}$

$$\mathcal{S}_{\text{pp}} = -\sum_{i=1,2} m_i \int \sqrt{-g_{\mu\nu}(x_i)\, dx_i^\mu dx_i^\nu}$$

Point particle action

$$\mathcal{S}_{\text{bulk}} = 2\Lambda^2 \int d^{d+1}x \sqrt{-g} \left[R(g) - \frac{1}{2}\Gamma_\mu \Gamma^\mu \right]$$

Einstein-Hilbert action & gauge fixing term

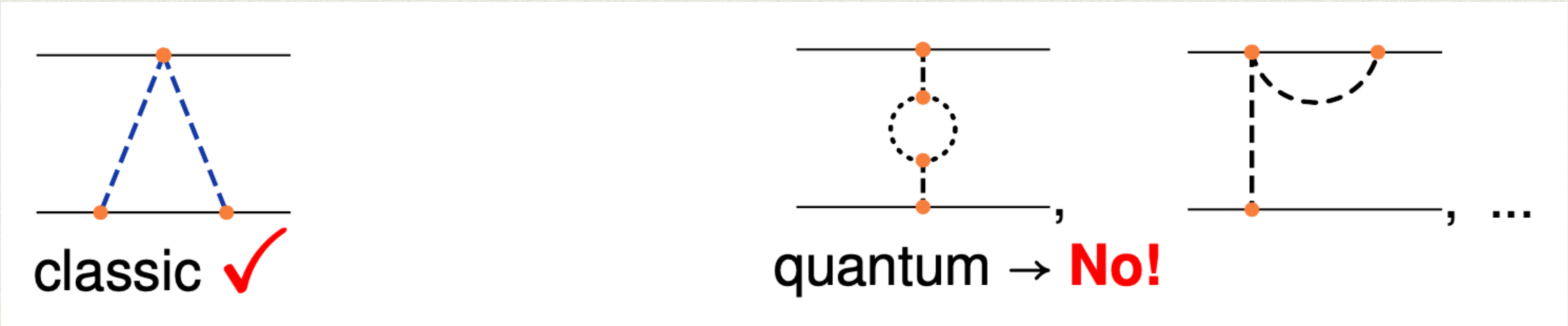
- Kaluza-Klein parametrisation of metric tensor

$$g_{\mu\nu} = \mathrm{e}^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_j/\Lambda & \mathrm{e}^{-c_d\phi/\Lambda}(\delta_{ij} + \sigma_{ij}/\Lambda) - A_i A_j/\Lambda^2 \end{pmatrix}$$

scalar field
vector field
tensor field

[Kol. Smolkin]
[Blanchet, Damour]

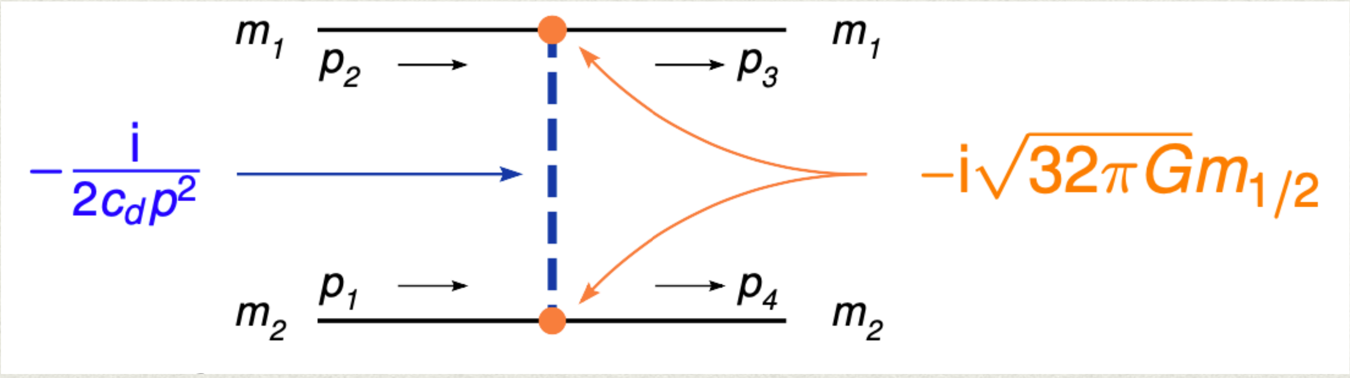
- Static limit :: $A_j \to 0$
- Perturbative expansion :: Feynman {rules, diagrams}
- Gravitational modes :: ϕ, σ —> Emitted/Absorbed by point particles
- Take classical contribution



William J. Torres Bobadilla

Gravitational potential — Post-Newtonian corrections

- Newton diagram



- Amplitude

$$\mathcal{A} = \left(-i\sqrt{32\pi G m_1}\right)\left(-\frac{i}{2c_d p^2}\right)\left(-i\sqrt{32\pi G m_2}\right)$$

$$= 32\pi i \frac{G m_1 m_2}{2 c_d p^2}, \qquad c_d = 2\frac{d-1}{d-2}$$

- Fourier transform

$$\mathcal{V}(r) = i \lim_{d\to 3} \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot r} \mathcal{A} = -\frac{G m_1 m_2}{\pi^2} \int d^3 p \frac{e^{ip\cdot r}}{p^2} = -\frac{G m_1 m_2}{r} \checkmark$$

Gravitational potential — Post-Newtonian corrections

Static PN corrections

Potential

$$\mathcal{V}_{\text{static}}(r) =$$

$-\frac{G m_1 m_2}{2r}$	Newton	N
$+\frac{1}{2}\frac{G^2 m_1^2 m_2}{r^2}$	Einstein, Infeld, Hoffmann,...	1PN
$-\frac{1}{2}\frac{G^3 m_1^3 m_2}{r^3} - \frac{3}{2}\frac{G^3 m_1^2 m_2^2}{r^3}$	Ohta, Okamura, Kimura, Hiida, Damour, Schäfer, Gilmore, Ross,...	2PN
$+\frac{3}{8}\frac{G^4 m_1^4 m_2}{r^4} + 6\frac{G^4 m_1^3 m_2^2}{r^4}$	Damour, Jaranowski, Schäfer, Blanchet, Faye, Itoh, Futamase, Esposito-Farese,...	3PN
$-\frac{3}{8}\frac{G^5 m_1^5 m_2}{r^5} - \frac{31}{3}\frac{G^5 m_1^4 m_2^2}{r^5} - \frac{141}{8}\frac{G^5 m_1^3 m_2^3}{r^5}$	Damour, Jaranowski, Schäfer, Foffa, Porto, Rothstein, Sturani, Mastrolia, C.S., Bernard, Blanchet, Bohé, Faye, Marsat,...	4PN
$+\frac{5}{16}\frac{G^6 m_1^6 m_2}{r^6} + \frac{91}{6}\frac{G^6 m_1^5 m_2^2}{r^6} + \frac{653}{6}\frac{G^6 m_1^4 m_2^3}{r^6}$		5PN

↑ ***New*** Foffa, Mastrolia, Sturani, C.S., Torres Bobadilla
confirmed by Blümlein, Maier, Marquard

$$+ \quad \cdots$$

$$+ (m_1 \leftrightarrow m_2)$$

(Coefficients are just rational numbers)

31 C. Sturm, Aachen, 30.01.2020 PN corrections to the gravitational 2-body potential

Gravitational potential — Post-Newtonian corrections

Static PN corrections

Potential

$$\mathcal{V}_{\text{static}}(r) =$$

$$-\frac{Gm_1 m_2}{2r}$$ Newton **N**

$$+\frac{1}{2}\frac{G^2 m_1^2 m_2}{r^2}$$

$$-\frac{1}{2}\frac{G^3 m_1^3 m_2}{r^3} - \frac{3}{2}\frac{G^3 m_1^2 m_2^2}{r^3}$$

$$+\frac{3}{8}\frac{G^4 m_1^4 m_2}{r^4} + 6\frac{G^4 m_1^3 m_2^2}{r^4}$$

$$-\frac{3}{8}\frac{G^5 m_1^5 m_2}{r^5} - \frac{31}{3}\frac{G^5 m_1^4 m_2^2}{r^5} - \frac{141}{8} G \ldots$$

$$+\frac{5}{16}\frac{G^6 m_1^6 m_2}{r^6} + \frac{91}{6}\frac{G^6 m_1^5 m_2^2}{r^6} + \frac{653}{6}\frac{G^6 m_1^4 m_2^3}{r^6}$$ **5PN**

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$$+ (m_1 \leftrightarrow m_2)$$

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PHYSICAL REVIEW LETTERS **122,** 241605 (2019)

Static Two-Body Potential at Fifth Post-Newtonian Order

Stefano Foffa,[1] Pierpaolo Mastrolia,[2] Riccardo Sturani,[3] Christian Sturm,[4] and William J. Torres Bobadilla[5]
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31 C. Sturm, Aachen, 30.01.2020 PN corrections to the gravitational 2-body potential

Gravitational potential — Post-Newtonian corrections

- A few diagrams @ 5L

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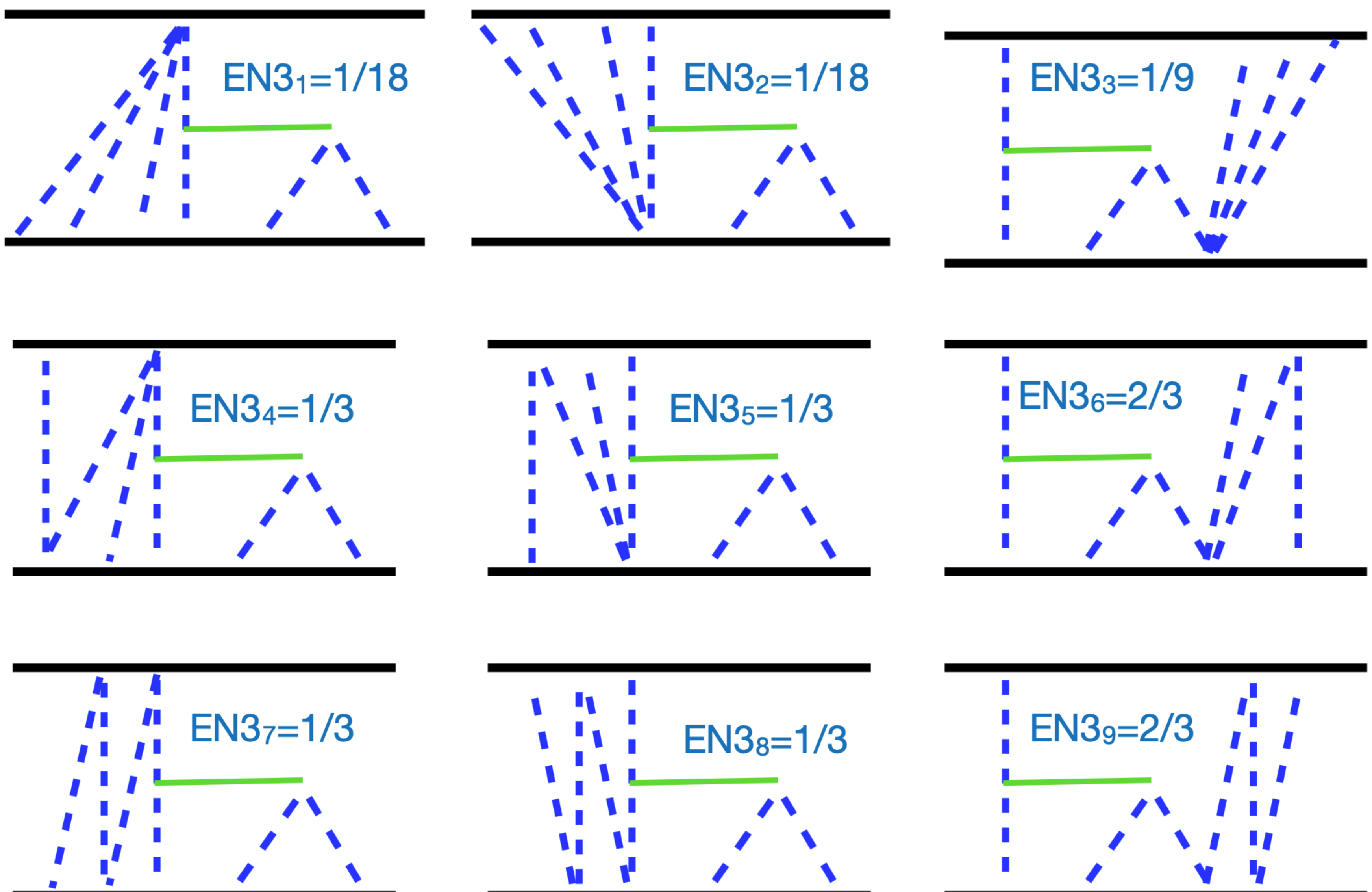
4) E_{2PN} x N^3: 29 elements



Some conclusions

- ☑ Understanding of mathematical properties of scattering amplitudes
- ☑ Computational techniques of scattering amplitudes
- ☑ Efficient techniques for tree- and multi-loop-level amplitudes
- ☑ Applications in quantum & classical theories

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- ☐ Completely treatment of IR and UV singularities is desirable
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