Systematically building the generalized Proca and SU(2) Proca theories of gravity and beyond

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Based on: *Phys. Lett., B198* [Gallego Cadavid and Rodriguez, 2019] & *Phys. Rev., D102* [Gallego Cadavid et al., 2020]

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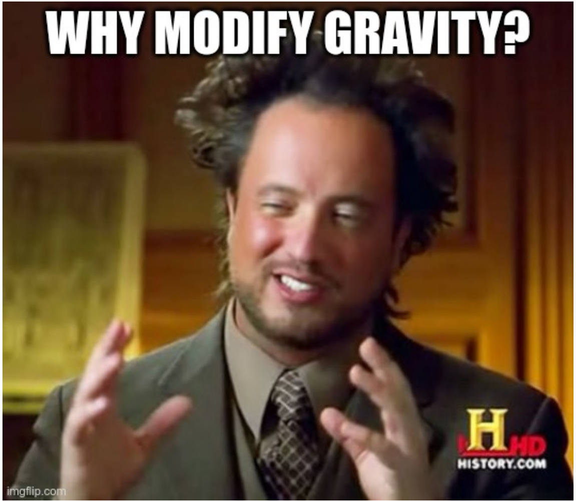
MAIN RESULTS

In Ref. [Gallego Cadavid and Rodriguez, 2019]

1. We present a systematic procedure to build Modified Gravity Theories (MGT) when adding scalar and vector fields.
2. The procedure naturally yields the "Beyond" terms of the MGT.
3. It introduces a **new** covariantization process to build the MGT.

In Ref. [Gallego Cadavid et al., 2020]

We build the Generalized SU(2) Proca theory and Beyond.


WHY MODIFY GRAVITY?
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General Relativity (GR)

- More than 100 years ago Einstein proposed the field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1}$$

- They still are our best description of how spacetime behaves on macroscopic scales.
- We can explain the expansion of the Universe, NS, BH, GW, and the formation of all structures in the Universe.

Problems with GR [A. Ashtekar et al., 2015, Heisenberg, 2019]

- "36% of unresolved problems in physics involve gravity", M. Zumalacárregui.
- Some observations might point to modifications of GR (DE and DM).
- Breakdown of GR at the infrared and ultraviolet scales.
- Why not modify it? In the process we will learn more about gravity.

How to build a theory of gravity?

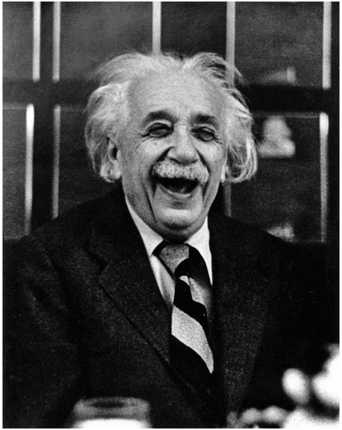
What do we know about gravity?

- Gravitational force (GF) is long-range $\Rightarrow$ force carrier mass should be $m \approx 0$.
- gravity exists as a classical theory $\Rightarrow$ bosonic particle with spin $s = 0, 1, 2, \dots$

Possibilities for integer spins [Heisenberg, 2019]

1. for $s > 2$ there are theoretical challenges to build the theories.
2. a spin-1 particle cannot be since we know that the GF is attractive.
3. a spin-0 particle cannot be since it naturally couples to the matter fields via $\phi T^{\mu}{}_{\mu}$, but since relativistic particles are traceless $\Rightarrow \phi$ would not interact with light.

Therefore, the most promising survivor is a spin-2 particle $\Rightarrow$ GR.



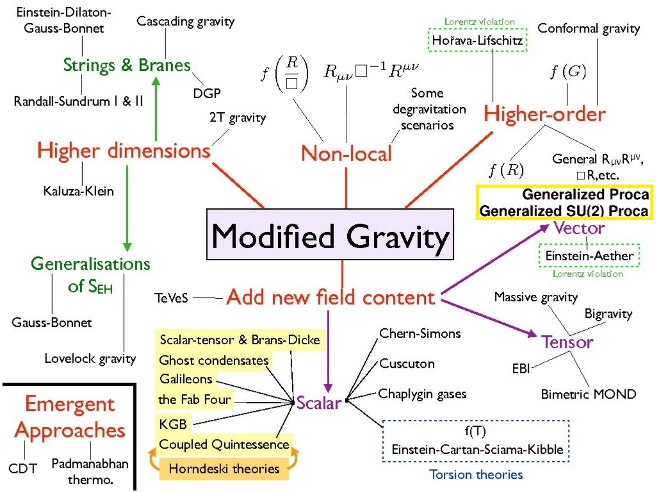
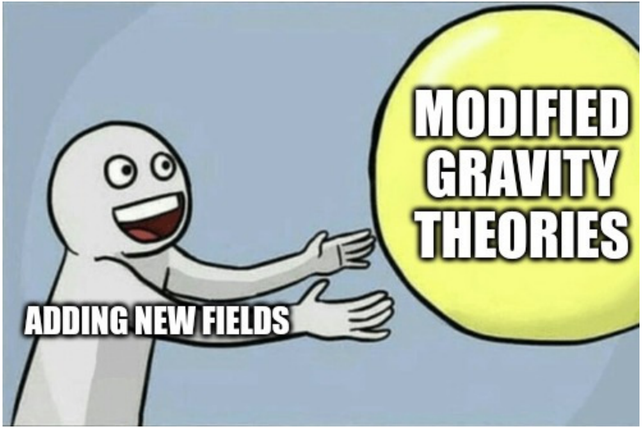


Figure: Tree diagram of modified theories of gravity. Image Credit: Tessa Baker.



- $\mathcal{L} = \mathcal{L}(m_A, A_a^\mu, g_{\mu\nu}) \Rightarrow$ (non-) Abelian Vector-Tensor theories

Ostrogradsky's theorem (non-degenerate theories) [Ostrogradsky, 1850]

Field equations higher than second order lead to an unbounded Hamiltonian from below.

Horndeski theory [Horndeski, 1974, Gleyzes et al., 2015]

$$\mathcal{L}_2^H = f_2(\phi, X)\,, \tag{2}$$

$$\mathcal{L}_3^H = f_3(\phi, X)\Box\phi\,, \tag{3}$$

$$\mathcal{L}_4^H = f_4(\phi, X)R - 2f_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right]\,, \tag{4}$$

$$\mathcal{L}_5^H = f_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} + \frac{1}{3}f_{5,X}(\phi, X)\left[(\Box\phi)^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\nu\rho}\phi^{\mu}_{\rho}\right]\,, \tag{5}$$

where $X \equiv \nabla_\mu\phi\nabla^\mu\phi$, $\phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor.

GR is recovered by setting $f_2 = f_3 = f_5 = 0$, and $f_4 = M_{\rm Pl}^2/2$.

Beyond Horndeski theory $\mathcal{L}^H + \mathcal{L}^{BH}$ [Gleyzes et al., 2015].

$$\mathcal{L}_4^{BH} = F_4(\phi, X)\Big[X\left((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right) - 2\left(\Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_\nu - \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\right)\Big], \quad (6)$$

$$\begin{aligned}\mathcal{L}_5^{BH} &= F_5(\phi, X)\Big[X\left((\Box\phi)^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho\right) \\ &- 3\Big((\Box\phi)^2\,\phi_\mu\phi^{\mu\nu}\phi_\nu - 2\,\Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho - \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma + \\ &2\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\Big)\Big]. \quad (7)\end{aligned}$$

Generalized Proca theory [Heisenberg, 2014, Allys et al., 2016a]

$$
\begin{aligned}
\mathcal{L}_2^{\mathrm{GP}} &= G_2(A_\mu, F_{\mu\nu})\,, \\
\mathcal{L}_3^{\mathrm{GP}} &= G_3(X)\nabla_\mu A^\mu, \\
\mathcal{L}_4^{\mathrm{GP}} &= G_4(X)R + G_{4,X}\left[(\nabla_\mu A^\mu)^2 - \nabla_\rho A_\sigma \nabla^\sigma A^\rho\right], \\
\mathcal{L}_5^{\mathrm{GP}} &= G_5(X)G_{\mu\nu}\nabla^\mu A^\nu - \frac{1}{6}G_{5,X}\Big[(\nabla\cdot A)^3 - 3(\nabla\cdot A)\nabla_\rho A_\sigma \nabla^\sigma A^\rho + \\
&\quad 2\nabla_\rho A_\sigma \nabla^\gamma A^\rho \nabla^\sigma A_\gamma\Big] - g_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu \nabla_\alpha A_\beta, \qquad (8) \\
\mathcal{L}_6^{\mathrm{GP}} &= -\frac{1}{2}G_6(X)L^{\mu\nu\alpha\beta}\nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{1}{2}G_{6,X}\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}\nabla_\alpha A_\mu \nabla_\beta A_\nu\,.
\end{aligned}
$$

Beyond Generalized Proca theory $\mathcal{L}^{\rm GP} + \mathcal{L}^{BP}$ [Heisenberg et al., 2016]

$$\mathcal{L}_4^{\rm N} = f_4 \hat{\delta}^{\beta_1\beta_2\beta_3\gamma_4}_{\alpha_1\alpha_2\alpha_3\gamma_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A_{\beta_2} \nabla^{\alpha_3} A_{\beta_3} \,, \tag{9}$$

$$\mathcal{L}_5^{\rm N} = f_5 \hat{\delta}^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A_{\beta_2} \nabla^{\alpha_3} A_{\beta_3} \nabla^{\alpha_4} A_{\beta_4} \,, \tag{10}$$

$$\tilde{\mathcal{L}}_5^{\rm N} = \tilde{f}_5 \hat{\delta}^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A^{\alpha_3} \nabla_{\beta_2} A_{\beta_3} \nabla^{\alpha_4} A_{\beta_4} \,, \tag{11}$$

$$\mathcal{L}_6^{\rm N} = \tilde{f}_6 \hat{\delta}^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} \nabla_{\beta_1} A_{\beta_2} \nabla^{\alpha_1} A^{\alpha_2} \nabla_{\beta_3} A^{\alpha_3} \nabla_{\beta_4} A^{\alpha_4} \,. \tag{12}$$

They were introduced later and not in a systematic way

In Ref. [Gallego Cadavid and Rodriguez, 2019] we present a systematic procedure to build the most general Proca and SU(2) Proca theories

i) Test Lagrangians

All possible Lorentz invariants in flat spacetime (FST) using only

$$A_\mu,\ \partial_\mu A_\nu,\ g_{\mu\nu},\ \text{and}\ \epsilon_{\mu\nu\rho\sigma}, \tag{13}$$

where $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$ are the metric and Levi-Civita tensors, respectively.

Primary Hessian condition [Heisenberg, 2014]

$$^{(1st)}\mathcal{H}^{\mu 0 de}_{\mathcal{L}_{\text{test}}} = \frac{\partial^2 \mathcal{L}_{\text{test}}}{\partial \dot{A}_{\mu d} \partial \dot{A}_{0e}} = 0. \tag{14}$$

Secondary Hessian condition [Errasti Díez et al., 2020]

$$^{(2nd)}\mathcal{H}^{de}_{\mathcal{L}_{\text{test}}} = \frac{\partial^2 \mathcal{L}_{\text{test}}}{\partial \dot{A}_{0d} \partial A_{0e}} - \frac{\partial^2 \mathcal{L}_{\text{test}}}{\partial \dot{A}_{0e} \partial A_{0d}} = 0. \tag{15}$$

In a curved spacetime (CST) the Hessian conditions are not sufficient to account for the ghost and Laplacian instabilities → Stability analysis must be performed [Gómez and Rodríguez, 2019, Gómez and Rodriguez, 2020].

Find constraints among the test $\mathcal{L}$'s

To this end, it is handy to use the identity

$$A^{\mu\alpha}\tilde{B}_{\nu\alpha} + B^{\mu\alpha}\tilde{A}_{\nu\alpha} = \frac{1}{2}(B^{\alpha\beta}\tilde{A}_{\alpha\beta})\delta^{\mu}_{\nu}, \tag{16}$$

valid for all antisymmetric tensors A and B.

Identify the test $\mathcal{L}$'s related by $\partial_\mu J^\mu$

In a FST we use expressions of the form

$$\partial_\mu J^\mu = \mathcal{L}_i + \mathcal{L}_j \qquad (17)$$

to eliminate $\mathcal{L}_j$ in terms of $\mathcal{L}_i$ since they yield the same EofM.

But in CST, coming from the same previous expression, we might have

$$\nabla_\mu J^\mu = \mathcal{L}_i + \mathcal{L}_j + \mathcal{F}(A^\mu, \nabla^\mu A^\nu)\,, \qquad (18)$$

hence the field equations for $\mathcal{L}_i$ and $\mathcal{L}_j$ *may not* be the same due to $\mathcal{F}$

Covariantize the resulting FST theory

- Follow the minimal coupling principle.
- Include possible coupling terms between A_μ and the curvature tensors [Allys et al., 2016a].

vi) Split A^μ into the pure scalar and vector modes

- Decompose

$$A^\mu = \nabla^\mu \phi + \hat{A}^\mu, \tag{19}$$

where ϕ is the Stuckelberg field and $\hat{A}^\mu$ is the divergence-free contribution.
- Verify that the field equations for all physical DoF fulfil Ostrogradsky's theorem.
- Add appropriate counterterms.

In the case of the $\mathcal{L}_4^{\rm GP}$ Proca Lagrangian

$$\mathcal{L}_4^{\rm GP} \sim (\partial_\mu A_\nu)(\partial_\rho A_\sigma). \tag{20}$$

i) The test Lagrangian is

$$\mathcal{L}_{\rm test} = \sum_{i=1}^{11} f_i(X)\mathcal{L}_i, \tag{21}$$

where $X \equiv \partial_\mu A \partial^\mu A$ and $f_i(X)$ are arbitrary functions.

After applying the Hessian condition and the constraints (steps ii and iii)

$$\mathcal{L}_{\rm test} = f_{2,3}(X)(\mathcal{L}_2 - \mathcal{L}_1) + f_7(X)(\mathcal{L}_7 - \mathcal{L}_5) \tag{22}$$

iv) Flat Space-Time Currents in the Lagrangian

The term $(\mathcal{L}_7 - \mathcal{L}_5)$ may be removed *in FST* since

$$f_7(X)(\mathcal{L}_7 - \mathcal{L}_5) = F_7(X)(\mathcal{L}_2 - \mathcal{L}_1 - A^\mu[\partial_\mu, \partial_\nu]A^\nu) + \partial_\mu J^\mu_\delta \,, \tag{23}$$

where $A^\mu[\partial_\mu, \partial_\nu]A^\nu \equiv A^\mu\partial_\mu\partial_\nu A^\nu - A^\mu\partial_\nu\partial_\mu A^\nu$.

This part is crucial since in CST the ∇_μ's do not commute

$$f_7(X)(\mathcal{L}_7 - \mathcal{L}_5) = F_7(X)(\mathcal{L}_2 - \mathcal{L}_1 - \underbrace{A^\mu[\nabla_\mu, \nabla_\nu]A^\nu}_{A^\mu R_{\mu\nu} A^\nu}) + \nabla_\mu J^\mu_\delta \,. \tag{24}$$

Final Lagrangian [Gallego Cadavid and Rodriguez, 2019]

After covariantization and taking the scalar limit (steps v and vi)

The most general Lagrangian for a Proca field theory in the $\mathcal{L}_4^{\rm GP}$ sector is

$$\mathcal{L}_4^{\rm GP} = G_4(X)R - G_{4,X}(X)\delta^{\mu_1\mu_2}_{\nu_1\nu_2}(\nabla_{\mu_1}A^{\nu_1})(\nabla^{\nu_2}A_{\mu_2}) + f_4^{\rm N}(X)\delta^{\beta_1\beta_2\beta_3\gamma_4}_{\alpha_1\alpha_2\alpha_3\gamma_4}A^{\alpha_1}A_{\beta_1}\nabla^{\alpha_2}A_{\beta_2}\nabla^{\alpha_3}A_{\beta_3}\,. \quad (25)$$

The "beyond" terms are generated in CST, terms that simply vanish in FST.

Generalized SU(2) Proca theory

In Ref. [Allys et al., 2016b] consider an SU(2) gauge field A_μ^a

$$\mathcal{L}^{\mathrm{GSU2P}} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \sum_{n=1}^{2} \mathcal{L}_{2n} + \sum_{m=1}^{5} \mathcal{L}_m^{\mathrm{cst}} . \tag{26}$$

- The theories have been successfully applied to inflation and DE scenarios [Rodríguez and Navarro, 2018].
- The ghost and Laplacian instabilities have been studied in Ref. [Gómez and Rodríguez, 2019, Gómez and Rodriguez, 2020].
- The "beyond" terms are missing.

The Generalized SU(2) Proca theory and beyond [Gallego Cadavid et al., 2020]

$$\mathcal{L}_2 = \mathcal{L}_2(A^a_{\mu\nu}, A^a_\mu)$$

$$\mathcal{L}_{4,0} = G_{\mu\nu} A^{\mu a} A^\nu_a$$

$$\mathcal{L}_{4,2} = \sum_{i=1}^{6} \frac{\alpha_i}{m_P^2} \mathcal{L}^i_{4,2} + \sum_{i=1}^{3} \frac{\tilde{\alpha}_i}{m_P^2} \tilde{\mathcal{L}}^i_{4,2}$$

$$\tilde{\mathcal{L}}_{5,0} = A^{\nu a} R^\sigma{}_{\nu\rho\mu} A^b_\sigma \tilde{A}^{\mu\rho c} \epsilon_{abc}$$

$$\mathcal{L}^1_{4,2} = (A_b \cdot A^b)[S^{\mu a}_\mu S^\nu_{\nu a} - S^{\mu a}_\nu S^\nu_{\mu a}] + 2(A_a \cdot A_b)[S^{\mu a}_\mu S^{\nu b}_\nu - S^{\mu a}_\nu S^{\nu b}_\mu]$$

$$\mathcal{L}^2_{4,2} = A^a_{\mu\nu} S^{\mu b}_\sigma A^\nu_a A^\sigma_b - A^a_{\mu\nu} S^{\mu b}_\sigma A^\nu_b A^\sigma_a + A^a_{\mu\nu} S^{\rho b}_\rho A^\mu_a A^\nu_b$$

$$\mathcal{L}^3_{4,2} = A^{\mu a} R^\alpha{}_{\sigma\rho\mu} A_{\alpha a} A^{\rho b} A^\sigma_b + \frac{3}{4}(A_b \cdot A^b)(A^a \cdot A_a) R$$

$$\mathcal{L}^4_{4,2} = [(A_b \cdot A^b)(A^a \cdot A_a) + 2(A_a \cdot A_b)(A^a \cdot A^b)] R$$

$$\mathcal{L}^5_{4,2} = G_{\mu\nu} A^{\mu a} A^\nu_a (A^b \cdot A_b)$$

$$\mathcal{L}^6_{4,2} = G_{\mu\nu} A^{\mu a} A^{\nu b} (A_a \cdot A_b)$$

$$\tilde{\mathcal{L}}^1_{4,2} = A^a_{\mu\nu} S^{\mu b}_\sigma A_{\alpha a} A_{\beta b} \epsilon^{\nu\sigma\alpha\beta} - \tilde{A}^{\alpha\beta}_a S^b_{\rho\alpha} A^{\rho a} A_{\beta b} + \tilde{A}^{\alpha\beta}_a S^\rho_{\rho b} A^a_\alpha A^b_\beta$$

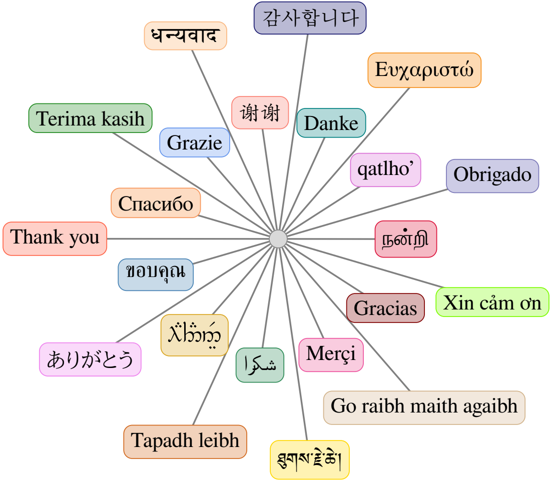
$$\tilde{\mathcal{L}}^2_{4,2} = A^b_\beta R^\alpha{}_{\sigma\rho\mu} A^a_\alpha (A_a \cdot A_b) \epsilon^{\mu\rho\sigma\beta}$$

$$\tilde{\mathcal{L}}^3_{4,2} = A_{\beta a} R^\alpha{}_{\sigma\rho\mu} A^a_\alpha (A^b \cdot A_b) \epsilon^{\mu\rho\sigma\beta}$$

- The method systematically produces the most general Lagrangian for a scalar- and vector-tensor theories.
- Total derivatives in FST may no longer be total derivatives in CST.
- The "Beyond terms" are hidden in terms that look as total derivatives in the Lagrangian.
- We obtain *brand new interaction terms*.
- The new terms could bring rich phenomenological features in cosmology and astrophysics.

Future projects

- The Extended SU(2) Proca theory (in collaboration with Carlos Nieto and Yeinzon Rodriguez.)



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Terima kasih
Grazie
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Спасибо
Thank you
நன்றி
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