Instituto Superior Técnico

Projecto MEFT

Stability of Circular Orbits in the Three Body Problem

Author:
João Cruz

Supervisor:
Prof. Rui Dilão

Research work performed for the Master in Engineering Physics

at

Nonlinear Dynamics Group
Department of Physics

June 29, 2020

1 Introduction

1.1 Brief Historical Background and Motivation

It is perhaps surprising how little it is known about the motion of many body systems under classical gravity, especially taking into account that these have been studied ever since Newton formulated the law of universal gravitation in the 17^{th} century and tried to describe the motion of the Earth and the Moon around the Sun. While the Keplerian two body problem is well known with closed exact solutions, the three body problem sees many difficulties arise, as non linearity and chaos prevent the same mathematical treatment that was used before. These hardships lead some of the most famous names in mathematics and classical physics, such as Euler, Lagrange and Poincaré to tackle the problem.

In order to more effectively study it, the three body problem is usually segmented into different parts, depending on what specific case is of interest. Out of these partitions, the circular restricted three body problem is the most studied and comprehended [1], since it is, fortunately for us, both one of the ways to simplify the equations and a very good approximation to many real cases, which makes it one of the most analytically treatable versions and also very useful at the same time.

On the other hand, the general three body problem, the foundation of the proposed work, isn't as easily dealt with. However, the focus here is once again on a specific partition of this problem - stability of planetary systems. This will be done by starting with the three body problem to establish first results and methodology, which will later be extended to an N body system.

As will be seen in the following section, the equations governing the three body problem depend only on the mass parameters of each body and it is clear that the planets' orbits around the star tend to become Keplerian as their masses get smaller compared to the star's. However, it is known that it is not easy to stabilise a three body problem with comparable masses between bodies. Therefore, there should be a limited interval for the mass parameters that allows the system to be stable. The fact that most planets in our solar system have orbits of negligible eccentricity and that in all known planetary systems planets have masses orders of magnitude lower than their stars is further evidence of this.

Other unexplained observations have been made throughout the years and are hints that organisation of planetary systems must have some dynamical origin. Amongst these is the empirical Titius-Bode law, discovered in the 18^{th} century, which establishes a geometric relation between the orbit radii of the planets in our solar system. A modern version of this law can be stated as $r_n = r_0 1.73^n$ [2], where r_n is the radius of the n^{th} planet in the solar system, with r_1 corresponding to Mercury's orbital radius. A similar expression is valid for the orbital periods.

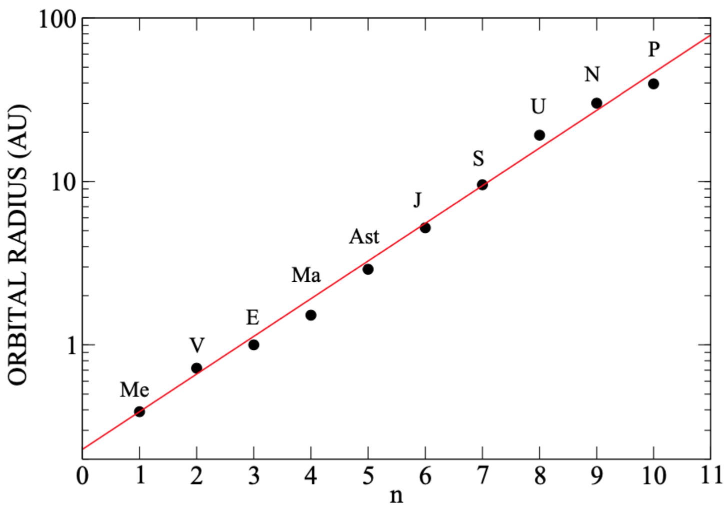

FIGURE 1: Titius-Bode law (red line) in a semi-logarithmic scale. The planets, the asteroid belt and Pluto are represented [3]

For some time, some astronomers have dismissed this law as a numerical coincidence. However, the fact that this law predicted a planet between Mars and Jupiter lead to the discovery of Ceres and, consequently, the asteroid belt in the 19^{th} century [4], which solidified the belief that there must be a physical meaning to it. Even though there is still no definite explanation for the Titius-Bode law, some work has been done recently that attempts to find an origin for it, and there is initial evidence that it is related to the stability of planetary systems, which will be mentioned in a later section.

The goal of this work is therefore to be able to obtain significant results regarding stability boundaries for the three body problem, then extending the methodology to an N body problem and searching for similar boundaries depending on masses and planetary configurations that might shed some light on the stability and final configurations of planetary systems as they are known today.

1.2 Main mathematical background [2]

A general N body problem is described by the Newtonian equations:

$$m_i \ddot{\vec{R}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} G \frac{m_i m_j}{|\vec{r_{ij}}|^3} \vec{r_{ij}} \,, \tag{1}$$

where $\vec{r_{ij}} = \vec{R_j} - \vec{R_i}$ is the vector pointing from mass i to mass j. These equations translate to a total of $3N$, 2^{nd} order, or $6N$, 1^{st} order equations, where N is the number of bodies considered.

There are a total of ten constants of motions in this problem, one resulting from conservation of energy, six resulting from the centre of mass having uniform motion, and three from conservation of angular momentum. As such, the total number of independent equations for any N body problem is $6N - 10$.

For the 3 body problem, equations (1) reduce to three vectorial 2^{nd} order equations

$$\begin{cases} m_1 \ddot{\vec{R}}_1 = G \frac{m_1 m_2}{|\vec{r_{12}}|^3} \vec{r_{12}} + G \frac{m_1 m_3}{|\vec{r_{13}}|^3} \vec{r_{13}} \\ m_2 \ddot{\vec{R}}_2 = G \frac{m_2 m_1}{|\vec{r_{21}}|^3} \vec{r_{21}} + G \frac{m_2 m_3}{|\vec{r_{23}}|^3} \vec{r_{23}} \\ m_3 \ddot{\vec{R}}_3 = G \frac{m_3 m_1}{|\vec{r_{31}}|^3} \vec{r_{31}} + G \frac{m_3 m_2}{|\vec{r_{32}}|^3} \vec{r_{32}} \,. \end{cases} \tag{2}$$

The constants of motion related to the centre of mass can be used to reduce the number of equations, since this means the 3 coordinate vectors $\vec{R_i}$ are linearly dependent. Noting also that it is better to analyse the system in the centre of mass reference frame, where the coordinates are denoted $\vec{R_i}'$, it is more useful to write the equations in terms of $\vec{r}_{ij} = \vec{r}'_{ij}$, and then calculate the coordinates $\vec{R_i}' = \vec{R_i} - R_{CM}$ in terms of them. To eliminate one of the vectors, one can write $\vec{r}_{23} = \vec{R_3} - \vec{R_2} = (\vec{R_3} - \vec{R_1}) - (\vec{R_2} - \vec{R_1}) = \vec{r}_{13} - \vec{r}_{12}$. The coordinates in the centre of mass are then given by

$$\begin{cases} \vec{R_1}' = -\frac{m_2 \vec{r}_{12} + m_3 \vec{r}_{13}}{m_1 + m_2 + m_3} \\ \vec{R_2}' = \frac{(m_1 + m_3)\vec{r}_{12} - m_3 \vec{r}_{13}}{m_1 + m_2 + m_3} \\ \vec{R_3}' = \frac{-m_2 \vec{r}_{12} + (m_1 + m_3)\vec{r}_{13}}{m_1 + m_2 + m_3} \,. \end{cases} \tag{3}$$

As such, equations (2) can be rewritten by diving both sides by m_i and subtracting the first equation to the second and the third, which yields

$$\begin{cases} \ddot{\vec{r}}_{12} = -G\frac{m_1+m_2}{|\vec{r}_{12}|^3}\vec{r}_{12} - G\frac{m_3}{|\vec{r}_{13}|^3}\vec{r}_{13} + G\frac{m_3}{|\vec{r}_{13}-\vec{r}_{12}|^3}(\vec{r}_{13}-\vec{r}_{12}) \\ \\ \ddot{\vec{r}}_{13} = -G\frac{m_1+m_3}{|\vec{r}_{13}|^3}\vec{r}_{13} - G\frac{m_2}{|\vec{r}_{12}|^3}\vec{r}_{12} + G\frac{m_2}{|\vec{r}_{12}-\vec{r}_{13}|^3}(\vec{r}_{12}-\vec{r}_{13}) \,. \end{cases} \tag{4}$$

One important thing to note is that, although they describe the same thing, the system of equations (2) is Hamiltonian, whereas the system (4) is not. In order words, the latter system of equations can't be derived from a Hamiltonian or a Lagrangian with the variational principle. This has some theoretical implications - methods developed for Hamiltonian systems are not applicable in equations (4).

It is sometimes useful to deal with normalised equations, and as such the parameters $\mu_i = m_i/(m_1 + m_2 + m_3)$ and the coordinates $\tau = \sqrt{G}t$ and $\vec{u}_{ij} = (m_1 + m_2 + m_3)^{-1/3}\vec{r}_{ij}$ can be introduced into the previous equations, yielding

$$\begin{cases} \ddot{\vec{u}}_{12} = -\frac{\mu_1+\mu_2}{|\vec{u}_{12}|^3}\vec{u}_{12} - \frac{\mu_3}{|\vec{u}_{13}|^3}\vec{u}_{13} + \frac{\mu_3}{|\vec{u}_{13}-\vec{u}_{12}|^3}(\vec{u}_{13}-\vec{u}_{12}) \\ \\ \ddot{\vec{u}}_{13} = -\frac{\mu_1+\mu_3}{|\vec{u}_{13}|^3}\vec{u}_{13} - \frac{\mu_2}{|\vec{u}_{12}|^3}\vec{u}_{12} + \frac{\mu_2}{|\vec{u}_{12}-\vec{u}_{13}|^3}(\vec{u}_{12}-\vec{u}_{13}) \,, \end{cases} \tag{5}$$

where the derivatives are now taken with respect to τ.

Equations (4) and (5) are the main equations that are used to describe and numerically integrate the problem. They are equivalent, and depending on the situation one might prefer to use regular units or normalised values, since the first might be easier to interpret when comparing with a real system and the latter might be more useful from a mathematical point of view when comparing orders of magnitude between the terms. Obtained solutions are then used to compute the orbits in the centre of mass reference frame through equations (3). The equations describing an N body problem can be easily obtained through the same method. In fact, the pattern in equations (5) is evident, from which it is simple to extend them for more bodies.

As a final note, it is important to highlight that, in the limit where the planets' masses are negligibly small comparing to the mass m_1 of the star, the parameters μ_2 and μ_3 become much smaller than μ_1 in (5), and therefore $\mu_1 + \mu_2 \approx \mu_1 + \mu_3 \approx \mu_1$ and the two final terms in both equations can be neglected, leading to

$$\ddot{\vec{u}}_{12} = -\frac{\mu_1}{|\vec{u}_{12}|^3}\vec{u}_{12} \quad \text{and} \quad \ddot{\vec{u}}_{13} = -\frac{\mu_1}{|\vec{u}_{13}|^3}\vec{u}_{13} \,, \tag{6}$$

which represent well known Keplerian motions of individual non interacting planets around the star. This is the support for the argument presented in the previous section as to why there should be a limited interval for the masses for which stable periodic orbits exist.

2 State of the Art

Over the course of the last 300 hundred years, it is expected that a lot of effort has been put into this problem. However, due to the chaotic and non linear nature of gravitational systems, which translates into a plethora of diverse possible cases and solutions, studies are usually divided into specific cases, as mentioned previously. Therefore, to avoid being lost in a maze of information, much of which wouldn't be relevant for this work, of the investigation of the state of the art's focus is in what periodic solutions are known to date in the 3 body problem, how have collisions been handled so far and lastly attempts at investigating stability specifically directed towards planetary systems.

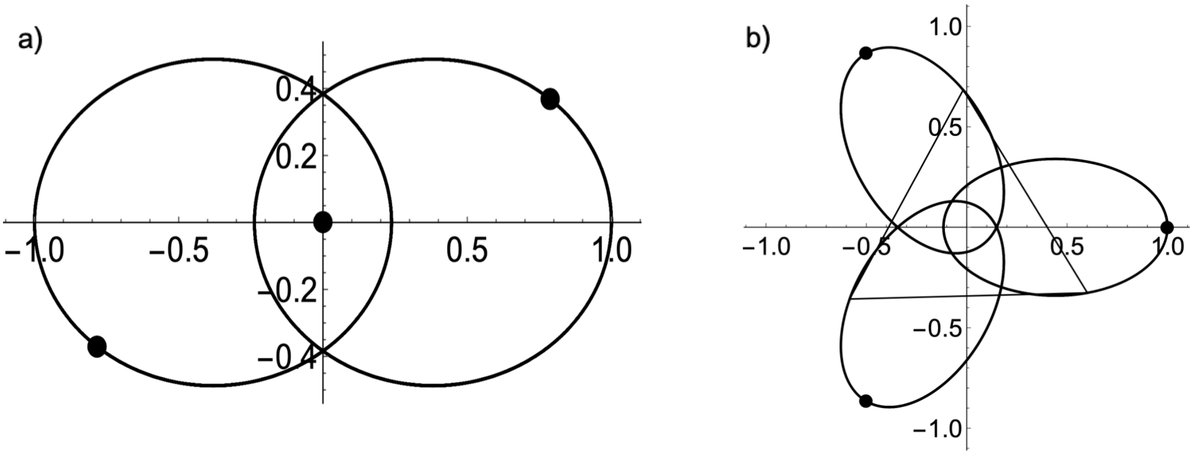

FIGURE 2: a) Example of an Euler periodic planar solution, with bodies arranged in a linear configuration and elliptic orbits. b) Example of a Lagrange periodic planar solution, with the bodies arranged as the vertices of an equilateral triangle. The drawn triangle corresponds to the initial configuration of the bodies [2].

2.1 Known Periodic Solutions

The periodic orbits found ever since this problem started to be studied can be essentially divided into three families, which are exposed here by chronological order of discovery. An important thing to keep in mind is that, evidently, collisions cannot happen in periodic solutions. Hence, the following motions are all collisionless.

2.1.1 Euler and Lagrange

Euler was the first to obtain a group of exact solutions for a periodic orbit (1767), followed closely by Lagrange (1772). These are usually considered in the same family since not only were they discovered at around the same time, but also the concept is similar for both cases. Euler discovered that by placing the three bodies in a linear configuration, given suitable initial conditions, they would stay aligned during the whole motion and orbit in ellipses about the centre of mass. Lagrange, on the other hand, demonstrated the same thing but for a configuration where the bodies are placed in the vertices of an equilateral triangle [2]. The triangle's area and orientation changes during the motion, but the shape is maintained at all times throughout the motion. The Euler and Lagrange solutions are exemplified in Figure 2.

In both cases, since the initial configurations of the bodies are maintained, they are known as central configurations of the three body problem and have appeared in more recent times. It is also relevant to note that the Euler orbits are unstable [1] under small displacements, whereas the Lagrange orbits have regions of stability and instability [1, 6].

2.1.2 Analytical continuation of the restricted three body problem

The restricted three body problem is a particular case of the general problem that is not within the scope of this work. However, the second family of solutions, which was consolidated only in the mid 1970s [7], stems from it, and so a brief description is needed. The restricted three body problem consists in considering two main bodies, the primaries of the system, that have a usual Keplerian two body motion, and a third one whose mass is small enough not to disturb the motion of the primaries. This is particularly useful since it simplifies the equations of the three body problem when studying motions of small celestial bodies.

[1] It has been shown in [5] that, for suitable perturbations, Euler solutions are structurally unstable.

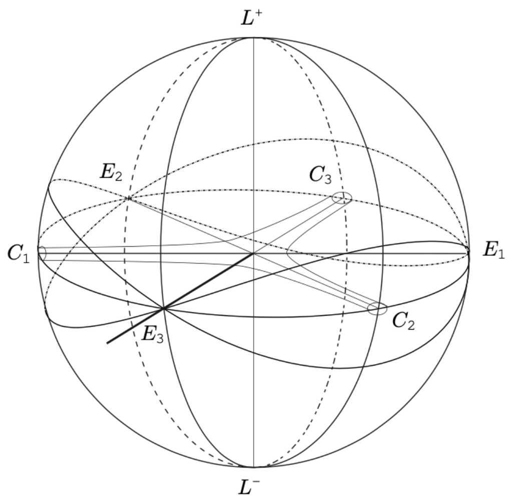

FIGURE 3: Chenciner and Montgomery's shape sphere. E_1, E_2 and E_3 correspond to Euler's central configurations, whereas L^+ and L^- correspond to Lagrange's. C_1, C_2 and C_3 correspond to 2 body collisions. The figure-8 orbit is represented in the surface of the sphere, going through the three Euler central configurations [10].

Poincaré established periodic solutions for the restricted problem, but only more recently, with advancements in the theory of dynamical systems, was it proven that those same periodic orbits could be analytically continued for small enough masses of the third body in both the planar and the 3D cases [1]. Due to the nature of these solutions, they are not exact as the Euler and Lagrangian solutions. Therefore, they are still obtained numerically and are not restricted to only a couple of configurations. This in turn is the reason why some members of this family have been discovered (or rediscovered) until at least as recently as 2004 [8].

It is also relevant to note that this family of orbits is, out of the three, the one that comes closer to the proposed work, since it also relies on establishing intervals for masses, even though the starting point for this work is not the restricted three body problem, but a more general case.

2.1.3 Figure-8 orbit and its extensions in the shape spheres

The last family of known periodic solutions was hinted at in 1993 [9], when a planar periodic motion was found where the three bodies describe an "8" shape with net zero angular momentum[2]. This motion was rediscovered through a new method developed by Chenciner and Montgomery [10] that became more widespread.

In a more purely mathematical approach to the problem, the usual real space was transformed into a spherical configuration space, called shape sphere, by resorting to group theory techniques. Each point in the sphere corresponds to a specific configuration, among which are the Euler and Lagrange's configurations as special cases. This represents therefore a way to extend the first family of solutions. The so called figure-8 orbit can then be represented in the surface of this sphere. An illustration of the shape sphere and the orbit is shown in Figure 3.

By using and further enhancing this method, the figure-8 orbit family was extended to 13 more classes of periodic solutions in 2013 [8], which in turn enabled the discovery of thousands of similar periodic orbits, including in systems with bodies of different masses [11, 12]. All these motions have similar representations on the shape spheres, are planar and have net zero angular momentum.

It is easily concluded from a quick overview of all three families that there is a lack of knowledge regarding periodic orbits in planetary systems since, apart from some specific cases where the second

[2]Note that this contrasts with the Kepler two body problem, where zero angular momentum movements are always either collision or escape orbits.

family might be applied (as in a binary star system with a planet orbiting them), none of these currently known solutions have realistic configurations for general cases. Moreover, these motions are all collisionless, which is also an unrealistic expectation in what concerns planetary systems - after all, planets form through successive collisions between planetesimals in the protoplanetary disk.

2.2 Collisions

Within the scope of the gravitational 3 body problem, collisions are a subject that have been lightly touched but mostly ignored. On the one hand, they are potential mathematical nuisances since they manifest as singularities in the equations. On the other hand, as there usually is more interest in describing stable systems or motions that do not continue after two bodies meet, not much has been done to obtain solutions after a collision occurs. However, it is necessary to deal with collisions to understand stability of planetary systems.

In fact, some attempts have been made to model both two and three body collisions in the three body problem. For triple collisions, the most notable results have been obtained by McGehee [13], who introduced coordinate transformations that allowed the singularity to be magnified, hence revealing a distorted surface, a collision manifold. This, however, is beyond the scope of this work and as such will not be discussed further.

In the case of two body collisions, the most successful attempt was achieved by Sundman, who regularised the equations to remove singularities at the time of collisions. His particular method lead to a complete solution of the three body problem [14], a remarkable result which comes, however, with caveats. Firstly, the solution is written in terms of a power series that has an extremely slow convergence, so slow that actually renders it unusable, since it requires the computation of too large an amount of terms to integrate an infinitesimal time [1]. Furthermore, even if it was usable, the regularisation performed, as well as the other methods to deal with collisions, rest on the the assumption that they are elastic, which is an unphysical assumption in planetary dynamics, for reasons already mentioned in the previously section. A more accurate way to describe a more realistic scenario would involve necessarily inelastic collisions resulting in merges of bodies, i.e., accretion.

2.3 Stability of planetary systems

On a final analysis of the field's state of the art, it is also important, considering the scope of this work, to mention recent attempts to address stability and organisation of planetary systems not restricted to three bodies. Stability of the solar system has been a debated topic over the years. Although it is obviously stable over short enough timescales, the solar system is chaotic and its stability over longer timescales is still a source of controversy [15].

As such, new results that establish an important proof of concept should be highlighted. In particular, a recent attempt was made at numerically obtaining self stabilising planetary systems, based purely on their dynamics, starting from configurations other than that of our solar system [3]. One main idea that resulted in a change to the equations was to assume a presence of a protoplanetary disk, which was modelled as a viscous fluid that interacted with the planets by means of a rather simplistic phenomenological non conservative force that would drain angular momentum from the planets up to a maximum pre-determined value, after which the equations would reduce to their usual form, (1). Moreover, collisions were also taken into account as non elastic, although not very explicitly how.

Due to the rather experimental and nature of this modelling, it might not come as a surprise that a system with as many planets as our solar system was not possible to stabilise. However, it was found that, from a uniform configuration, the planets would either collide or be ejected until a stable configuration was achieved. Furthermore, all these stable configurations would follow Titius-Bode like laws ranging from $r_n \sim 1.63^n$ to $r_n \sim 1.69^n$, very close to the value in our solar system, which is now very convincing evidence that this law is a consequence of planetary dynamics.

3 Preliminary Results

Some initial results, which showcase the problem at hand, have already been obtained. The system of equations (4) was used to analyse and numerically integrate the motion for different parameters. In this section, all three bodies are considered to be arranged as a planetary system, where the first body is considered to be the star and the other two are (less massive) planets orbiting it, with the initial orbit radius of the third planet being larger than that of the second. Moreover, the entire motion is considered to be planar.

On a first note, it is important to note that the absolute length scales of the system do not affect the dynamics in any way, only the timescale. This was observed numerically and can be confirmed by evaluating the initial position of the centre of mass. If, for a moment, the system is considered in the reference frame where the star is at the origin, we have, for a given initial configuration of the planets (i.e., $x_2 = r_{12}\cos\phi$ and $x_3 = \alpha x_2$, with $\alpha > 1$ and ϕ the initial angle of the first planet's position vector), at the initial time,

$$X_{CM} = \frac{x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} = \frac{m_2 + \alpha m_3}{m_1 + m_2 + m_3} x_2 \Rightarrow \frac{X_{CM}}{r_{12}} = \text{constant}\,, \tag{7}$$

and similarly for Y_{CM} and the y coordinates. This means that the initial position of the centre of mass is proportional to the distance of the first planet to the star, and so changing it won't change the dynamic. What will influence the motion are the masses and the initial configuration (i.e., α and ϕ).

For these preliminary results, the initial configuration was fixed and different values for the planets' mass were tested. Moreover, a system would be defined unstable if any two bodies collided or if any body was ejected from the system. For the configuration and initial conditions tested, however, no ejections were observed, and thus any instability is related solely with collisions. The time until an instability happens, henceforth referred to as collision time, was then measured, and an analysis was conducted using a Padé Approximant to estimate a stability interval for the system.

3.1 Numerical Results

All numerical integrations were performed using the Runge-Kutta method of 4^{th} order, since it is a proven robust method for the three body problem that conserves both energy and angular momentum [16]. The integration was performed over a reasonably long amount of time, constrained only by computational time. The initial configuration in every simulation was set to have all bodies aligned in the X-axis, with the second planet being a distance from the star double that of the first. This corresponds to, using the previous notation, $\alpha = 2$ and $\phi = 0$. In an attempt to produce initially stable circular orbits, the initial velocities of the planets were set to be those of a uniform circular motion about the centre of mass,

$$v_{x_i}(t=0) = 0, \qquad v_{y_i}(t=0) = \sqrt{\frac{GM}{x'_i(t=0)}}\,, \tag{8}$$

where M is the total mass of the system, i.e., of the centre of mass. The masses of the planets were set to be equal to each other and a fraction of the mass of the star, which was normalised to $m_1 = 1$.

Simulations were performed for several different mass parameters of the planets. As an example, two of them are shown in Figures 4 and 5, where a stable and unstable dynamic, respectively, occur.

It is clearly observable in Figure 4 the presence of a mean quasi-circular orbit for each planet. The system remains stable despite the evident non negligible precession, which is expected as it is usual and observed, for example, in the solar system. On the other hand, Figure 5 showcases an unstable system, since at some point there is a collision. Note that in this case the integration is not shown in its entirety, but only the time up to the instability, because the motion after that does not have any physical meaning, since no attempt was made to model collisions. It is worth mentioning that in these

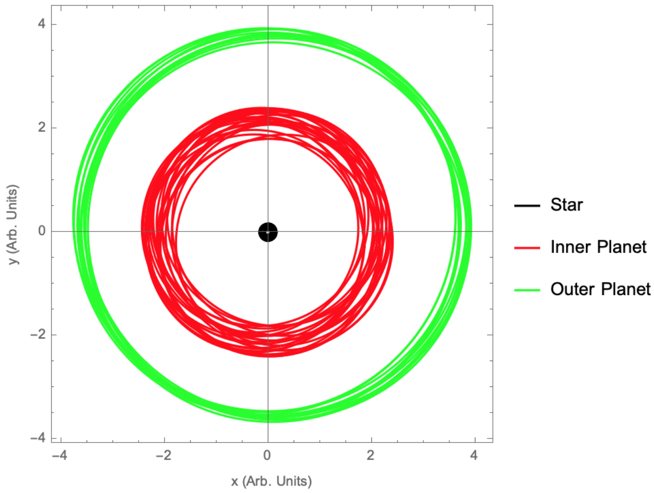

FIGURE 4: Stable orbits. The mass of each planet is 3% of the mass of the star.

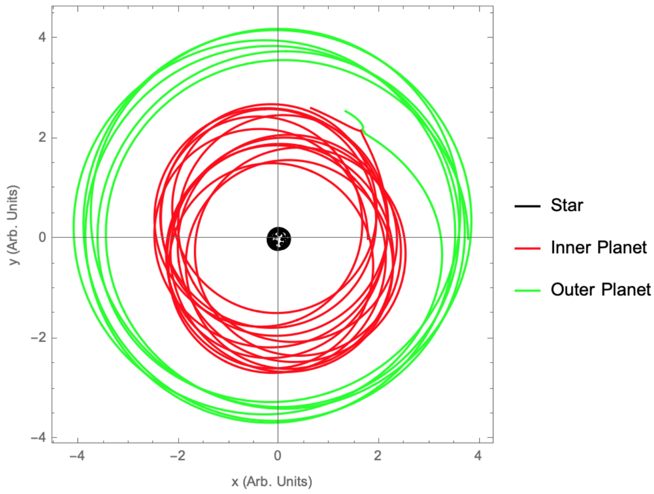

FIGURE 5: Unstable orbits. The mass of each planet is 4% of the mass of the star.

simulations, since the bodies are being treated as point particles, collisions occur when the positions of the two planets are the same. In reality, a collision occurs when the distance between the planets is less than the sum of their mean radii.

The subsequent analysis consisted in extracting the collision time for each simulated system. This measurement was taken by resorting to the velocities, since they are discontinuous at the time of a collision and thus yielded more precise values.

A very clear behaviour was observed following this method - there was a value of the mass parameters under which the system was stable, i.e., the collision time was infinite, whereas above that value the system would always be unstable. In other words, evidence of the existence of a stability region was found. The following section describes how it was estimated.

3.2 Padé Approximant

After obtaining relevant data, the collision time was plotted against the planet mass parameter. This is shown in Figure 6, where the points were obtained from the numerical simulations.

The interest is in obtaining the singularity that seems to appear around 3.2%. A good way to approximate general functions with unknown behaviours is with a power series up to some order. However, regular polynomials don't describe singularities accurately. So, for this purpose, a Padé approximant is ideal [17]. This technique consists of approximating a function by the ratio [18]

$$f(x) \approx \frac{P^{(m)}(x)}{Q^{(n)}(x)}\,,$$

where $P^{(m)}(x)$ and $Q^{(n)}(x)$ are polynomials of degrees m and n, respectively. The Padé approximant is said to be of degree (m,n). By construction, the zeros of f are given by the zeros of P, whereas its singularities are given by the zeros of Q. It was empirically found that (m, n) = (3,3) worked well in this case to fit a ratio of the two polynomials to the points. The result is the blue line in Figure 6.

It is observable that, although the approximant doesn't exactly describe the behaviour, it successfully models the singularity. The denominator, being a cubic polynomial, has three roots, two of which are complex. The real one, corresponding to the critical mass, takes the value of $m_c = 3.21\%$ of m_1. One thing that is also observed is that the fit function does not tend to zero as the mass gets higher. In reality, this is not what happens with the data, since it was found that for higher masses, the collision

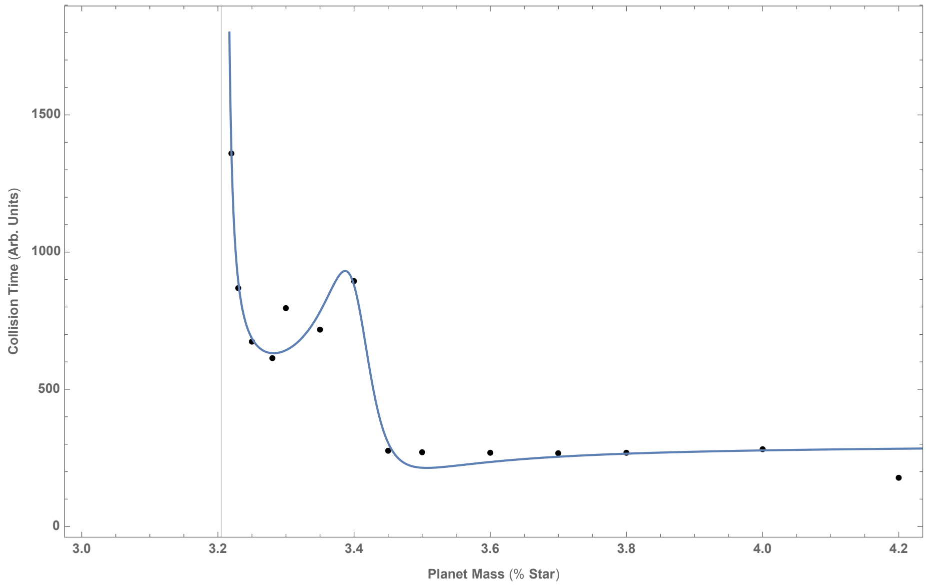

FIGURE 6: Obtained data and Padé Approximant. The denominator of the approximant is a cubic polynomial of the form $y \approx -18.446 + 16.573x - 4.962x^2 + 0.495x^3$.

time diminished drastically. However, the focus was in the study near the singularity; terms of higher order would be needed to correctly model the behaviour in a wider range.

In any case, this is initial evidence that stability limits exist. While they may or not be different for other configurations, they might be able to shed some light into the final organisation of planets in planetary systems, in particular to help explain observations such as the Titius-Bode law.

4 Thesis goals and workplan

As a conclusion for this report, it is important that the objectives of the proposed work are summarised and a plan containing the main stages of the thesis is laid out.

This thesis will focus on identifying stability conditions for the parameters that allow the existence of stable periodic orbits in the three body problem, configured as a planetary system, i.e., 2 bodies (planets) orbiting the 3rd one (a star). As a result of this analysis, the stability problem will be extended to the study of accretion effects in the formation of N body protoplanetary systems. In order to accomplish this, four main tasks have been defined and are laid out in a timeline in Figure 7.

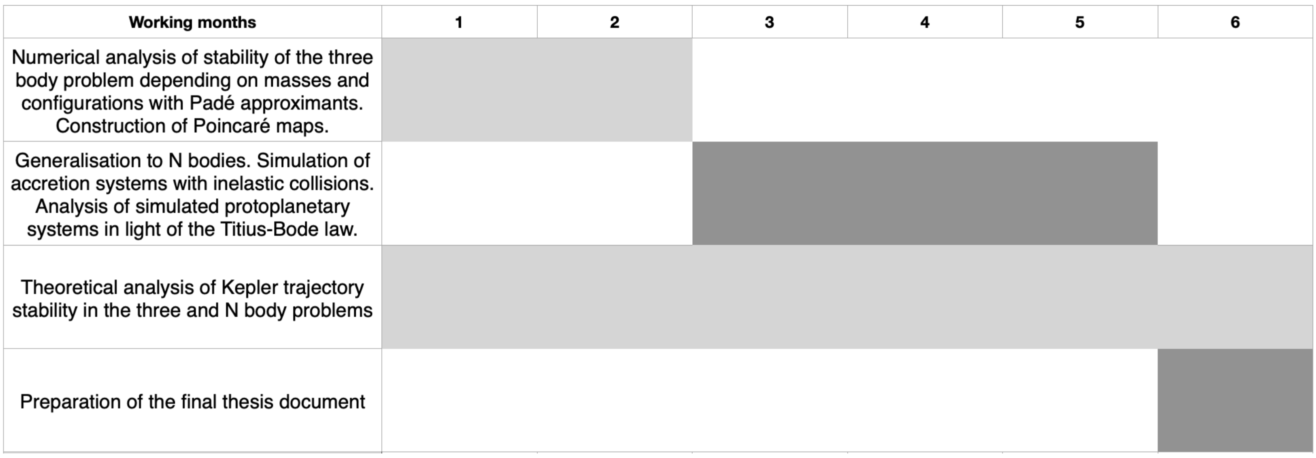

FIGURE 7: Proposed timeline of tasks for the development of the thesis

On a first stage, the focus will be on refining the approach showcased in the previous section and obtaining relevant information on stability in the three body problem. Other tools, such as Poincaré maps, will be essential to draw reliable conclusions. After this stage, the generalisation to N bodies will be made and mechanisms like accretion from inelastic collisions will be introduced. By the end of this task, the final configurations of protoplanetary systems will be able to be analysed, in particular taking into consideration the Titius-Bode law. At the same time, the problem will be analysed from a theoretical standpoint to complement the understanding of the problem. In the final stage of the work the focus will be on organising and establishing all the relevant conclusions and writing the thesis.

References

[1] Z.E. Musielak and B. Quarles. "The three-body problem". In: *Rep. Prog. Phys.* 77 (2014).

[2] Rui Dilão. "Introdução à mecânica celeste". In: *Uma Introdução à Teoria dos Sistemas Dinâmicos e do Caos*. 2018.

[3] R. Pakter and Y. Levin. "Stability and self-organization of planetary systems". In: *Phys. Rev. E* 97 (2018).

[4] M. M. Nieto. "The Titius-Bode Law of Planetary Distances". In: *International Series of Monographs in Natural Philosophy*. Vol. 47. 1972.

[5] R. Dilão and M. Murteira. "Principal Periodic Orbits of the Keplerian Dumbbell System". In: *SIAM Applied Dynamical Systems* 19 (2020), pp. 181–207.

[6] J. Mansilla. "Stability of Hamiltonian Systems with Three Degrees of Freedom and the Three Body Problem". In: *Celestial Mechanics and Dynamical Astronomy* 94 (2006), pp. 249–269.

[7] G. Bozis and J. D. Hadjidemetriou. "On the Continuation of Periodic Orbits from the Restricted to the General Three-Body Problem". In: *Celestial Mechanics* 13 (1976).

[8] M. Šuvakov and V. Dmitrašinović. "Three Classes of Newtonian Three-Body Planar Periodic Orbits". In: *Phys. Rev. Lett.* 110 (2013).

[9] C. Moore. "Braids in classical gravity". In: *Phys. Rev. Lett.* 70 (1993).

[10] A. Chenciner and R. Montgomery. "A remarkable periodic solution of the three-body problem in the case of equal masses". In: *Annals of Mathematics* 150 (2000).

[11] Xiaoming Li et al. "More than six hundred new families of Newtonian periodic planar collisionless three-body orbits". In: *Sci. China-Phys. Mech. Astron.* 60 (2017).

[12] Xiaoming Li et al. "The 1223 new periodic orbits of planar three-body problem with unequal mass and zero angular momentum". In: *Publications of the Astronomical Society of Japan* 70 (2018), pp. 181–207.

[13] R. McGehee. "Triple Collision in the Collinear Three-Body Problem". In: *Inventiones Mathematicae* 27 (1974).

[14] J.Barrow-Green. "Poincaré and the Three Body Problem". In: *History of Mathematics*. Vol. 11. American Math. Soc., 1997.

[15] J. Laskar. "Michel Hénon and the stability of the solar system". In: (2014). arXiv: 1411.4930.

[16] Ernst Hairer et al. "Conservation of First Integrals and Methods on Manifolds". In: *Geometric Numerical Integration*. 2nd ed. Springer, 2006.

[17] G. A. Baker Jr. et al. "An Investigation of the Applicability of the Padé Approximant Method". In: *Journal of Mathematical Analysis and Applications* 2 (1961), pp. 405–418.

[18] A. V. M. Ferris-Prabhu. "Padé Approximant Calculation of the Singularity in the Magnetic Susceptibility of an Ising Square Lattice". In: *NASA TN* D2822 (1965).