Introduction to Lattice QCD

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CP3

SDU

DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE

COVID-19 Disclaimer

Thanks to the organisers for making this possible despite the current situation!

This is...

- my **first remote talk**
- my **first remote audience**

...so please bare with me.

This is intended to be an accessible introduction - please

interrupt and ask questions!

Outline

Section 1

Motivation

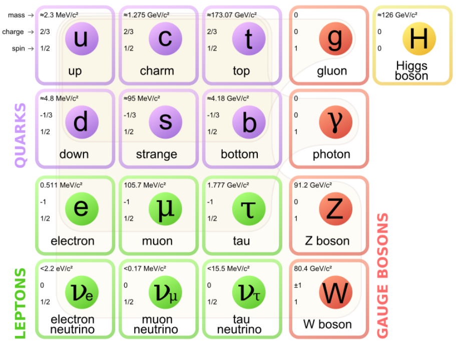
Motivation

The SM is very successful, but...

- Matter/Antimatter asymmetry?
- Why hierarchy of masses?
- Why three generations?
- What is dark matter?
- What is dark energy?
- ...

$\cdots$ **not the end of the story!**

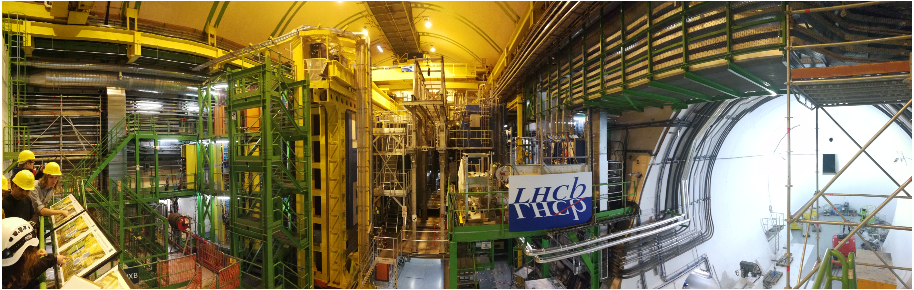
The Standard Model

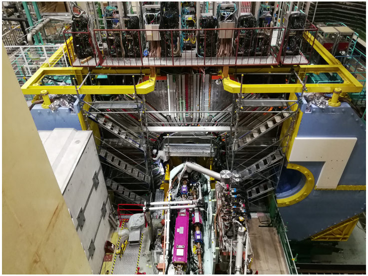


Source: Wikipedia

$\Rightarrow$ Search for New Physics!

Search for new physics: (Flavour) Experiments





top: LHCb at LHC, CERN

left: Belle II at SuperKEKb, KEK

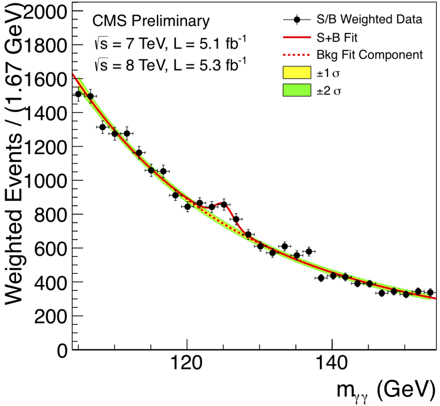
$\Rightarrow$ Huge experimental efforts!
+ BESIII and other LHC experiments

$\Rightarrow$ ***B*-factory** vs **hadron machine**

- "Old" data from BaBar, Belle, Cleo, ...

How can we see New Physics?

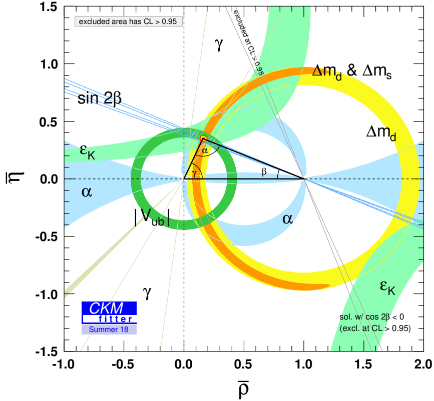
Direct searches



[CERN]

"NP is directly observed and shows as bump in the spectrum"

Indirect searches



[CKM fitter]

"NP modifies processes and shows as discrepancy between Exp and The"

(quark) flavour physics: CKM matrix

experiment $\approx$ CKM $\times$ **nonperturbative** $\times$ known factors

CKM Matrix

- when quarks change flavour
- 3 generations

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

- **in the SM: unitary**

$$VV^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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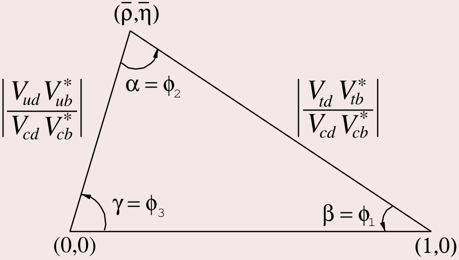
Unitarity Triangle

Source: PDG



$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$$

$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = -1.$$

$\Rightarrow$ Determine CKM matrix elements **from many different processes**

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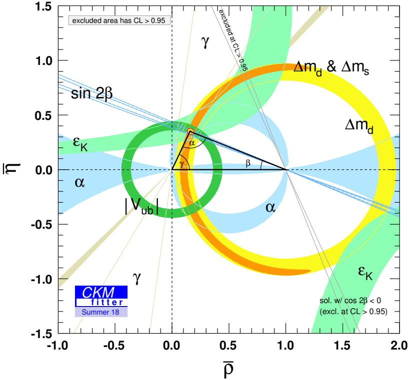

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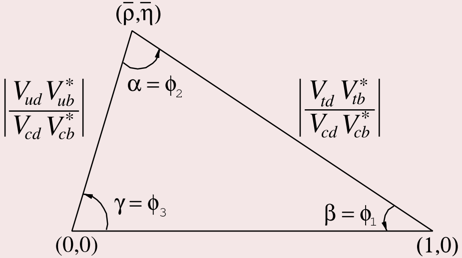

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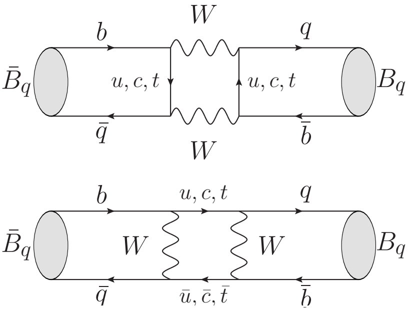
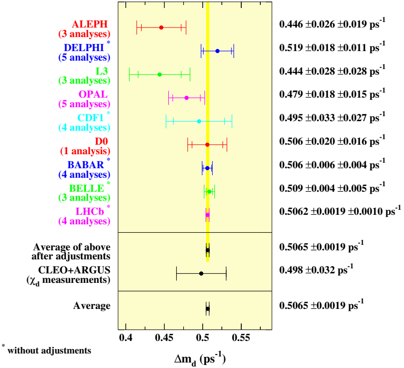
Example: Neutral $B_{(s)}$ meson mixing



where $q = d, s$

Neutral mesons oscillate:
Mass eigenstate $\neq$ flavour eigenstate



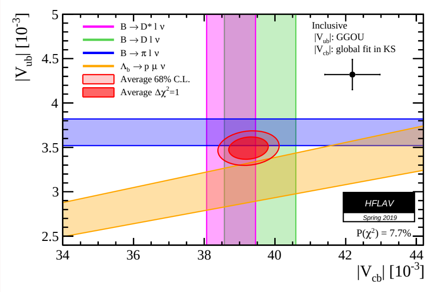
$$\Delta m_d = 0.5065(19)\mathrm{ps}^{-1}$$
$$\Delta m_s = 17.757(21)\mathrm{ps}^{-1}$$ [HFLAV]

below percent level

$$\Delta m_q \propto |V_{tb}^* V_{tq}| \times \textbf{nonperturbative} \times \text{known factors}$$

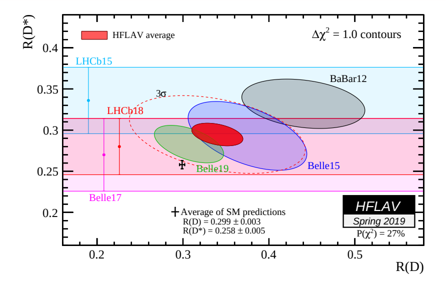
Indirect searches: tensions?

Inclusive vs Exclusive



exclusive: specific (e.g. $B \to D\ell\nu$)
inclusive: general (e.g. $B \to X_c\ell\nu$)

Lepton Flavour Universality



$$R(D^{(*)}) = \frac{\mathcal{B}(B \to D^{(*)}\tau\nu_\tau)}{\mathcal{B}(B \to D^{(*)}\ell\nu_\ell)}$$

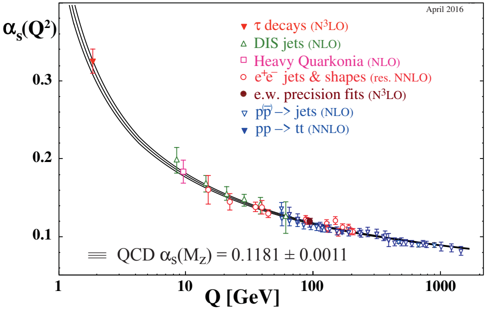
Most stringent test if **EXP and THEO uncertainties are comparable**!

⇒ More experimental data soon - **need to sharpen theory predictions**!

Section 2

Setting up Lattice QCD (QFT)

Theory predictions for non-perturbative physics



Source: PDG

At *low energy scales*: perturbative methods fail. Require non-perturbative methods, e.g.

- Effective theories
- 'AdS/CFT-like' correspondences
- Sum rules
- Lattice QFT

- Lattice QCD simulations provide **first principle precision predictions** for phenomenology
- Calculations need to be improved for observables where the error is dominated by **non-perturbative physics**...

Formulating LQCD I: Minkowski to Euclidean

Recall the *Path Integral* in Minkowski space-time

$$\mathcal{Z} = \int \mathcal{D}[\psi, \bar{\psi}]\, \mathcal{D}[U]\, e^{i\left(S_G[U] + S_F[\psi,\bar{\psi},U]\right)}$$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi, \bar{\psi}]\, \mathcal{D}[U]\, \mathcal{O}[\psi, \overline{\psi}, U]\, e^{i\left(S_G[U] + S_F[\psi,\bar{\psi},U]\right)}$$

Highly oscillatory, infinite dimensional integral...

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Highly oscillatory, infinite dimensional integral...
$\Rightarrow$ Wick rotating to Euclidean (i.e. imaginary) time ($t \to i\tau$)...

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Exponentially decaying, infinite dimensional integral...

Formulating LQCD II: Interpretation as a probability weight

Abusing notation[1], we write the fermion action as

$$S_F[\psi, \overline{\psi}, U] = \int \mathrm{d}^4 x \sum_{f=1}^{N_f} \overline{\psi}\, D_f[U]\, \psi$$

S_F is quadratic in the fields and $\psi, \overline{\psi}$ are Grassmannian, so

$$\int \mathcal{D}[\psi, \bar{\psi}] e^{-\int \mathrm{d}^4 x \overline{\psi}(D_f[U])\psi} = \det\left(D_f[U]\right)$$

Now

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}[\psi, \bar{\psi}]\, \mathcal{D}[U]\, e^{-\left(S_G[U] + S_F[\psi, \bar{\psi}, U]\right)} \\ &= \int \mathcal{D}[U] \left[\prod_{n=1}^{N_f} \det\left(D_f[U]\right)\right] e^{-S_G[U]} \end{aligned}$$

[1]Dirac operator for flavour f: $D_f = \gamma_\mu D_\mu + m_f$ where D_μ is the covariant derivative

Formulating LQCD II: Interpretation as a probability weight

$$\mathcal{Z} = \int \mathcal{D}[U] \underbrace{\left[\prod_{n=1}^{N_f} \det\left(D_f[U]\right)\right] e^{-S_G[U]}}_{\text{probability weight}}$$

Assuming $\left[\prod_{n=1}^{N_f} \det\left(D_f[U]\right)\right] > 0$ we can interpret this as a **probability weight** and sample this probability distribution.[2]

- A collection of *configurations* which samples this probability distribution is called an *ensemble*.
- Neglecting the fermionic vacuum bubbles (determinant factors in Z) is called the *quenched approximation*.
- Accounting for fermionic vacuum bubbles from two, mass-degenerate flavours only, i.e. $\det D_u = \det D_d \equiv \det D_l$ is called "$N_f = 2$".

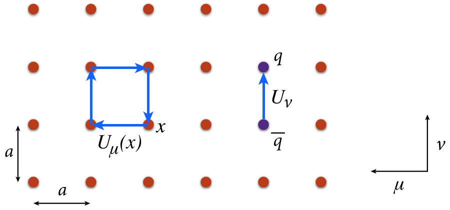
[2]Assuming γ_5 hermiticity ($\gamma_5 D \gamma_5 = D^\dagger$) we can show that $\det D$ is real ($\det D^* = \det D^\dagger = \det \gamma_5 D \gamma_5 = \det D$)

Formulating LQCD III: Make finite dimensional

We still have the issue of an infinite dimensional integral...

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi, \bar{\psi}]\, \mathcal{D}[U]\, \mathcal{O}[\psi, \overline{\psi}, U]\, e^{-\left[S_G[U] + S_F[\psi,\bar{\psi},U]\right]}$$

Now we introduce the **lattice** for make the PI **finite** dimensional:



PDG

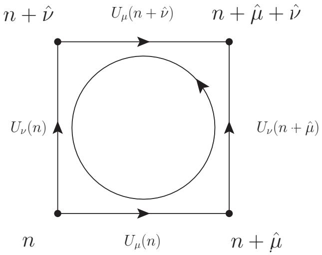
- Finite lattice spacing a
  $\Rightarrow$ UV regulator
- Finite Box of length $L^3 \times T$
  $\Rightarrow$ IR regulator
- $\int \to \sum$, $\partial \to$finite differences
- Choose boundary conditions.

$\Rightarrow$ **The Path Integral is now (large but) <u>finite dimensional</u>.**

$\Rightarrow$ Need to discretise the action (S_G and S_F) and any operators $\mathcal{O}$...

Formulating LQCD IVa: Discretising the gauge action

- Gauge fields A_μ enter through *link variables*, i.e. $U_\mu(n)$ is on the link from n to $n+\hat{\mu}$. They are related to the gauge field by $U_\mu(n) = e^{iaA_\mu(n)}$.
- A plaquette $U_{\mu\nu}(n)$ is the oriented product of link-fields as depicted to the right.



One can verify that

$$S_G[A] = \frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right)$$
$$= \frac{a^4}{2g^2}\left(\sum_{n\in\Lambda}\sum_{\mu<\nu}\mathcal{R}\left[\mathrm{tr}\left(\mathbb{1} - U_{\mu\nu}(n)\right)\right] + O(a^2)\right)$$

This reproduces the correct **continuum limit**, i.e. behaviour as $a \to 0$.

Formulating LQCD IVb: Discretising fermions

- When naively discretising the Dirac operator, the propagator has $2^4 = 16$ poles, i.e. 16 copies of the desired particle ("*doublers*")

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Note: We are free to add higher dimensional operators to the action by multiplying the appropriate powers of a.

$$S_{\text{eff}} = a^4 \sum_x \left[\mathcal{L}_0(x) + a\mathcal{L}_1(x) + a^2\mathcal{L}_2(x) + \cdots \right]$$

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- Adding a dimension-5 operator, called the *Wilson term*, adds a mass $\propto \frac{2}{a}$ to the unwanted poles, i.e. they disappear in the continuum limit. $\Rightarrow$ simulate *single fermion of mass m*!
- **but this term explicitly breaks chiral symmetry...**

ASIDE: Lattice QCD and chiral symmetry

. . . this turns out to be more than just bad luck:

The **Nielsen-Ninomiya theorem** states that it is impossible to simultaneously achieve

- a doubler free theory
- an ultra-local theory
- Chiral symmetry ($\gamma_5 D = D\gamma_5$)
- the correct continuum limit

Overcome by modifying the above to the Ginsparg-Wilson equation

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D$$

i.e. replacing γ_5 by $\hat{\gamma}_5 = \gamma_5(\mathbb{1} - aD)$.

Formulating LQCD IVb: Discretising fermions

So one has to make a choice about the fermion action - **MANY** choices:

- Naive Wilson fermions - (have doublers)
- Wilson-like fermions - (no chiral symmetry)
- Ginsparg-Wilson-type fermions: Overlap, Domain Wall - (not ultra-local)
- Staggered (involves rooting of the determinant, so not-local)
- Many more choices, e.g. inspired by effective theories to solve particular problems: NRQCD (non-relativistic QCD), HQET, static quarks, relativistic heavy quarks (RHQ), $\cdots$

"Improvement": one can tune and add higher dimensional operators to cancel the leading discretisation effects by smartly choosing terms in

$$S_{\rm eff} = a^4 \sum_x \left[\mathcal{L}_0(x) + a\mathcal{L}_1(x) + a^2\mathcal{L}_2(x) + \cdots \right]$$

$\Rightarrow$ **UNIVERSALITY**: physical predictions need to agree in the CL.

Formulating LQCD V: Correlation functions

Operators induce <u>all states</u> with the correct **quantum numbers**:

$$
\begin{aligned}
&\frac{1}{Z}\int \mathcal{D}[U] \prod_{n=1}^{N_f} \det\left(D_f[U]\right)\, e^{-S_G[U]} \sum_{\mathbf{x}} \mathrm{tr}\left(\left[\gamma_5 S_d(\mathbf{0},0|\mathbf{x},t)\gamma_5 S_u(\mathbf{x},t|\mathbf{0},0)\right]\right) \\
&= \frac{1}{Z}\int \mathcal{D}[U] \prod_{n=1}^{N_f} \det\left(D_f[U]\right) e^{-S_G[U]} \sum_{\mathbf{x}} \overline{u}(\mathbf{x},t)\gamma_5 d(\mathbf{x},t)\overline{d}(\mathbf{0},0)\gamma_5 u(\mathbf{0},0) \\
= C_2(t)|_{\mathbf{p}=0} &= \sum_{\mathbf{x}} \left\langle 0 \right| \mathcal{T}\, [\overline{u}\gamma_5 d](\mathbf{x},t)\ [\overline{d}\gamma_5 u]^\dagger(\mathbf{0},0) \left| 0 \right\rangle \\
&= \sum_{\mathbf{x}} \sum_{n} \frac{1}{2m_n} \left\langle 0 \right| [\overline{d}\gamma_5 u](\mathbf{x},t) \left| n \right\rangle \left\langle n \right| [\overline{d}\gamma_5 u]^\dagger(\mathbf{0},0) \left| 0 \right\rangle \\
&= \sum_{n} \sum_{\mathbf{x}} \frac{1}{2m_n} \left\langle 0 \right| e^{Ht}[\overline{d}\gamma_5 u](\mathbf{0},0)e^{-Ht} \left| n \right\rangle \left\langle n \right| [\overline{d}\gamma_5 u]^\dagger(\mathbf{0},0) \left| 0 \right\rangle \\
&= \sum_{n} \frac{1}{2m_n} \left| \left\langle 0 \right| \overline{d}\gamma_5 u \left| n(\mathbf{p}=\mathbf{0}) \right\rangle \right|^2 e^{-m_n t}
\end{aligned}
$$

- We compute and contract the propagators S on each configuration.
- We extract masses and matrix elements by fitting to the last line.

Parameters of QCD and scale setting

A lattice QCD computation starts from **first principles**, i.e. the **action**. So the parameters that are *inputs* to a lattice QCD computation are the bare **coupling constant** g and the bare **quark masses** m_f.

$$S_{\mathrm{QCD}}[\psi,\overline{\psi},U] = S_G[U] + S_F[\psi,\overline{\psi},U]$$
$$= \int \mathrm{d}^4x \, \frac{1}{2g^2} \mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] + \sum_{N_f} \overline{\psi}_f \left(\gamma_\mu D_\mu + m_f\right) \psi_f$$

So we *do not know the lattice spacing of the quark masses a priori*, these are determined from the scale-setting. To make contact with "the real world", we need to sacrifice as many predictions as we have parameters.

EXAMPLE: scale setting (rough idea)

Consider a (set of) $N_f = 2 + 1$ ensemble(s).

- need to determine 3 parameters: $m_l (= m_u = m_d)$, m_s and a
- need to sacrifice 3 predictions: e.g. m_π, m_K and m_Ω

Note that the output of a lattice computation is always dimensionless, i.e. we only obtain numbers such as am_π from the simulation.

1. Vary the quark masses m_l and m_s until

$$\left(\frac{am_\pi}{am_\Omega}\right)_{\rm lat} \equiv \left(\frac{m_\pi}{m_\Omega}\right)_{\rm PDG} \qquad \text{and} \qquad \left(\frac{am_K}{am_\Omega}\right)_{\rm lat} \equiv \left(\frac{m_K}{m_\Omega}\right)_{\rm PDG}$$

2. Require $(m_\Omega)_{\rm lat} = (m_\Omega)_{\rm PDG}$, i.e.

$$a^{-1} = \frac{(m_\Omega)_{\rm PDG}}{(am_\Omega)_{\rm lat}}$$

Size of the numerical problem

Using $N_{\text{spin}} = 4$, $N_{\text{colour}} = 3$ and defining $|\Lambda| = (X/a)^3 \times (T/a)$

- Gauge fields → *link variables* $U_\mu(n)$: $SU(3)$ matrix linking sites: $9 \times |\Lambda|$ complex entries.
- Fermion fields turn vectors of size $4 \times 3 \times |\Lambda|$
- Operators → matrices. E.g. Dirac operator: $(4 \times 3 \times |\Lambda|)^2$ entries.

(former) BG/Q in Edinburgh

Small volume : $L = 16a$, $T = 32a$: 2.5×10^{12} entries

State of the art : $L = 96a$, $T = 196a$: 4.2×10^{18} entries

Propagators are the inverse of the Dirac operator...

...**clearly not possible to do by hand or on your laptop**

A Lattice Computation

Lattice vs Continuum

We simulate:

- at finite lattice spacing a
- in finite volume L^3
- lattice regularised
- Some bare input quark masses
  am_l, am_s, am_h
  In general: $m_\pi \neq m_\pi^{\mathrm{phys}}$

We want:

- $a = 0$
- $L = \infty$
- some continuum scheme
- $m_l = m_l^{\mathrm{phys}}$
- $m_s = m_s^{\mathrm{phys}}$
- $m_h = m_c^{\mathrm{phys}}, m_b^{\mathrm{phys}}$

$\Rightarrow$ Need to control all limits!

$\rightarrow$ particularly simultaneously control FV and discretisation

$\Rightarrow$ Decide on a fermion action (suitable to your problem):
Wilson, Staggered, Twisted Mass, **Domain Wall fermions**, $\cdots$

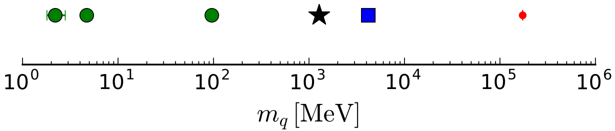
Section 3

Challenges in *b*-physics on the lattice

Multiple scale problem on the lattice: back of the envelope

Control effects of IR (finite volume) and UV (discretisation) regulators:

$$m_\pi L \gtrsim 4 \qquad\qquad a^{-1} \gg \text{Mass scale of interest}$$



For $m_\pi = m_\pi^{\mathrm{phys}} \sim 140\,\mathrm{MeV}$ and $\overline{m_b}(m_b) \approx 4.2\,\mathrm{GeV}$:

$$L \gtrsim 5.6\,\mathrm{fm} \qquad\qquad a^{-1} \gg 4.2\,\mathrm{GeV} \approx (0.05\,\mathrm{fm})^{-1}$$

Requires $N \equiv L/a \gg 120 \Rightarrow N^3 \times (2N) \gg 4 \times 10^8$ lattice sites.

VERY EXPENSIVE to satisfy both constraints simultaneously...

... needs to be repeated for different values of a.

How to simulate the b-quark?

For now choose between:

Effective action for b

- Can tune to m_b
- comes with systematic errors which are hard to estimate/reduce

Relativistic action for b

- Theoretically cleaner and systematically improvable
- Need to control extrapolation in heavy quark mass

Different properties:

- computational cost
- chirality
- tuning errors
- systematic errors
- cut off effects
- renormalisation

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Different properties:

- computational cost
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BUT SOON:

Huge efforts in the community to produce **very fine lattice spacings**:

$\Rightarrow$ **Direct simulation of $\approx m_b^{\text{phys}}$ will become possible!**

Signal-to-noise estimates

- The signal decays with the lowest lying energy in the spectrum: $e^{-E_0 t}$.
- The variance decays with the lightest combination of states with the correct quantum numbers that can be built from the square of the operator. The noise (statistical error) is the square root of the variance.

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Consider a B meson, i.e. $\mathcal{O} = \overline{b}\gamma_5 u$: From the "square" of the operator we can build two B-mesons or a π and a η_b. From the PDG we find:

$$m_\pi \approx 0.140\,\mathrm{GeV} \qquad m_B \approx 5.3\,\mathrm{GeV} \qquad m_{\eta_b} \approx 9.4\mathrm{GeV}$$

So $(m_\pi + m_{\eta_b})/2 \approx 4.77\,\mathrm{GeV} \approx 0.9 m_B$

$\Rightarrow$ The noise decays slower than the signal.

$\Rightarrow$ The **noise-to-signal ratio grows exponentially** with $e^{+0.1 m_B t}$

Signal extraction and excited state contamination

Assume you have an operator $\mathcal{O}$ that induces a ground state $|P_0\rangle$ with energy m_0 and first excited state $|P_1\rangle$ with energy m_1.

$$C(t) = \sum_n \frac{|\langle 0|\,\mathcal{O}\,|n\rangle|^2}{2E_n} e^{-E_n t}$$

$$\approx \frac{|\langle 0|\,\mathcal{O}\,|P_0\rangle|^2}{2E_0} e^{-E_0 t} \times \left(1 + \frac{E_0}{E_1}\frac{|\langle 0|\,\mathcal{O}\,|P_1\rangle|^2}{|\langle 0|\,\mathcal{O}\,|P_0\rangle|^2} e^{-(E_1-E_0)t}\right)$$

For heavy mesons there are many excited states which might lie close together making it hard to unambiguously isolate the ground state.
$\Rightarrow$ Can only extract the ground state at "late times", i.e. fit for larger values of t.
$\Rightarrow$ BUT **Signal-to-noise problem from above**!

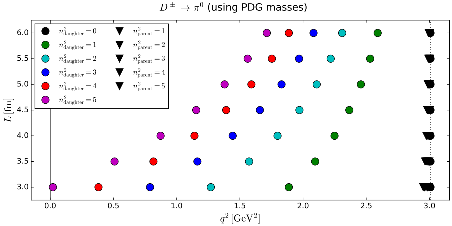
Semi-leptonic decays I: q^2-coverage

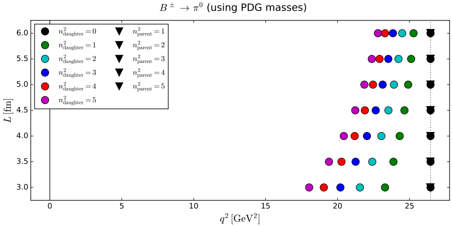
$$\frac{d\Gamma(P \to D\ell\nu_\ell)}{dq^2} \approx |V_{q_2 q_1}|^2 \times \left[\left| f_+(q^2) \right|^2 \mathcal{K}_1 + \left| f_0(q^2) \right|^2 \mathcal{K}_2 \right]$$

For a semi-leptonic decay we are interested in $f(q^2)$ for

$$q^2 = (E_P - E_D)^2 - (\mathbf{p}_P - \mathbf{p}_D)^2 \in [0, (m_P - m_D)^2]$$

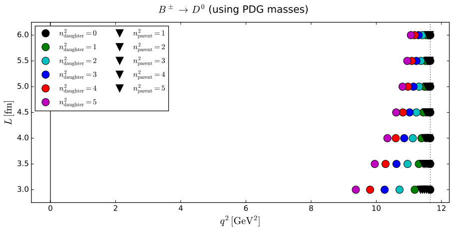
In a finite volume, momenta are discretised: $\mathbf{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$.

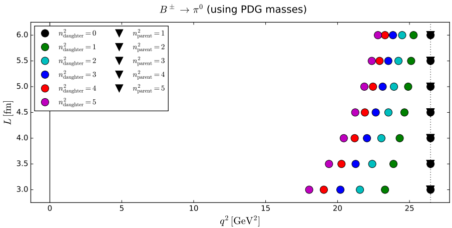




Semi-leptonic decays II: q^2-coverage

$$q^2 = (E_P - E_D)^2 - (\mathbf{p}_P - \mathbf{p}_D)^2 \in [0, (m_P - m_D)^2]$$





- Lattice is easiest near $q^2_{\rm max}$, experiment easiest near $q^2 = 0$.
- Covering q^2 becomes harder as
  - $m_P - m_D$ becomes larger
  - m_P, m_D become heavier ($E \approx m$)
  - Volume increases (but large V needed for physical pions...)
- Noise increases as $n^2 = n_x^2 + n + y^2 + n_z^2$ increases.
- Discretisation errors increase as n^2 increases.

Section 4

Example of a Lattice QCD computation (arXiv:1812.08791)

Work-flow of a Lattice QCD computation

1. Settle on the desired ensemble properties (number of flavours in the sea, fermion discretisation, gauge discretisation).
2. Generate multiple gauge ensembles for desired choices of (a^{-1}, V, m_q) to allow for all necessary inter/extrapolations.
    - "Set the scale", i.e. determine parameters of your ensemble (lattice spacing and quark masses).
3. Settle on a measurement strategy (valence quark masses, valence fermion actions, included operators and currents, source types and positions, contractions).
4. Generate correlation functions and extract observables (masses, energies, matrix elements) from them.
5. Extrapolate data to the "physical point", i.e. $a \to 0$, $V \to \infty$, $m_q \to m_q^{\mathrm{phys}}$, ...
6. Full **statistical** and **systematic** error analysis.

Example of a Lattice QCD computation (arXiv:1812.08791)

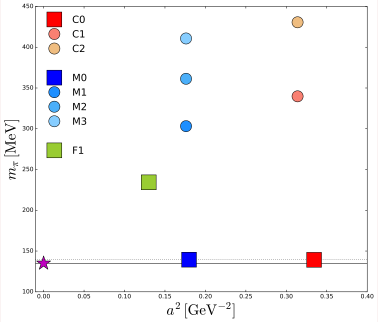
[1812.08791]

Edinburgh - Southampton - Boulder - BNL (RBC/UKQCD)

Peter Boyle, Luigi Del Debbio, Nicolas Garron, Andreas Jüttner, Amarjit Soni, JTT, Oliver Witzel

EXAMPLE - Steps 1-3: RBC/UKQCD's ensembles

ensembles (steps 1 and 2)



- $N_f = 2 + 1$
- Iwasaki gauge action
- Domain Wall Fermion action
- **2 ensembles with physical pion masses** [PRD **93** (2016) 074505]
- 3 Lattice spacings [JHEP **12** (2017) 008]
- Heavier m_π ensembles guide small chiral extrapolation of F1

measurement strategy (steps 3) - arXiv:1812.08791

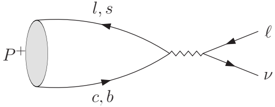
- Keep light and strange quark like in the sea
- Simulate multiple $c(b)$-like quarks with suitably chosen DWFs
- Decide to measure decay constants and $B_{(s)} - \bar{B}_{(s)}$ mixing

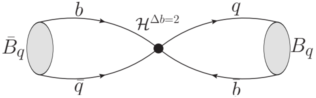
EXAMPLE - Steps 4: Correlators - arXiv:1812.08791

Leptonic decays:

$P^0 - \bar{P}^0$-mixing

quark flow

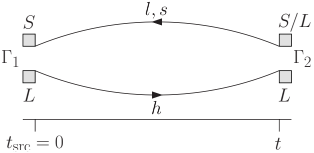


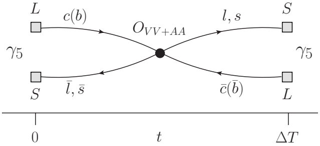


$$\mathcal{Z}_A \left\langle 0 \right| \overline{b}\gamma_4\gamma_5 q \left| B_q(0) \right\rangle = f_{B_q} m_{B_q}$$

$$B_P = \frac{\left\langle \bar{P}^0 \right| O_{VV+AA} \left| P^0 \right\rangle}{8/3 f_P^2 m_P^2}$$

lattice





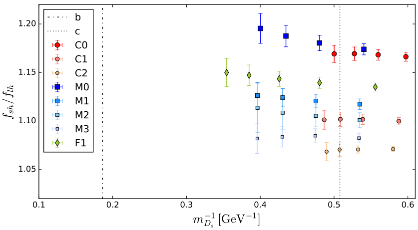
$\Rightarrow$ Fit data - to get masses (m_P), decay constants (f_P) and bag parameters (B_p) for each ensemble and each choice of valence masses.

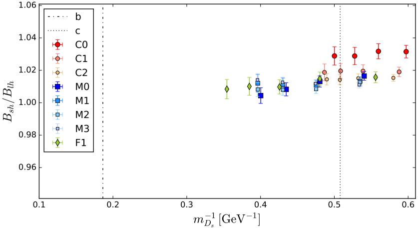
EXAMPLE - Steps 5: back to the continuum

arXiv:1812.08791

f_{hs}/f_{hl}

B_{hs}/B_{hl}



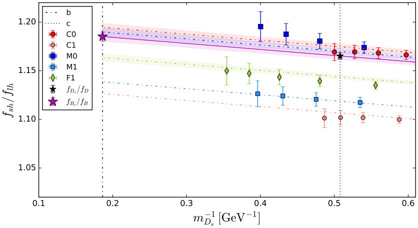


Data from correlator fits

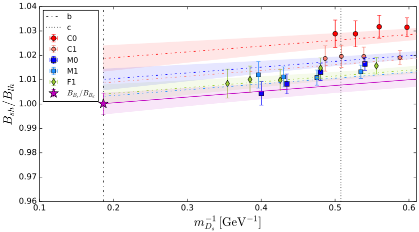
EXAMPLE - Steps 5: back to the continuum

arXiv:1812.08791

f_{hs}/f_{hl}



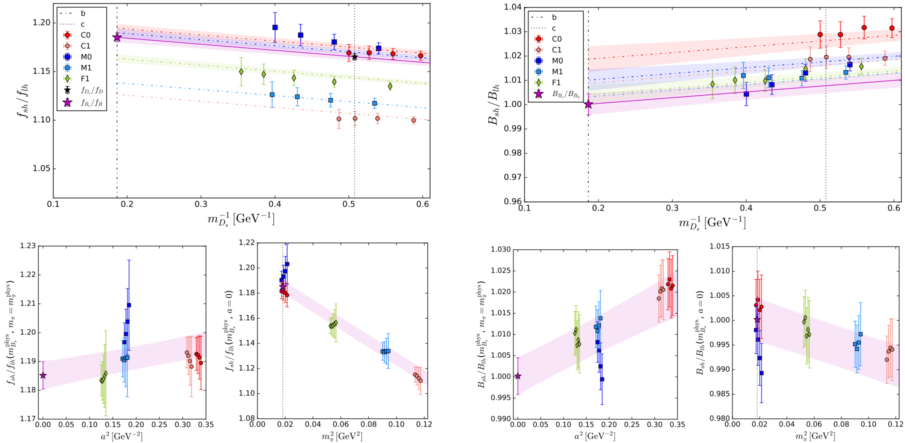
B_{hs}/B_{hl}



Global fit to m_π, a and m_H behaviour

EXAMPLE - Steps 5: back to the continuum

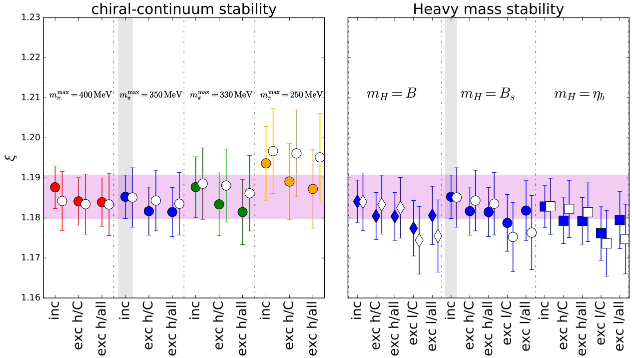
f_{hs}/f_{hl} arXiv:1812.08791 B_{hs}/B_{hl}



Exposing the continuum limit (left) and the chiral extrapolation (right).

EXAMPLE - Steps 6: Error budget I (1812.04981)

- Global fits all correlated with satisfying p-values.
- sys error: includes chiral-CL (left), heavy mass (right), H.O. terms, $m_u \neq m_d$ and FV.



$$\xi = 1.1853(54)_{\mathrm{stat}} \begin{pmatrix} +116 \\ -156 \end{pmatrix}_{\mathrm{sys}}$$

EXAMPLE - step 6: Error budget II (1812.04981)

Experimental precision on $\Delta m_s \sim 0.1\%$ and $\Delta m_d \sim 0.4\%$.
Theoretical precision on $\xi \sim 1.3\%$

	f_{D_s}/f_D		f_{B_s}/f_B		ξ		B_{B_s}/B_{B_d}	
	absolute	relative	absolute	relative	absolute	relative	absolute	relative
central	1.1652		1.1852		1.1853		1.0002	
stat	0.0035	0.30%	0.0048	0.40%	0.0054	0.46%	0.0043	0.43%
fit chiral-CL	$^{+0.0112}_{-0.0031}$	$^{+0.96}_{-0.26}$ %	$^{+0.0110}_{-0.0045}$	$^{+0.93}_{-0.38}$ %	$^{+0.0084}_{-0.0038}$	$^{+0.71}_{-0.32}$ %	$^{+0.0020}_{-0.0044}$	$^{+0.20}_{-0.44}$ %
fit heavy mass	$^{+0.0003}_{-0.0000}$	$^{+0.02}_{-0.00}$ %	$^{+0.0000}_{-0.0081}$	$^{+0.00}_{-0.69}$ %	$^{+0.0000}_{-0.0091}$	$^{+0.00}_{-0.77}$ %	$^{+0.0012}_{-0.0031}$	$^{+0.12}_{-0.31}$ %
H.O. heavy	0.0000	0.00%	0.0054	0.45%	0.0049	0.41%	0.0021	0.21%
H.O. disc.	0.0009	0.07%	0.0009	0.07%	0.0021	0.18%	0.0016	0.16%
$m_u \neq m_d$	0.0009	0.08%	0.0009	0.07%	0.0010	0.08%	0.0001	0.01%
finite size	0.0021	0.18%	0.0021	0.18%	0.0021	0.18%	0.0018	0.18%
total systematic	$^{+0.0114}_{-0.0039}$	$^{+0.98}_{-0.34}$ %	$^{+0.0125}_{-0.0137}$	$^{+1.06}_{-1.16}$ %	$^{+0.0102}_{-0.0146}$	$^{+0.86}_{-1.24}$ %	$^{+0.0041}_{-0.0070}$	$^{+0.41}_{-0.70}$ %
total sys+stat	$^{+0.0120}_{-0.0052}$	$^{+1.03}_{-0.45}$ %	$^{+0.0134}_{-0.0145}$	$^{+1.13}_{-1.22}$ %	$^{+0.0116}_{-0.0156}$	$^{+0.97}_{-1.32}$ %	$^{+0.0060}_{-0.0082}$	$^{+0.60}_{-0.82}$ %

$\Rightarrow$ **Systematically Improvable** with finer lattices at (near) physical m_π.

Section 5

Status of the field

What we can do well

I.e. full calculations from multiple groups and different formulations.

- $m_q \to m_q^{\text{phys}}$, e.g. "chiral extrapolation"
- $a \to 0$, i.e. the continuum limit has been taken
- $V \to \infty$, i.e. infinite volume limit has been taken or accounted for

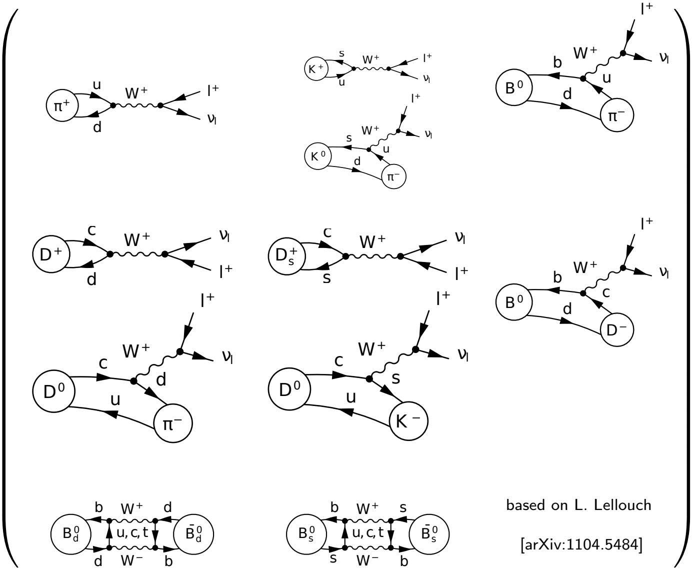
mature calculations

(Mostly) single hadron "stable states", mostly mesons

- Quark masses
- Spectrum - i.e. extracting masses and energies
- Leptonic decay constants (f_π, f_K, f_{D_s}, f_{B_s}, ...)
- Semi-leptonic form factors (mostly PS final state) near q^2_{max}
- Short-distance matrix elements of OPE's (e.g. $B_{(s)} - \bar{B}_{(s)}$ mixing).
- Determination of LECs (we often simulate at $m_\pi > m_\pi^{\text{phys}}$)

What we can do well

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \sim$$



based on L. Lellouch

[arXiv:1104.5484]

Some more things we can do

advanced or advancing, but fewer complete calculations

- Hadronic vacuum polarisation $(g-2)$
- Scattering of two hadrons in the final state - mature field but costly: $\Rightarrow$ Information is gained from FV $\Rightarrow$ many volumes $\equiv$ $£€...
- Possible to use this to treat vector final states, i.e. $B \to K^*(\to K\pi)$, but only if decays to only 2 hadrons
- Including isospin breaking effect into simulations:
  - $m_u \neq m_d$
  - $\alpha_{QED} \neq 0$
- baryons (signal to noise problems)
- Hadronic light-by-light scattering
- Testing strongly coupled BSM theories (e.g. composite Higgs)

Many more topics

"*Lattice*" is a ***BIG FIELD*** with many topics I have not touched upon...

- Algorithms and Machines
- Applications Beyond QCD
- Chiral Symmetry
- Hadron Spectroscopy and Interactions
- Hadron Structure
- QCD at nonzero Temperature and Density
- Physics Beyond the Standard Model
- Standard Model Parameters and Renormalisation
- Theoretical Developments
- Vacuum Structure and Confinement
- Weak Decays and Matrix Elements
- Code Development

Relevant for particle physicists (theorists, phenomenologists and experimentalist) but also computer science, applied mathematics, algorithm developments, high performance computing, statistics, statistical mechanics, condensed matter,...

Section 6

Jargon, Literature and Summary

Literature I: Text books and lecture notes

Many great lecture notes and textbooks - some examples below:

- "*Quantum Chromodynamics on the Lattice: An Introductory Presentation*" - Gattringer and Lang, 2010
- "*Lattice methods for quantum chromodynamics*" - DeGrand and Detar, 2006
- "*Quarks, gluons and lattices*", M. Creutz, 1985.

- "*Modern perspectives in lattice QCD: Quantum field theory and high performance computing*" - L. Lellouch, R. Sommer, B. Svetitsky, A. Vladikas and L. F. Cugliandolo, eds., proceedings for Les Houches Summer School in Theoretical Physics 2009) (many chapters on the arXiv)
- hep-lat/9802029 and references within (Advanced lattice QCD - M, Luescher) (Part of the proceedings for Les Houches Summer School in Theoretical Physics 1997)

Literature II: FLAG - Flavour Lattice Averaging Group

In any (research) community it is hard to assess results from the outside.

The Flavour Lattice Averaging Group (FLAG) aims to

- Summarise lattice results
- Assess their quality
- Provide averages of different results


FLAG Review 2019

The 2019 edition of the FLAG review can be downloaded here. The version dated March 4, 2020 is the one accepted for publication in EPJC. The EPJC version can be obtained from here.

The separate sections can be downloaded as separate pdf-files following the links in the table of contents below, or via the menu in the sidebar. Clicking on the FLAG logo in the upper left corner links back to this main page.

The latest figures can be downloaded in eps, pdf and png format, together with a bib-file containing the bibtex-entries for the calculations which contribute to the FLAG averages and estimates. The downloads are available via the menu in the sidebar.

In the notes we compile detailed information on the simulations used to calculate the quantities discussed in the review. Here we provide the complete tables, in contrast to the paper version of the review which contains this information only for results that have appeared since FLAG 16.

The original complete 2015/2016 review is still accessible from EPJC. The 2016/2017 updates are available from here. The 2013/2014 review is accessible here or from EPJC.

Contents

FLAG-webpage

⇒ This is a very useful and handy tool, BUT...

...**please cite the original papers** and not just FLAG.

...be aware that more recent results might not yet be included.

Jargon I

- Ensemble, configuration, trajectory, measurement, correlator
  An **ensemble** is a collection of representative gauge fields with the same input parameters
  A **trajectory** is an individual step in the (hybrid) Monte Carlo chain
  configuration is an element of the ensemble. Configurations are "de-correlated" from each other, e.g. only take every Nth trajectory.
  A **measurement** is an observable evaluated on a specific configuration.
  A **correlator** or **correlation function** is the ensemble (i.e. gauge) average of a measurement.
- **Continuum Limit (CL)**: $\lim_{a\to 0} \mathcal{O}(a, m_q, V, ...)$
- $O(a)$-**improved**: Discretisation effects of $O(a)$ are absent.
- **Infinite Volume Limit** (IV): $\lim_{L\to\infty} \mathcal{O}(a, m_q, V, ...)$

Jargon II

- Sea vs valence:
  Sea quarks or dynamical fermions: These are included **in the ensemble**. For each choice you need a new ensemble.
  Valence quarks: The quarks from which operators are constructed. They can - but do not have to - agree with the sea quarks (see below).
- Number of flavours: (IN THE SEA)
  $N_f = 0 = \textit{quenched} = \textit{pure gauge}$: No vacuum bubbles are included in the ensemble
  $N_f = 2$: Vacuum bubbles from two degenerate light (u, d) quarks are included.
  $N_f = 2 + 1(+1)$: Vacuum bubbles from two degenerate light (u, d) quarks + a strange quark (+ a charm quark) are included.
- Unitary vs *partially quenched*
  - uni *Unitary* data points have the same parameters in valence and sea
  - pq *Partially quenched* data points differ between valence and sea.

Summary

I hope I have convinced you that Lattice QCD

- provides a **first principles** and **systematically improvable** method to compute non-perturbative observables.
- requires **large computational resources**, several steps of **data analysis** and **human time** (Sound familiar...?).

I hope you have gained some insight into how LQCD works - if anything is unclear **please ask**!

- If **you** need lattice input for a particular process, or
- If there are particular analysis choices that would help you (momentum ranges, choice of bins, ...),

please **email me**.