A generalized Vlasov solver for e-cloud instabilities

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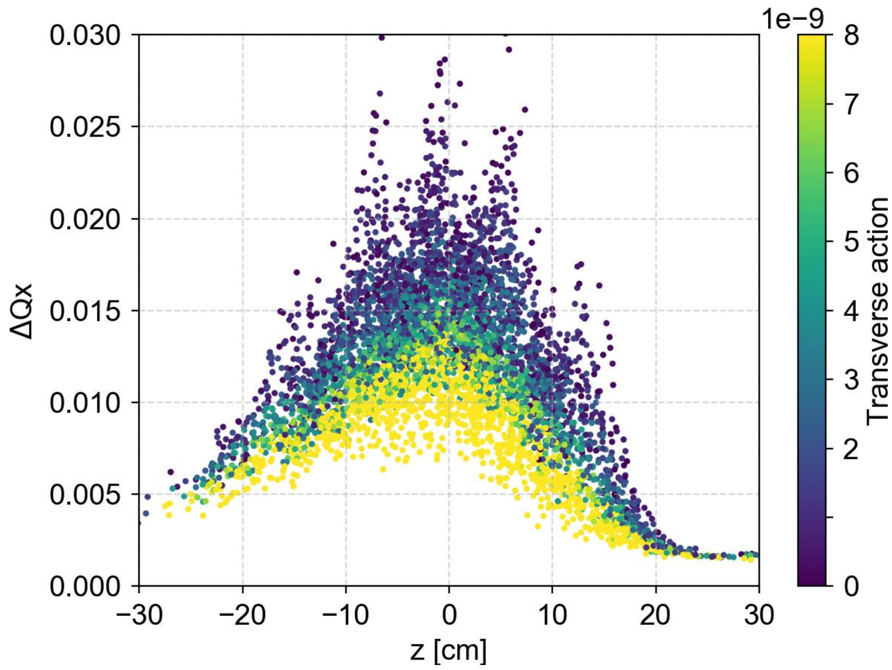
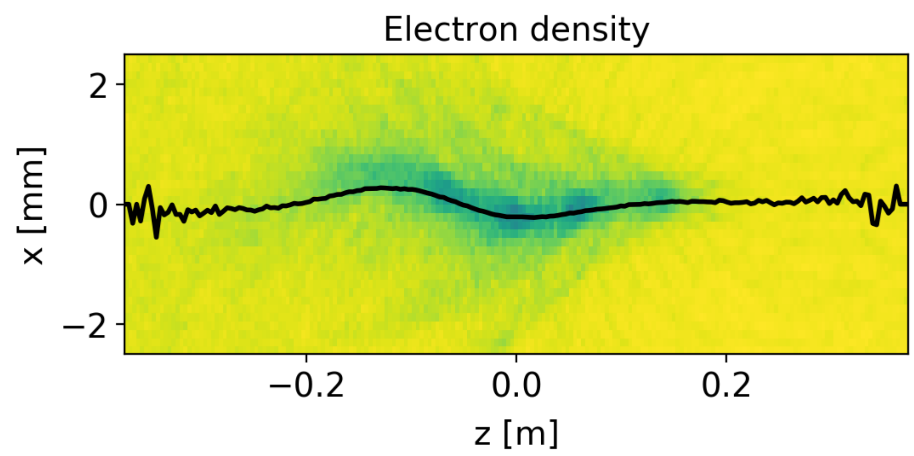
Many thanks to:
H. Bartosik, E. Métral, G. Rumolo, M. Schenk

Outline

- **Introduction**
- **Synthetic modelling of e-cloud dipolar and quadrupolar forces**
  - Modelling the detuning along the bunch
  - Response to sinusoidal distortions
- **Generalized Vlasov solver**
  - Generalization to an arbitrary detuning along the bunch
  - Handling the head-tail phase shift
  - Expressing the coherent force
  - Integral equation
  - Orthogonal polynomials and eigenvalue problem
  - Handling the detuning with longitudinal amplitude
  - Implementation
- **Application to LHC e-cloud instabilities**
  - Comparison against PyHEADTAIL
- **Checks for the case of a dipolar impedance**

- To model the effect of the the e-cloud on beam stability, **two aspects** need to be taken into account:
  - The alterations introduced by the e-cloud on the **single particle motion**, responsible in particular for a **detuning along the bunch** which can affect the stability properties
  - The **coherent forces** introduced by the e-cloud along the bunch, which can be responsible for driving an instability





The plots refer to the effect of **the e-cloud in the quadrupole magnets for the case of the LHC at 450 GeV** (studied extensively with PIC simulations by L. Sabato), which will be used for all examples

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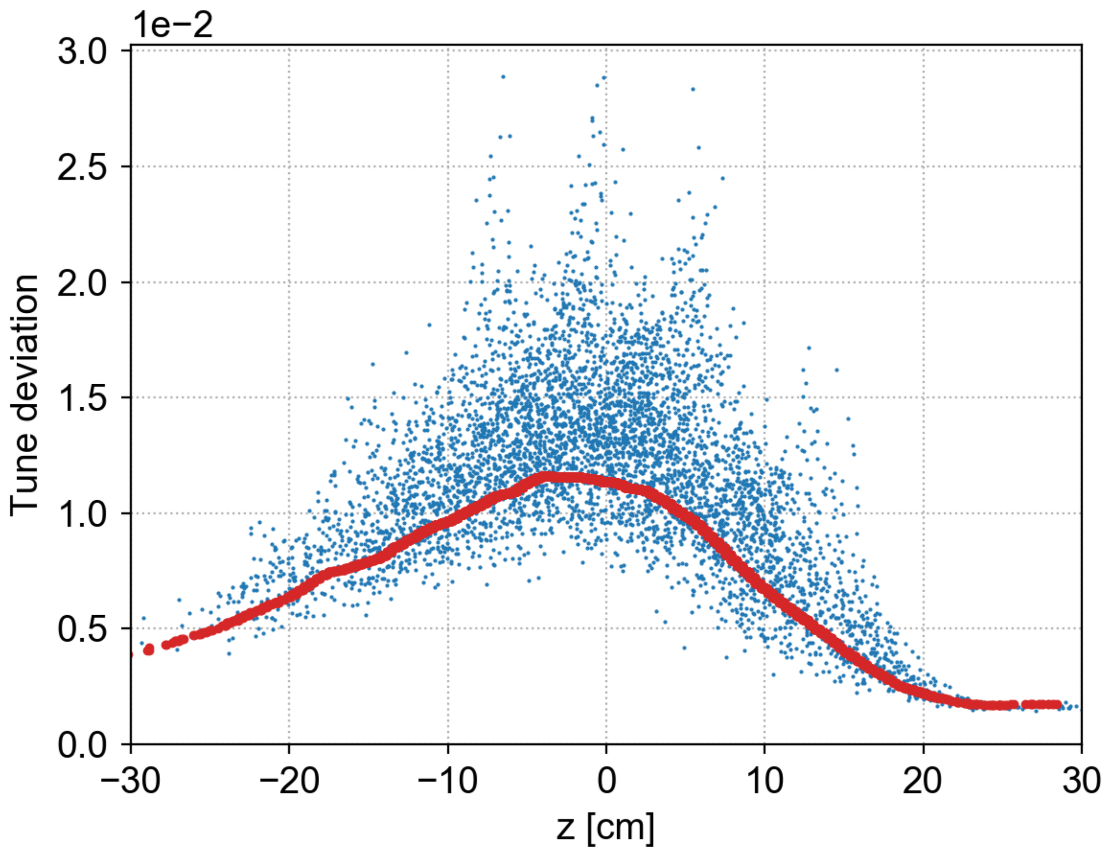
The **detuning introduced by the e-cloud** along the bunch is a non-trivial function of the longitudinal position and of the transverse action

- We **isolate the quadrupolar term** (in red) by measuring at each z the average focusing gradient over the transverse distribution



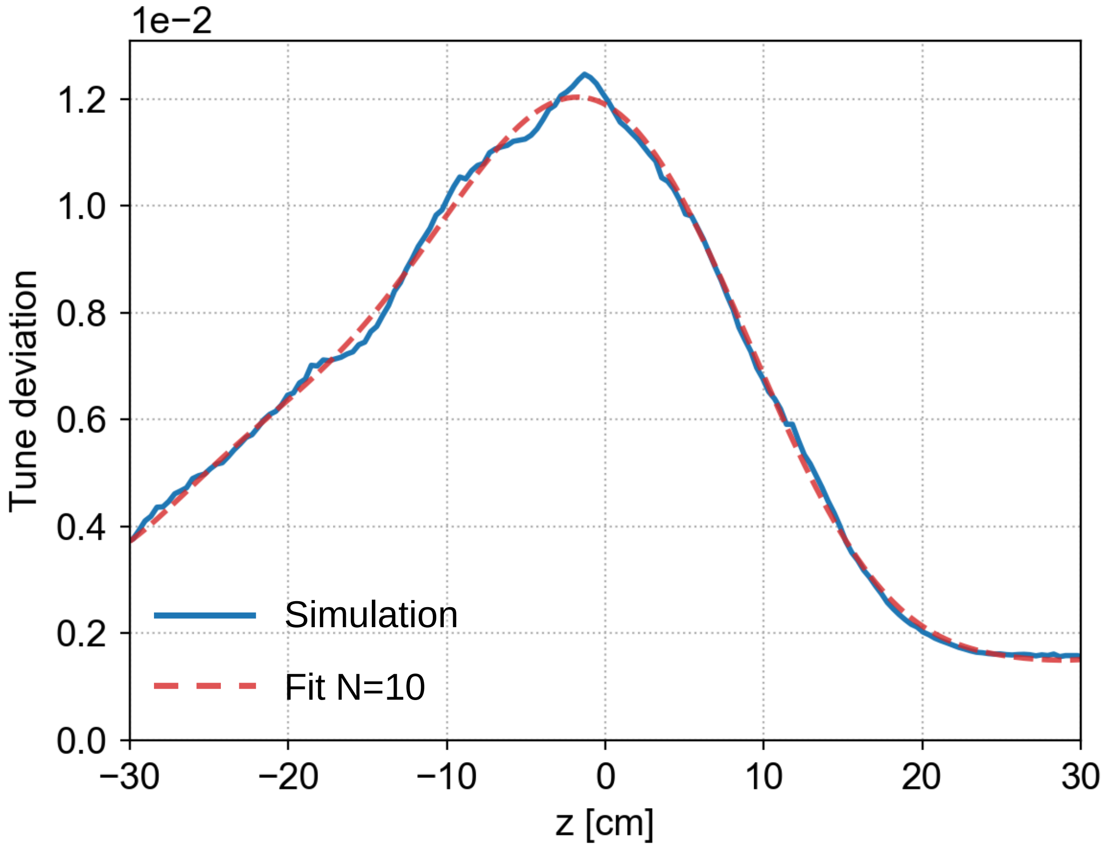
The **detuning introduced by the e-cloud** along the bunch is a non-trivial function of the longitudinal position and of the transverse action

- We **isolate the quadrupolar term** (in red) by measuring at each z the average focusing gradient over the transverse distribution
- This can be realistically **modelled with a polynomial**

Polynomial expansion:

$$\Delta Q(z) = \sum_{n=0}^{N} A_n z^n$$

This model of the detuning vs z **will be used in the analytical derivations in the following**



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Due the electron pinch during the bunch passage, the **e-cloud cannot be considered time-invariant system**

→ It cannot be characterized by a conventional wakefield (as highlighted by Perevedentsev et al.)

→ We need to use a **more general characterization of the coherent force**

We follow the approach illustrated at the HSC meeting on 17 Feb . We consider a **set of orthogonal functions**:

$$h_n(z) = \begin{cases} A_n \cos\left(2\pi \frac{n}{2} \frac{z}{L_{\mathrm{bkt}}}\right), & \text{if } n \text{ is even} \\ A_n \sin\left(2\pi \frac{n-1}{2} \frac{z}{L_{\mathrm{bkt}}}\right), & \text{if } n \text{ is odd} \end{cases}$$

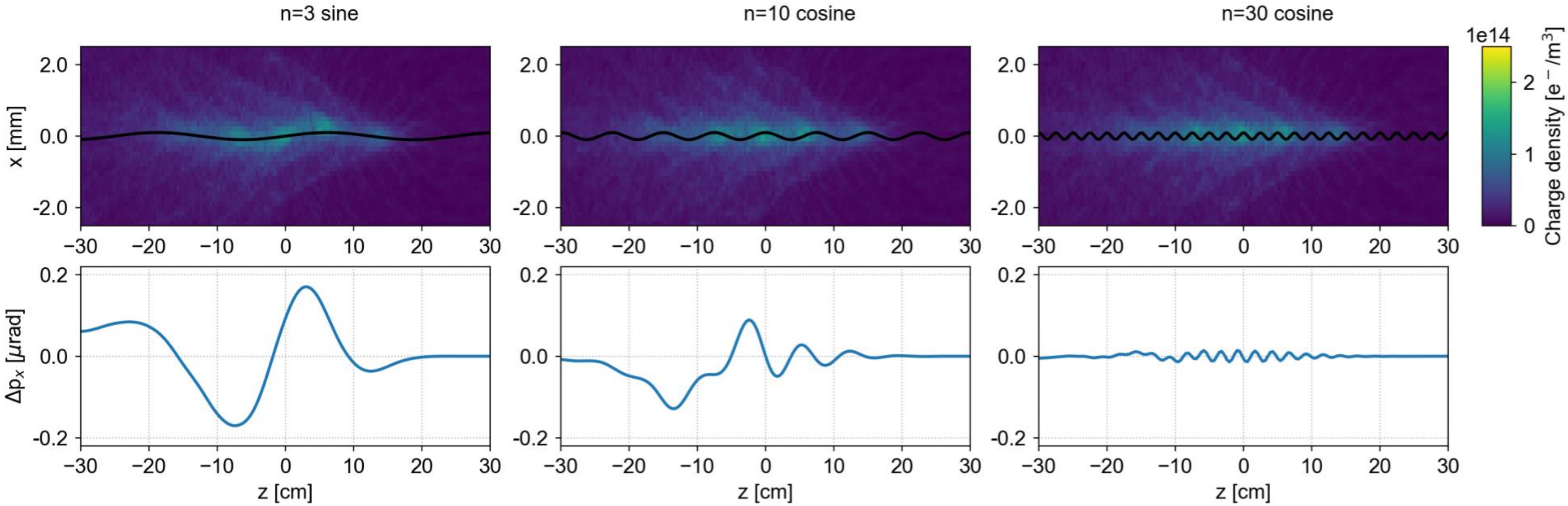
Orthogonality condition:

$$\int_{-L_{\mathrm{bkt}/2}}^{L_{\mathrm{bkt}}/2} h_n(z) h_{n'}(z) dz = H_n^2 \delta_{n,n'}$$

and we use them to **probe the response of the e-cloud to a distortion of the bunch** shape using short PyECLOUD simulations (single pass) by **measuring the corresponding transverse kick**, which we call $k_n(z)$:

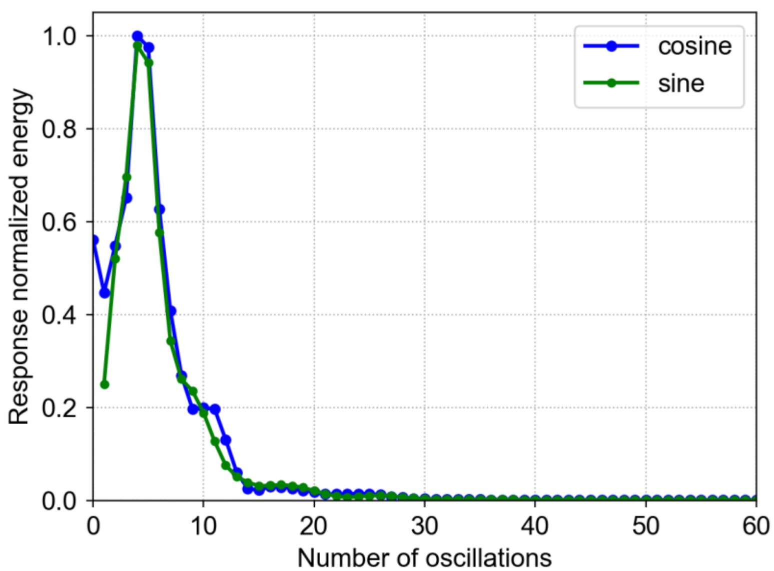
- this is conceptually equivalent to computing the impedance of a device using an electromagnetic simulation (but more general)

These **mini-simulations** look like this, the single particle focusing (quadrupolar fotce) is subtracted



Due to the inertia of the electrons, **the response becomes smaller at higher excitation frequencies**

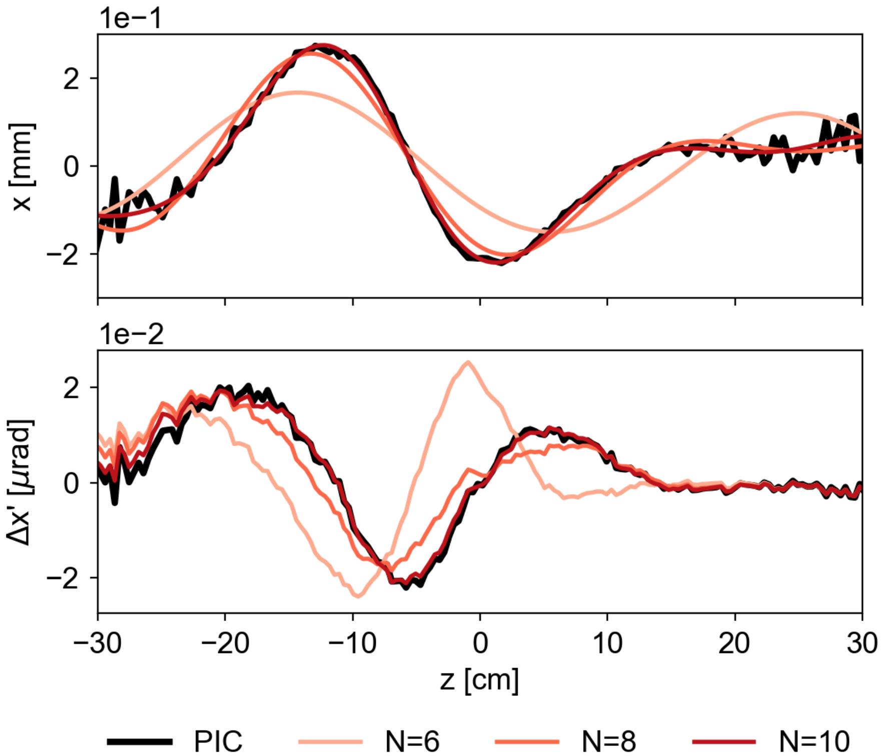
→ A full characterization can be obtained with a small number of test sinusoids



Dipolar forces

We can verify that the computed set of responses can be used to compute the dipolar effect resulting from a **generic bunch distortion**

Test on an oscillation from a simulated instability



We expand the bunch distortion:

$$\bar{x}(z) = \sum_{n=0}^{N} a_n h_n(z)$$

The expected kick is expressed as

$$\Delta x'(z) = \sum_{n=0}^{N} a_n k_n(z)$$

The **agreement is found to be very good** and the **convergence is quite fast**

The coefficient are obtained using the orthogonality: $$a_n = \frac{1}{H_n^2} \int \bar{x}(z)\, h_n(z)\, dz$$

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A generalized DELPHI solver

We investigated the possibility of **including the effects described above in a Vlasov solver**, which should allow identifying all the eigenmodes of the motion. General strategy:

- We used as a **starting point** the approach used in **DELPHI**. **Advantages**:
  - It's known to have good **convergence properties** for typical LHC cases (impedance)
  - **Document very clearly** in [1] and [2]
  - Profited of great **discussions and guidance from DELPHI's author**

- The method is built as an **extension of DELPHI's equations** and **falls back exactly on DELPHI's equations** for the case of an **impedance**:

  → Proved to be extremely useful to make checks during development

Let's dive into the equations → Brace yourself ☺

[1] N. Mounet, ``Direct Vlasov solvers'', proceedings of the CERN Accelerator School on Numerical Methods for Analysis, Design and Modelling of Particle Accelerators, Thessaloniki, Greece, 2018.

[2] N. Mounet, "Vlasov solvers and macroparticle simulations'", proceedings of the ICFA Mini-Workshop on Impedances and Beam Instabilities in Particle Accelerators, Benevento, Italy, 2018.

Vlasov equation (and notation)

We consider the **linearized Vlasov equation** as used in DELPHI (in vertical, sorry for the change pf notation!):

$$\frac{\partial \Delta\psi}{\partial t} - \omega_0 \left(Q_{y0} + Q'\delta\right) \frac{\partial \Delta\psi}{\partial \theta_y} + \omega_s \frac{\partial \Delta\psi}{\partial \phi} = -\frac{df_0}{dJ_y} g_0(r) \sqrt{\frac{2J_y R}{Q_{y0}}} \sin\theta_y \frac{F_y^{coh}(z,t)}{m_0 \gamma v}$$

where we **use polar coordinates** for transverse and longitudinal phase spaces:

$$z = r\cos\phi \qquad y = \sqrt{\frac{2J_y R}{Q_{y0}}} \cos\theta_y$$

$$\delta = \frac{\omega_S}{v\eta} r \sin\phi \qquad y' = \sqrt{\frac{2J_y Q_{y0}}{R}} \sin\theta_y$$

$F_y^{coh}(z,t)$ is the **coherent driving force** (e-cloud, or impedance)

$\Delta\psi$ is the **perturbation to the phase space** distribution which describes the instability **(our unknown!)**

The **unperturbed phase space distribution** is factorized as:

$$\psi_0 = f_0\left(J_y\right) g_0\left(r\right)$$

Other quantities:

ω_0	revolution frequency
ω_s	synchrotron frequency
Q_{y0}	betatron tune
Q'	linear chromaticity
R	machine radius
m_0, γ, v	particle mass, relativistic gamma and velocity

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Generalization to an arbitrary detuning

We consider the **linearized Vlasov equation** as used in DELPHI:

$$\frac{\partial \Delta\psi}{\partial t} - \omega_0 \left(Q_{y0} + Q'\delta\right) \frac{\partial \Delta\psi}{\partial \theta_y} + \omega_s \frac{\partial \Delta\psi}{\partial \phi} = -\frac{df_0}{dJ_y} g_0(r) \sqrt{\frac{2J_y R}{Q_{y0}}} \sin\theta_y \frac{F_y^{coh}(z,t)}{m_0 \gamma v}$$

We need to handle a **more general detuning term**, so we consider a **generalized version of the equation**:

$$\frac{\partial \Delta\psi}{\partial t} - \omega_0 \left(Q_{y0} + \Delta Q(r,\phi)\right) \frac{\partial \Delta\psi}{\partial \theta_y} + \omega_s \frac{\partial \Delta\psi}{\partial \phi} = -\frac{df_0}{dJ_y} g_0(r) \sqrt{\frac{2J_y R}{Q_{y0}}} \sin\theta_y \frac{F_y^{coh}(z,t)}{m_0 \gamma v}$$

In DELPHI we search for solutions in the form:

$$\Delta\psi(J_y, \theta_y, r, \phi; t) = e^{j\Omega t} \sum_{p=-\infty}^{+\infty} f^p(J_y) e^{jp\theta_y} \cdot e^{-\frac{jpQ'_y z}{\eta R}} \cdot \sum_{l=-\infty}^{+\infty} R_l^p(r) e^{-jl\phi}$$

We need to **generalize** also the form of the solution:

$$\Delta\psi(J_y, \theta_y, r, \phi; t) = e^{j\Omega t} \sum_{p=-\infty}^{+\infty} f^p(J_y) e^{jp\theta_y} \cdot e^{-jp\Delta\Phi(r,\phi)} \cdot \sum_{l=-\infty}^{+\infty} R_l^p(r) e^{-jl\phi}$$

Still to be defined

Decomposition of the detuning

Following Schenk et al. we **decompose the detuning** in two terms:

$$\Delta Q(r,\phi) = \Delta Q_R(r) + \Delta Q_\Phi(r,\phi)$$

Detuning with longitudinal amplitude

Head-tail phase shift

$$\Delta Q_R(r) = \frac{1}{2\pi}\int_0^{2\pi} \Delta Q(r,\phi)d\phi$$

Independent on ϕ

$$\frac{1}{2\pi}\int_0^{2\pi} \Delta Q_\Phi(r,\phi)d\phi = 0$$

Has zero average over the synchrotron motion

*M. Schenk et al., "Vlasov description of the effects of nonlinear chromaticity on transverse coherent beam instabilities", Phys. Rev. Accel. Beams **21**, 084402*

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Handling the head-tail phase shift

We recall the equation:

$$\frac{\partial \Delta\psi}{\partial t} - \omega_0(Q_{y0} + \Delta Q_\Phi(r,\phi) + \Delta Q_R(r))\frac{\partial \Delta\psi}{\partial \theta_y} + \omega_s \frac{\partial \Delta\psi}{\partial \phi} = -\frac{df_0}{dJ_y} g_0(J_z)\sqrt{\frac{2J_y R}{Q_{y0}}}\sin\theta_y \frac{F_y^{coh}(z;t)}{m_0\gamma v}$$

where we are looking for solutions in the form:

$$\Delta\psi(J_y,\theta_y,r,\phi;t) = e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p(J_y)e^{jp\theta_y}\cdot e^{-jp\Delta\Phi(r,\phi)}\cdot \sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-jl\phi}$$

we replace the expression in the equation and compute the derivatives:

$$e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p(J_y)e^{jp\theta_y}\cdot \sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-j(p\Delta\Phi(z,\delta)+l\phi)}\left(j\Omega - jp\omega_s\frac{\partial\Delta\Phi}{\partial\phi} - jl\omega_s - jp\omega_0(Q_{y0} + \Delta Q_\Phi + \Delta Q_R)\right)$$

$$= -\frac{df_0}{dJ_y} g_0(J_z)\sqrt{\frac{2J_y R}{Q_{y0}}}\sin\theta_y \frac{F_y^{coh}(z;t)}{m_0\gamma v}$$

By **choosing $\Delta\Phi$** so that $\Delta Q_\Phi(r,\phi) = -\frac{\omega_s}{\omega_0}\frac{\partial\Delta\Phi}{\partial\phi}$ we can cancel the two circled terms:

$$e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p(J_y)e^{jp\theta_y}e^{-jp\Delta\Phi(r,\phi)}\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-jl\phi}(j\Omega - jp\omega_0(Q_{y0}+\Delta Q_R) - jl\omega_s)$$

$$= -\frac{df_0}{dJ_y} g_0(r)\sqrt{\frac{2J_y R}{Q_{y0}}}\left(\frac{e^{j\theta_y} - e^{-j\theta_y}}{2j}\right)\frac{F_y^{coh}(z;t)}{m_0\gamma v}$$

Explicit expression of the phase shift

For the above to be valid **we need to find $\Delta\Phi$** so that

$$\Delta Q_{\Phi}(r,\phi) = -\frac{\omega_s}{\omega_0}\frac{\partial \Delta\Phi}{\partial \phi}$$

We consider a **detuning in polynomial form** (including for example the e-cloud and arbitrary order chromaticity):

$$\Delta Q(z,\delta) = \sum_{n=1}^{N} A_n z^n + B_n \delta^n$$

we can find an **explicit expression for $\Delta\Phi$**:

$$\Delta\Phi(r,\phi) = -\frac{\omega_0}{\omega_s}\sum_{n=1}^{N} r^n \left[A_n\left(C_n(\phi) - \overline{C}_n\frac{\phi}{2\pi}\right) + \left(\frac{\omega_s}{v\eta}\right)^n B_n\left(S_n(\phi) - \overline{S}_n\frac{\phi}{2\pi}\right)\right]$$

where:

$$C_0(\phi) = \phi \quad C_1(\phi) = \sin\phi$$
$$S_0(\phi) = \phi \quad S_1(\phi) = -\cos\phi$$

$$C_n(\phi) = \frac{\cos^{n-1}\phi\sin\phi}{n} + \frac{n-1}{n}C_{n-2}(\phi)$$
$$S_n(\phi) = -\frac{\sin^{n-1}\phi\cos\phi}{n} + \frac{n-1}{n}S_{n-2}(\phi)$$

and:

$$\overline{C}_n = C_n(2\pi) - C_n(0)$$
$$\overline{S}_n = S_n(2\pi) - S_n(0)$$

Simplifying the equation

We go back to our Vlasov equation

$$e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p(J_y)e^{jp\theta_y}e^{-jp\Delta\Phi(r,\phi)}\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-jl\phi}(j\Omega - jp\omega_0(Q_{y0}+\Delta Q_R) - jl\omega_s)$$
$$= -\frac{df_0}{dJ_y}g_0(r)\sqrt{\frac{2J_yR}{Q_{y0}}}\left(\frac{e^{j\theta_y}-e^{-j\theta_y}}{2j}\right)\frac{F_y^{coh}(z;t)}{m_0\gamma v}$$

with: $\Delta\psi(J_y,\theta_y,r,\phi;t) = e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p(J_y)e^{jp\theta_y}\cdot e^{-jp\Delta\Phi(r,\phi)}\cdot\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-jl\phi}$

we can make **the usual approximations**, assuming that the coherent force is purely dipolar and that the betatron tune is larger than the synchrotron tune (see Chao, or Nicolas @ CAS) obtaining:

$$\sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}(\Omega - Q_{y0}\omega_0 - \omega_0\Delta Q_R - l\omega_s) = e^{-j\Omega t}e^{j\Delta\Phi(r,\phi)}g_0(r)\frac{F_y^{coh}(z;t)}{2m_0\gamma v}$$

with: $\Delta\psi(J_y,\theta_y,r,\phi;t) = e^{j\Omega t}e^{j\theta_y}\frac{df_0}{dJ_y}\sqrt{\frac{2J_yR}{Q_{y0}}}\cdot e^{-j\Delta\Phi(r,\phi)}\cdot\sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}$

Our **unknowns** are the frequency (Ω) and the structure of the unstable mode (R_l).

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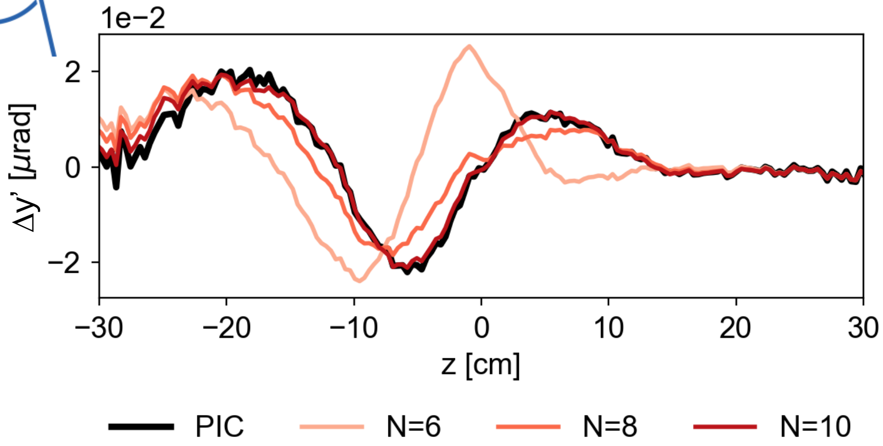
Expressing the coherent force



To describe the coherent force, we use the **response functions introduced before**:

$$\Delta y'(z) = \sum_{n=0}^{N} a_n k_n(z)$$

$$a_n = \frac{1}{H_n^2} \int \bar{y}(z) h_n(z) dz$$

Combining the two:

$$\Delta y'(z) = \sum_{n=0}^{N} k_n(z) \int \bar{y}(\tilde{z}) \frac{h_n(\tilde{z})}{H_n^2} d\tilde{z}$$

We go from a lumped kick to a **distributed force along the machine**:

$$F_y^{coh} = \frac{m_0 \gamma v^2}{2\pi R} \Delta y' = \frac{m_0 \gamma v^2}{2\pi R} \sum_{n=0}^{N} k_n(z) \int \bar{y}(\tilde{z}) \frac{h_n(\tilde{z})}{H_n^2} d\tilde{z}$$

Expressing the coherent force

$$F_y^{coh} = \frac{m_0 \gamma v^2}{2\pi R} \sum_{n=0}^{N} k_n(z) \int \bar{y}(\tilde{z}) \frac{h_n(\tilde{z})}{H_n^2} d\tilde{z}$$

The position along the bunch changes during the instability (head-tail motion), so it d**epends on the perturbation Δψ**:

$$\bar{y}(z) = \frac{1}{\lambda_0(z)} \iint d\tilde{y} d\tilde{y}' \int d\tilde{\delta} \tilde{y} \Delta\psi\left(\tilde{y}, \tilde{y}' z, \tilde{\delta}\right)$$

Replacing above we obtain:

$$F_y^{coh}(z,t) = \frac{1}{\lambda_0(z)} \frac{m_0 \gamma v^2}{2\pi R} \iint d\tilde{y} d\tilde{y}' \iint d\tilde{z} d\tilde{\delta}' \tilde{y} \Delta\psi\left(\tilde{y}, \tilde{y}', \tilde{z}, \tilde{\delta}\right) \sum_{n=0}^{N} k_n(z) \frac{h_n(\tilde{z})}{H_n^2}$$

We know that Δψ has this form:

$$\Delta\psi(J_y, \theta_y, r, \phi; t) = e^{j\Omega t} e^{j\theta_y} \frac{df_0}{dJ_y} \sqrt{\frac{2J_y R}{Q_{y0}}} \cdot e^{-j\Delta\Phi(r,\phi)} \cdot \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi}$$

With (quite) some manipulations we obtain:

$$F_y^{coh}(r,\phi,t) = -\frac{N m_0 \gamma v \omega_s}{2\pi \eta Q_{y0}} e^{j\Omega t} \iint \tilde{r} d\tilde{r} d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

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$$\sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}(\Omega - Q_{y0}\omega_0 - \omega_0\Delta Q_R - l\omega_s) = e^{-j\Omega t}e^{j\Delta\Phi(r,\phi)}g_0(r)\frac{F_y^{coh}(z;t)}{2m_0\gamma v}$$

$$F_y^{coh}(z,t) = \frac{1}{\lambda_0(z)}\frac{m_0\gamma v^2}{2\pi R}\iint d\tilde{y}d\tilde{y}'\iint d\tilde{z}d\tilde{\delta}'\tilde{y}\Delta\psi\left(\tilde{y},\tilde{y}',\tilde{z},\tilde{\delta}\right)\sum_{n=0}^{N}k_n(z)\frac{h_n(\tilde{z})}{H_n^2}$$

We combine them:

$$\sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}(\Omega - Q_{y0}\omega_0 - \omega_0\Delta Q_R - l\omega_s) = -\frac{N\omega_s}{4\pi\eta Q_{y0}}e^{j\Delta\Phi(r,\phi)}g_0(r)$$

$$\times\iint \tilde{r}d\tilde{r}d\tilde{\phi}\cdot e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}\sum_{l'=-\infty}^{+\infty}R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}}\sum_{n=0}^{N}k_n(r\cos\phi)\frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

We use the orthogonality condition $\int_0^{2\pi} d\phi\, \mathbf{e}^{jl\phi}\mathbf{e}^{-jl'\phi} = 2\pi\delta_{l,l'}$

$$R_l(r)(\Omega - Q_{y0}\omega_0 - \omega_0\Delta Q_R - l\omega_s) = -\frac{Nv}{8\pi^2 Q_{y0}}e^{j\Delta\Phi(r,\phi)}\hat{g}_0(r)$$

$$\times\int d\phi e^{jl\phi}\iint \tilde{r}d\tilde{r}d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}\sum_{l'=-\infty}^{+\infty}R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}}\sum_{n=0}^{N}k_n\left(r\cos\phi\right)\frac{\hat{h}_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2}$$

This is a **generalized form of Sacherer's integral equation**

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Orthogonal polynomials

$$\sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi}(\Omega - Q_{y0}\omega_0 - \cancel{\omega_0 \Delta Q_R} - l\omega_s) = -\frac{N\omega_s}{4\pi\eta Q_{y0}} e^{j\Delta\Phi(r,\phi)} g_0(r)$$
$$\times \iint \tilde{r}\, d\tilde{r}\, d\tilde{\phi} \cdot e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

We expand the radial distribution using a **set of orthogonal polynomial f_{lm}**:

$$R_l(r) = W_l(r) \sum_{m=0}^{+\infty} b_{lm} f_{lm}(r)$$

Orthogonality condition:

$$\int f_{lm}(r) f_{lm'}(r) w_l(r) dr = F_{lm}\delta_{m,m'}$$

where *W*(*r*) and *w*(*r*) are regular functions that we can choose to ease convergence.

By applying to both sides the operator $\int dr\, w_l(r) f_{lm}(r) \frac{(*)}{W_l(r)}$ and using the orthogonality:

$$b_{lm} F_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) = -\frac{N\upsilon}{8\pi^2 Q_{y0}} \int dr\, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{\hat{g}_0(r)}{W_l(r)}$$
$$\times \int d\phi\, e^{jl\phi} \iint \tilde{r}\, d\tilde{r}\, d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

(here we have assumed **ΔQ_R=0**, we will generalize later)

$$b_{lm}F_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) = -\frac{Nv}{8\pi^2 Q_{y0}} \int dr\, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{\hat{g}_0(r)}{W_l(r)}$$

$$\times \int d\phi\, e^{jl\phi} \iint \tilde{r}\, d\tilde{r}\, d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

We replace the expansion: $R_l(r) = W_l(r) \sum_{m=0}^{+\infty} b_{lm} f_{lm}(r)$

And we obtain:

$$b_{lm}F_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) = -\frac{Nv}{8\pi^2 Q_{y0}}$$

$$\times \sum_{l'm'} b_{l'm'} \sum_{n=0}^{N} \iint dr d\phi\, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{\hat{g}_0(r)}{W_l(r)} e^{jl\phi} k_n(r\cos\phi)$$

$$\times \iint \tilde{r}\, d\tilde{r}\, d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \frac{W_{l'}(\tilde{r})}{\lambda_0(\tilde{r}\cos\tilde{\phi})} f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} \frac{h_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

These are known quantities!

$$b_{lm} F_{lm} (\Omega - Q_{y0}\omega_0 - l\omega_s) = -\frac{Nv}{8\pi^2 Q_{y0}} \times \sum_{l'm'} b_{l'm'} \sum_{n=0}^{N} \iint dr d\phi \, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{\hat{g}_0(r)}{W_l(r)} e^{jl\phi} k_n(r\cos\phi) \times \iint \tilde{r} d\tilde{r} d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \frac{W_{l'}(\tilde{r})}{\lambda_0(\tilde{r}\cos\tilde{\phi})} f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} \frac{h_n(\tilde{r}\cos\bar{\phi})}{H_n^2}$$

This can be rewritten synthetically as:

These are our unknowns

$$b_{lm} (\Omega - Q_{y0}\omega_0 - l\omega_s) = \sum_{l'm'} M_{lm,l'm'} b_{l'm'}$$

Complex frequency (tune shift and risetime)

"shape" of the mode

The structure of **eigenvalue problem** is clearly visible

where: $$M_{lm,l'm'} = -\frac{Nv}{8\pi^2 Q_{y0} F_{lm}} \sum_{n=0}^{N} \iint dr d\phi \, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{\hat{g}_0(r)}{W_l(r)} e^{jl\phi} k_n(r\cos\phi) \times \iint \tilde{r} d\tilde{r} d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \frac{W_{l'}(\tilde{r})}{\lambda_0(\tilde{r}\cos\tilde{\phi})} f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} \frac{h_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

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We go back to our full integral equation:

$$R_l(r)(\Omega - Q_{y0}\omega_0 - l\omega_s - \Delta Q_R(r)\omega_0) = -\frac{Nv}{8\pi^2 Q_{y0}} e^{j\Delta\Phi(r,\phi)}\hat{g}_0(r) \times \int d\phi\, e^{jl\phi} \iint \tilde{r}\, d\tilde{r}\, d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}$$

We project it over our set of orthogonal polynomials applying the integral $\int dr\, w_l(r) f_{lm}(r) \frac{(*)}{W_l(r)}$

$$b_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) - \frac{\omega_0}{F_{lm}} \int dr\, w_l(r) f_{lm}(r) \frac{R_l(r)\Delta Q_R(r)}{W_l(r)} = \sum_{l'm'} M_{lm,l'm'} b_{l'm'}$$

We replace the expansion: $R_l(r) = W_l(r) \sum_{m=0}^{+\infty} b_{lm} f_{lm}(r)$

Matrix found before

$$b_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) - \frac{\omega_0}{F_{lm}} \sum_{m'=0}^{+\infty} b_{lm'} \int dr\, w_l(r) \Delta Q_R(r) f_{lm}(r) f_{lm'}(r) = \sum_{l'm'} M_{lm,l'm'} b_{l'm'}$$

$$b_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) - \frac{\omega_0}{F_{lm}} \sum_{m'=0}^{+\infty} b_{lm'} \int dr\, w_l(r) \Delta Q_R(r) f_{lm}(r) f_{lm'}(r) = \sum_{l'm'} M_{lm,l'm'} b_{l'm'}$$

These are known quantities!

We define: $\tilde{M}_{lm,l'm'} = \delta_{l,l'} \frac{\omega_0}{F_{lm}} \int dr\, w_l(r) \Delta Q_R(r) f_{lm}(r) f_{lm'}(r)$

obtaining again an **eigenvalue problem**:

$$b_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) = \sum_{l'm'} \left(M_{lm,l'm'} + \tilde{M}_{lm,l'm'} \right) b_{l'm'}$$

where the effect of the **detuning with longitudinal amplitude** just introduces an **additional term in the matrix**

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As in DELPHI we choose:

Gaussian bunch (in longitudinal):

$$\lambda_0(z) = \frac{N}{\sqrt{2\pi}\sigma_b} e^{-\frac{z^2}{2\sigma_b^2}}$$

$$\hat{g}_0(r) = \frac{1}{2\pi\sigma_b^2} e^{-\frac{r^2}{2\sigma_b^2}}$$

$$R_l(r) = W_l(r) \sum_{m=0}^{+\infty} b_{lm} f_{lm}(r)$$

The radial distribution is expanded using the **Associated Laguerre polynomials**:

$$f_{lm}(r) = L_m^{|l|}(ar^2)$$

making this choice for the shape function W_l

$$W_l(r) = \left(\frac{r}{r_b}\right)^{|l|} e^{-ar^2}$$

where: $a = \frac{8}{r_b^2}$ $\quad r_b = 4\sigma_b$

We can get **explicit expressions for our matrices**:

$$M_{lm,l'm'} = -\frac{va}{4\pi^2\sqrt{2\pi}Q_{y0}\sigma_b} \frac{m!}{(|l|+m)!} \times \sum_{n=0}^{N} \int \tilde{r}\, d\tilde{r} \left(\frac{\tilde{r}}{r_b}\right)^{|l'|} L_{m'}^{|l'|}(a\tilde{r}^2) \int d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} e^{-a\tilde{r}^2\left(1-\frac{\cos^2\tilde{\phi}}{2a\sigma_b^2}\right)} \frac{h_n(\tilde{r}\cos\tilde{\phi})}{H_n^2} e^{-jl'\tilde{\phi}} \times \int r\, dr\, (ar_b r)^{|l|} L_m^{|l|}(ar^2) e^{-\frac{r^2}{2\sigma_b^2}} \int d\phi\, e^{j\Delta\Phi(r,\phi)} k_n(r\cos\phi) e^{il\phi}$$

$$\tilde{M}_{lm,l'm'} = \delta_{l,l'} \frac{\omega_0}{F_{lm}} \int dr\, 2are^{-ar^2} (ar^2)^{|l|} \Delta Q_R(r) L_m^{|l|}(ar^2) L_{m'}^{|l|}(ar^2)$$

So practically **we needed to write a (python) code that computes these integrals** (the structure of the matrix can be exploited to speed-up the calculation):

$$M_{lm,l'm'} = -\frac{va}{4\pi^2\sqrt{2\pi}Q_{y0}\sigma_b}\frac{m!}{(|l|+m)!}$$

$$\times \sum_{n=0}^{N}\int \tilde{r}\,d\tilde{r}\left(\frac{\tilde{r}}{r_b}\right)^{|l'|}L_{m'}^{|l'|}(a\tilde{r}^2)\int d\tilde{\phi}e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}e^{-a\tilde{r}^2\left(1-\frac{\cos^2\tilde{\phi}}{2a\sigma_b^2}\right)}\frac{h_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}e^{-jl'\tilde{\phi}}$$

$$\times \int r\,dr\,(ar_b r)^{|l|}L_m^{|l|}(ar^2)e^{-\frac{r^2}{2\sigma_b^2}}\int d\phi\,e^{j\Delta\Phi(r,\phi)}k_n(r\cos\phi)e^{il\phi}$$

$$\tilde{M}_{lm,l'm'} = \delta_{l,l'}\frac{\omega_0}{F_{lm}}\int dr\,2are^{-ar^2}(ar^2)^{|l|}\Delta Q_R(r)L_m^{|l|}(ar^2)L_{m'}^{|l|}(ar^2)$$

And that **solves this eigenvalue problem**:

$$b_{lm}(\Omega - Q_{y0}\omega_0 - l\omega_s) = \sum_{l'm'}\left(M_{lm,l'm'} + \tilde{M}_{lm,l'm'}\right)b_{l'm'}$$

giving us the **tune shift and risetime of the unstable modes** (Ω), and **the shape of the intra-bunch motion** (b_{lm})

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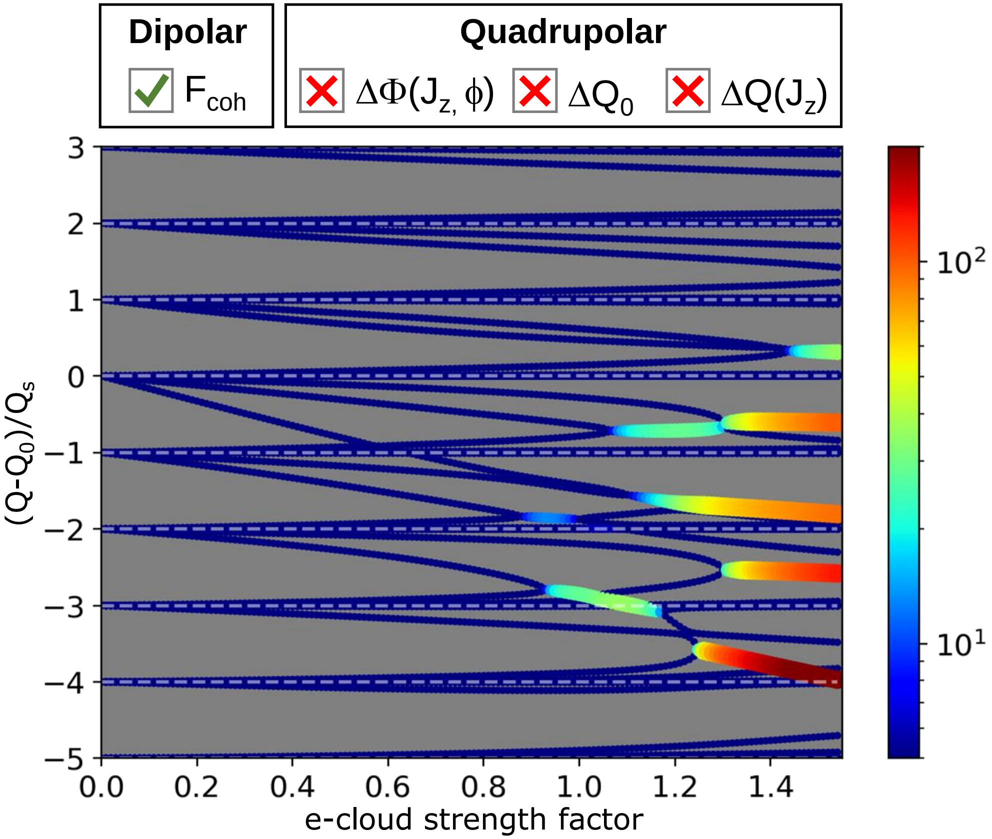
Application to LHC e-cloud instabilities

We **introduce the different effects one by one**:

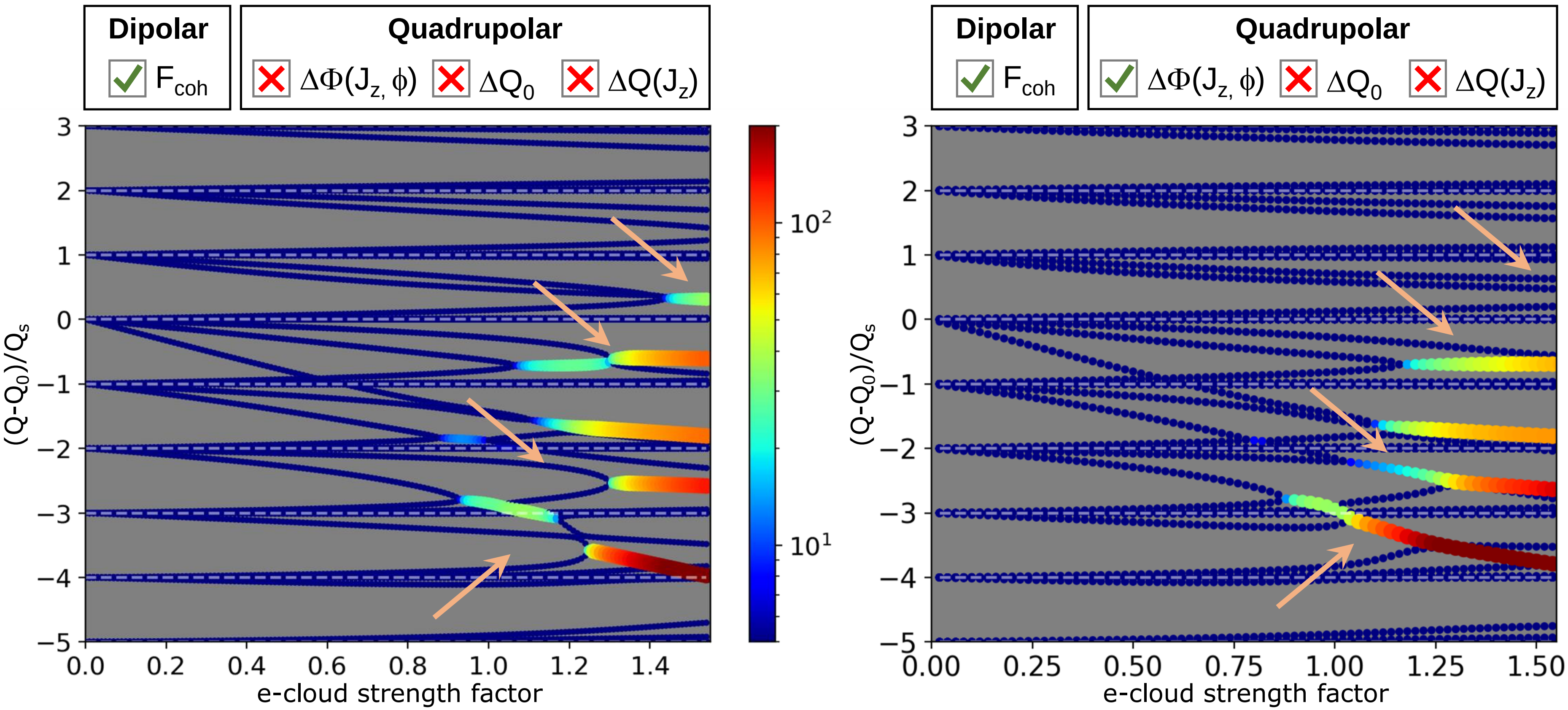
- With the e-cloud **dipolar force alone** (response functions)
  - **Negative tune shifts** are observed when increasing the cloud strength
  - Instabilities are triggered by **transverse mode coupling**



Application to LHC e-cloud instabilities

We **introduce the different effects one by one**:

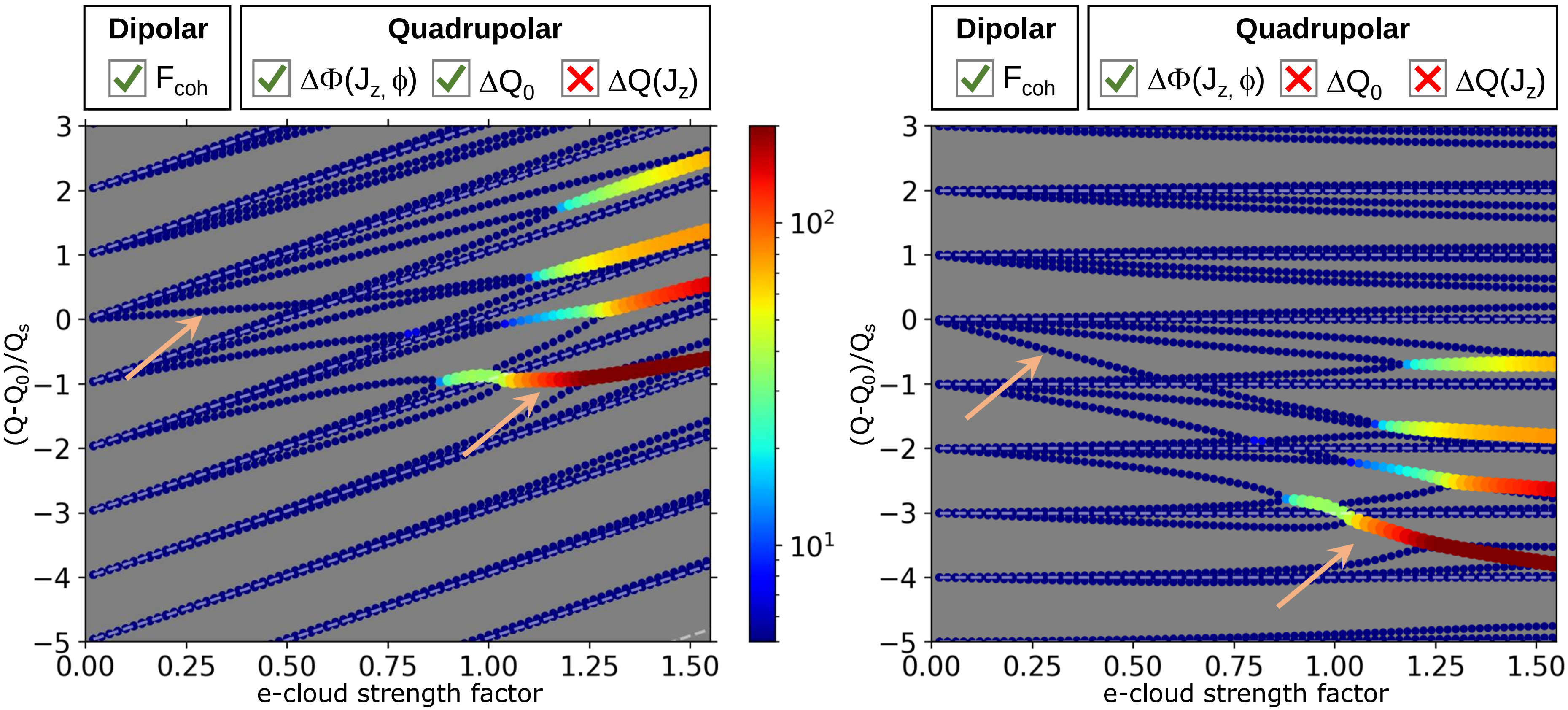
- With the e-cloud **dipolar force alone** (response functions)
  - **Negative tune shifts** are observed when increasing the cloud strength
  - Instabilities are triggered by **transverse mode coupling**
- When introducing the **head-tail phase shift** due to the e-cloud quadrupolar force
  - The **mode coupling behavior is affected**



Application to LHC e-cloud instabilities

We now introduce just a **tune shift (independent on z) due to the e-cloud quadrupolar forces:**

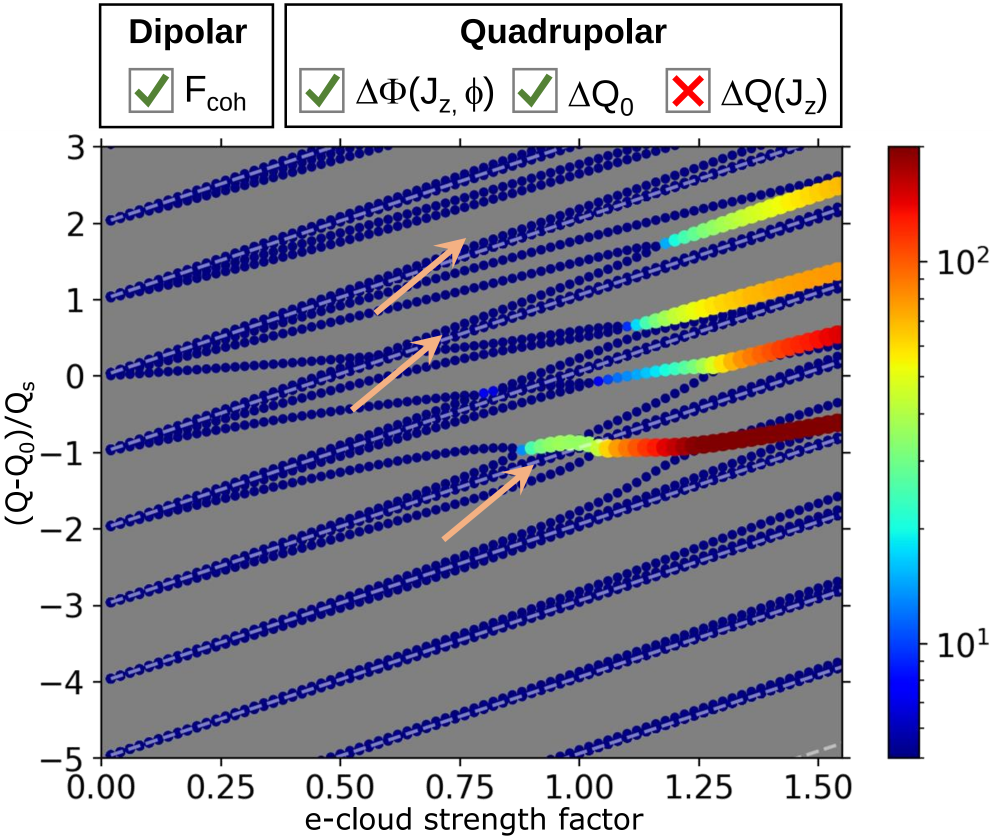
- All **frequencies are shifted upwards**
- The m**ost unstable mode** is shifted up to $(Q-Q_0)/Q_s \approx -1$, as **found in PIC macroparticle simulations**
- On the **mode-0 tune shift** a **cancellation between dipolar and quadrupolar** forces is observed

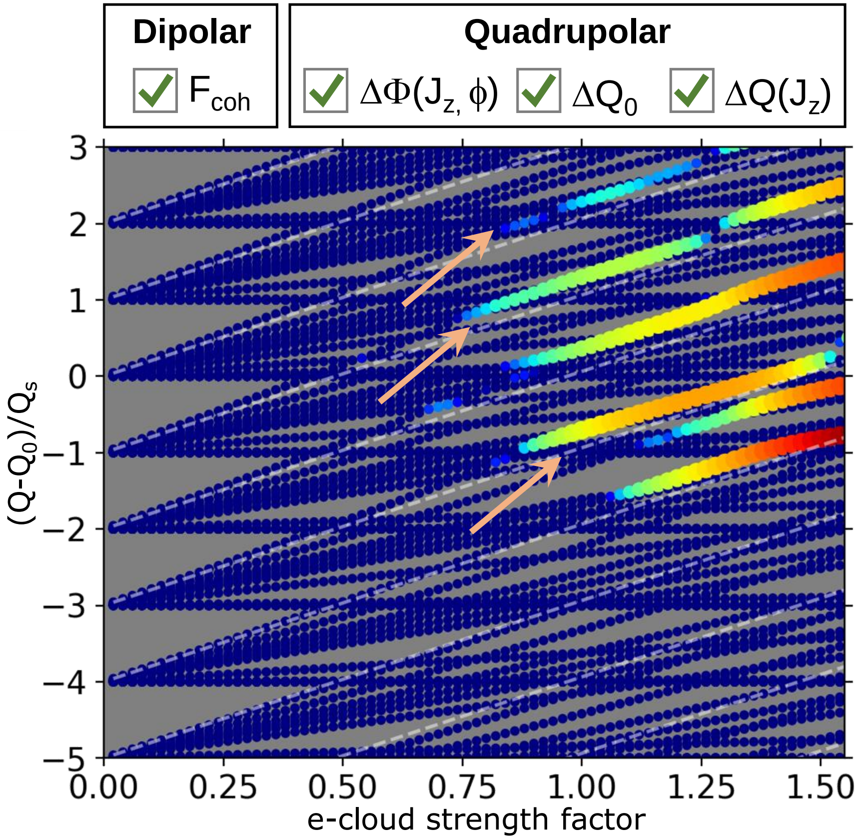


Application to LHC e-cloud instabilities

We finally introduce the **detuning with longitudinal amplitude**

- A forest of modes with different radial structure appears
- The **most unstable mode becomes less strong**
- **Other weaker instabilities** appear





Application to LHC e-cloud instabilities

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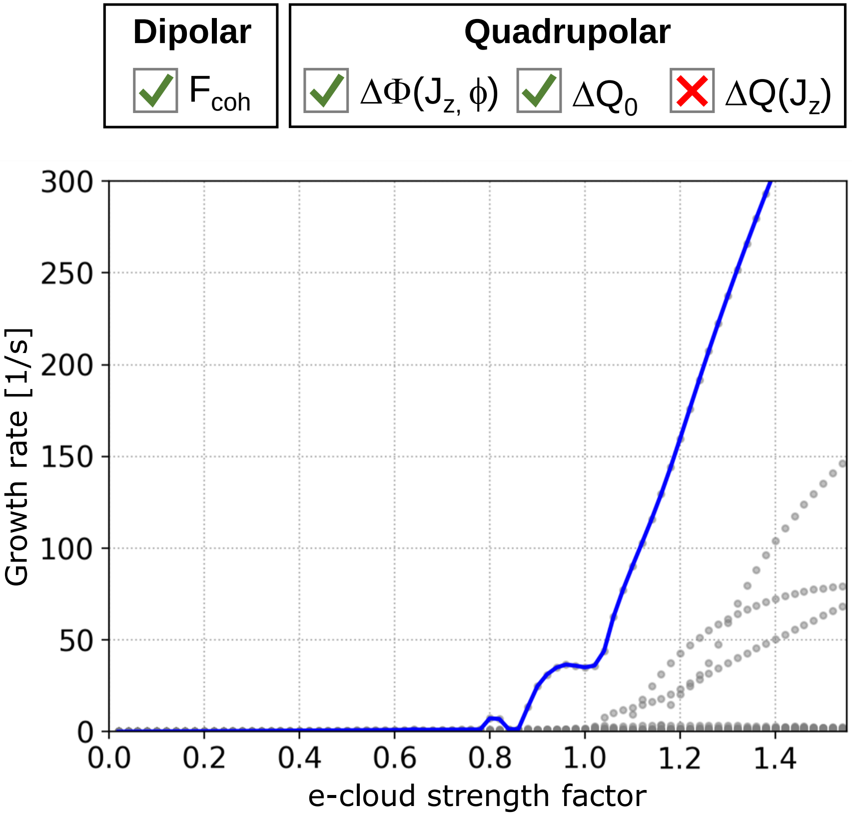
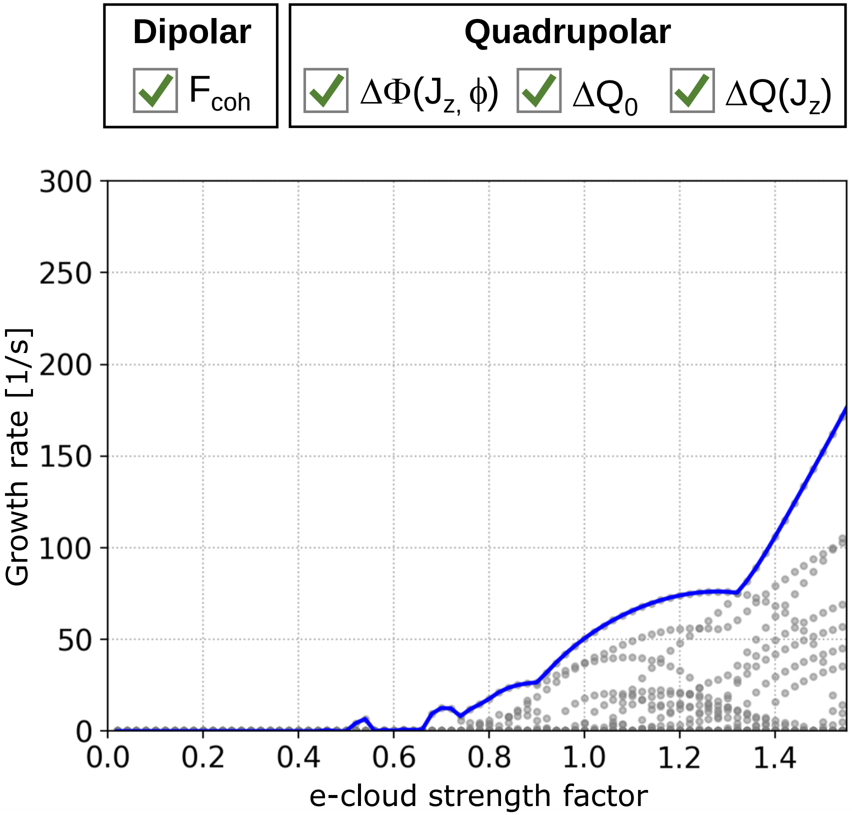
→ Clearly visible on **the imaginary parts** of the complex tune-shifts





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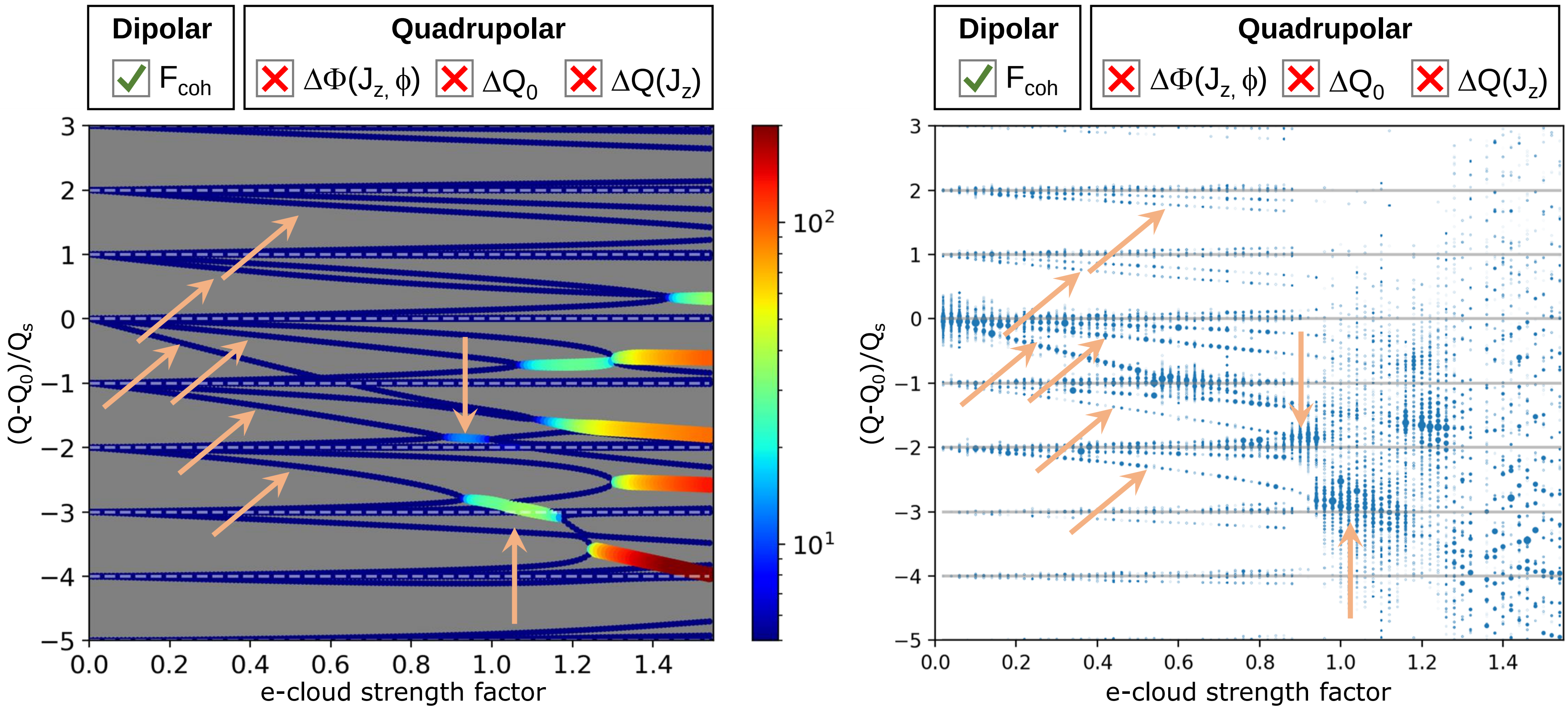
Comparisons against PyHEADTAIL

The different effects were **implemented in PyHEADTAIL to validate the Vlasov solver**

- The Vlasov solver **agrees very well** with the macroparticle simulations in all simulated conditions



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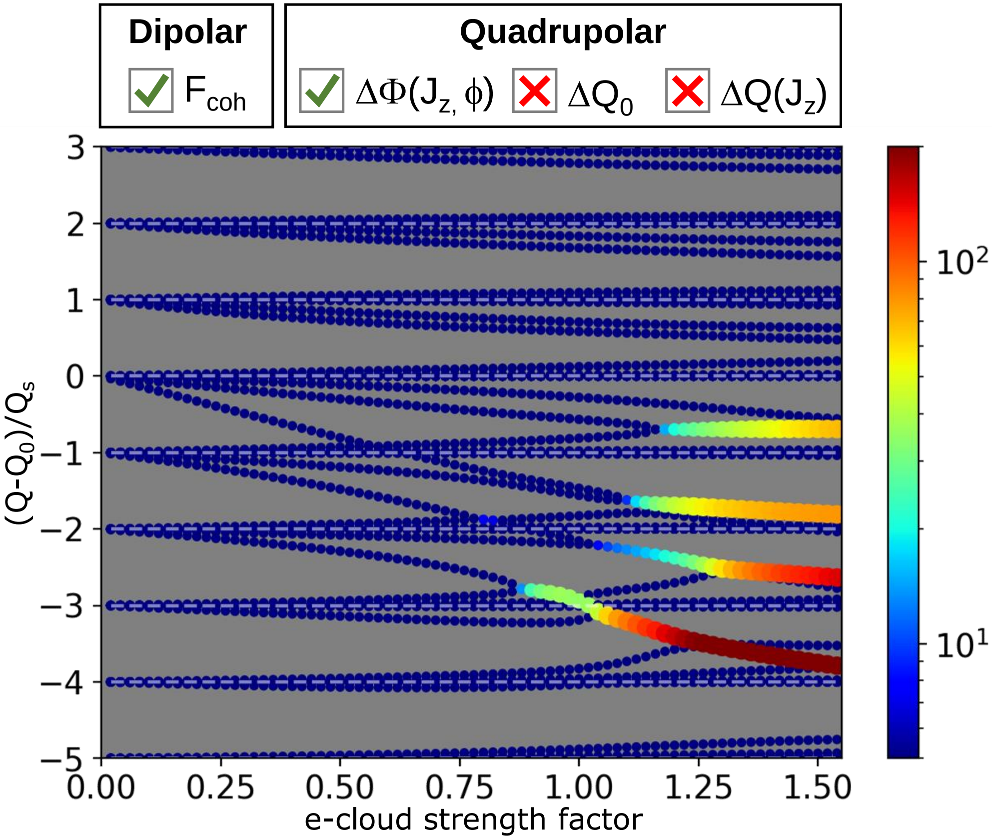

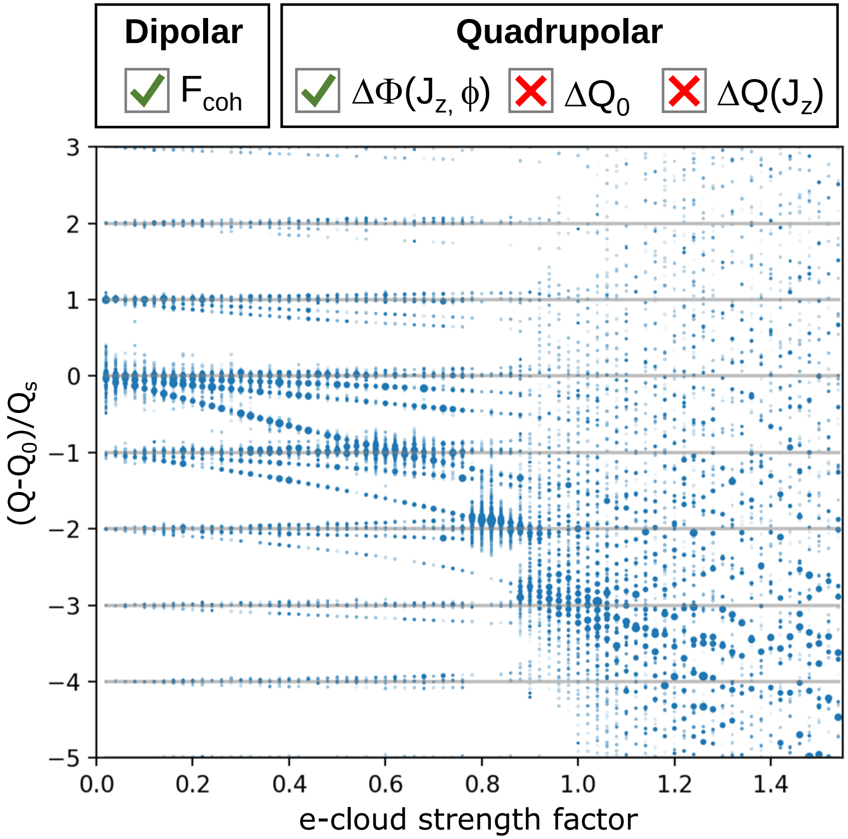

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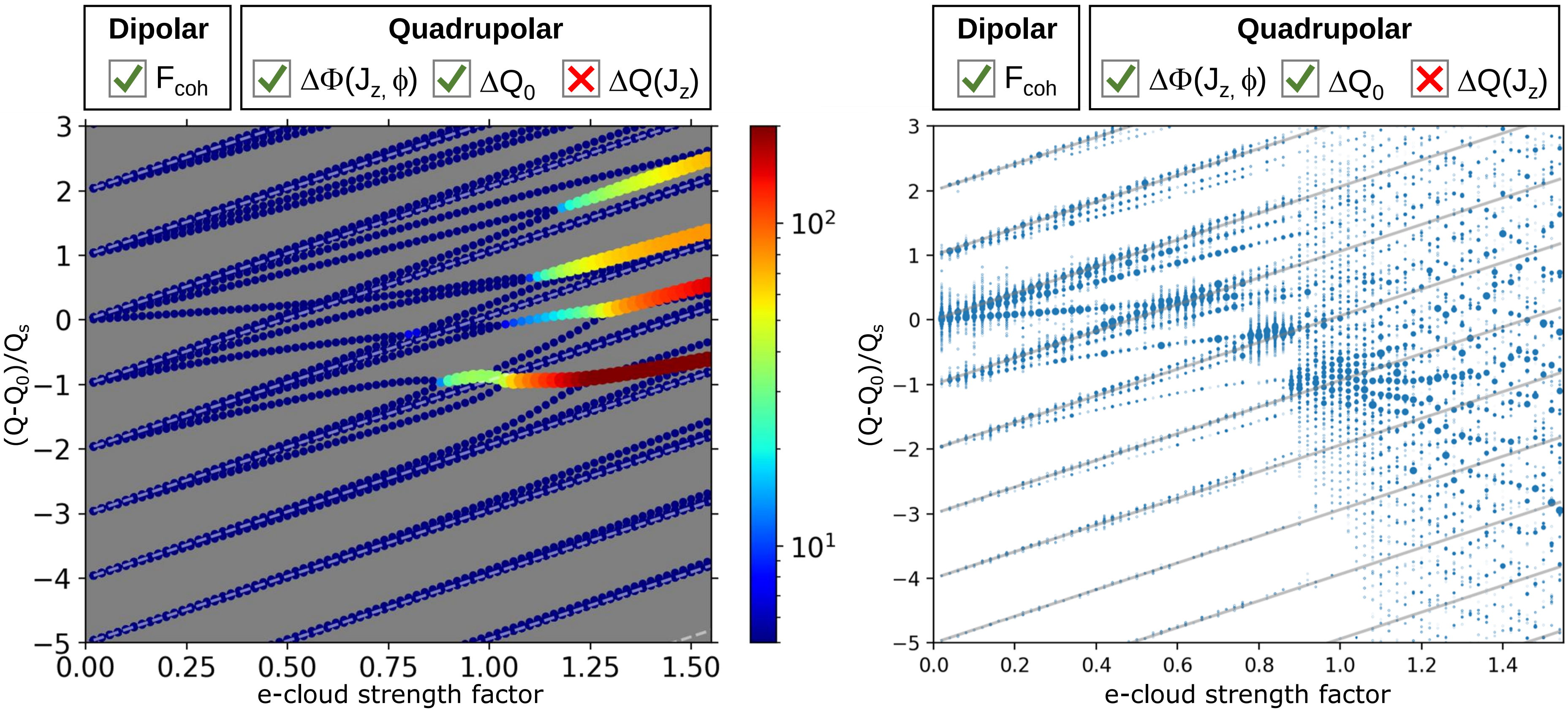
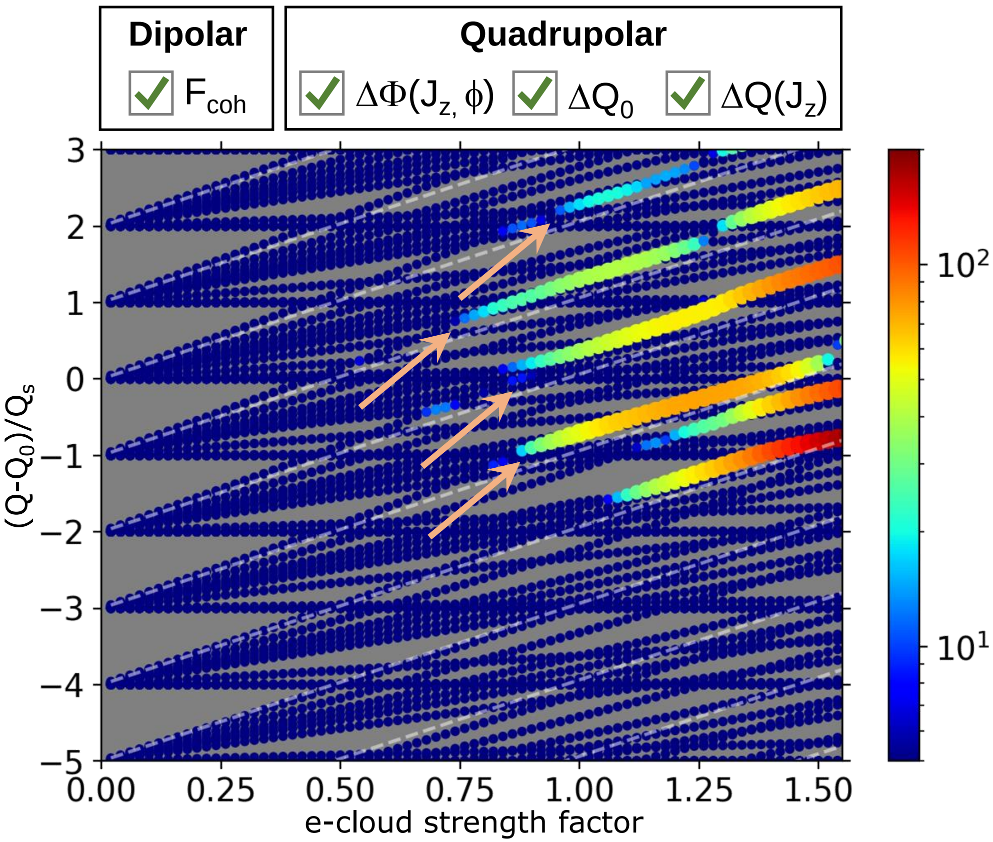
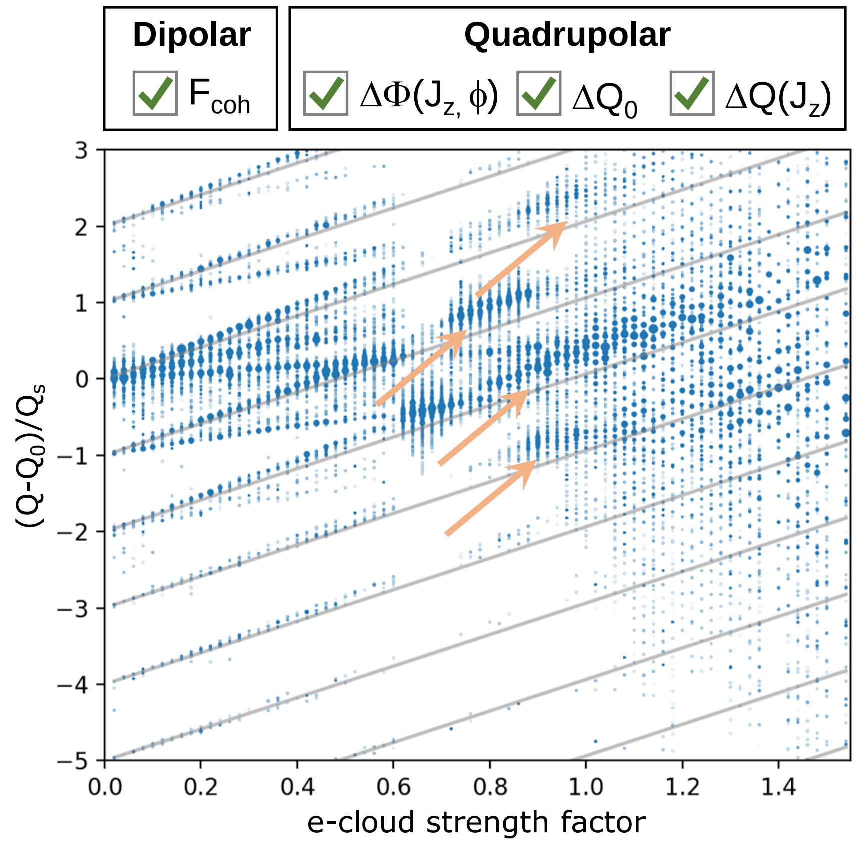


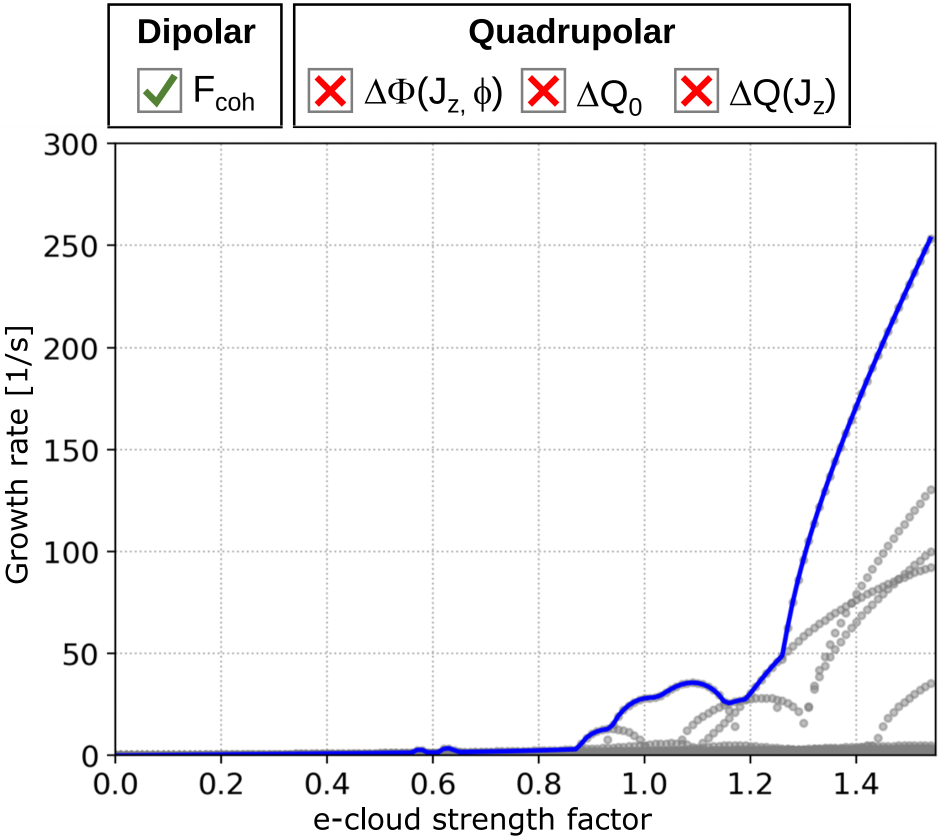
The different effects were **implemented in PyHEADTAIL to validate the Vlasov solver**

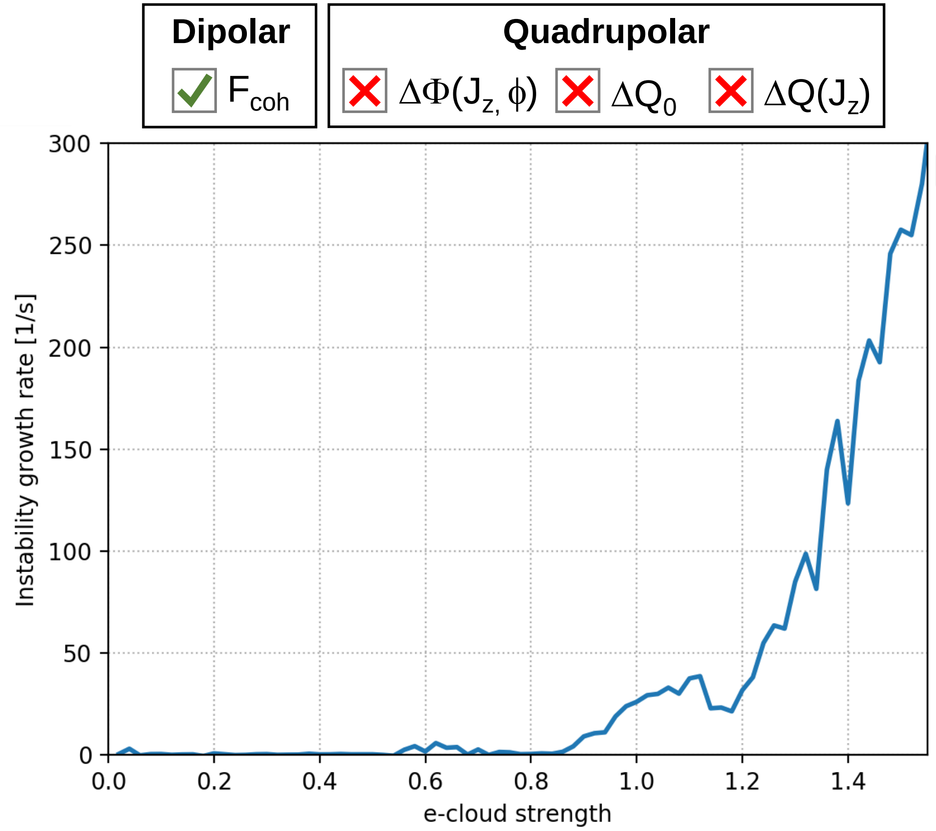
- The Vlasov solver **agrees very well** with the macroparticle simulations in all simulated conditions
- When **detuning with longitudinal amplitude** is introduced, only a **subset of the modes are visible on the centroid frequency spectrum**





Good agreement between PyHEADTAIL and the Vlasov solver is observed **also on the instability risetimes**





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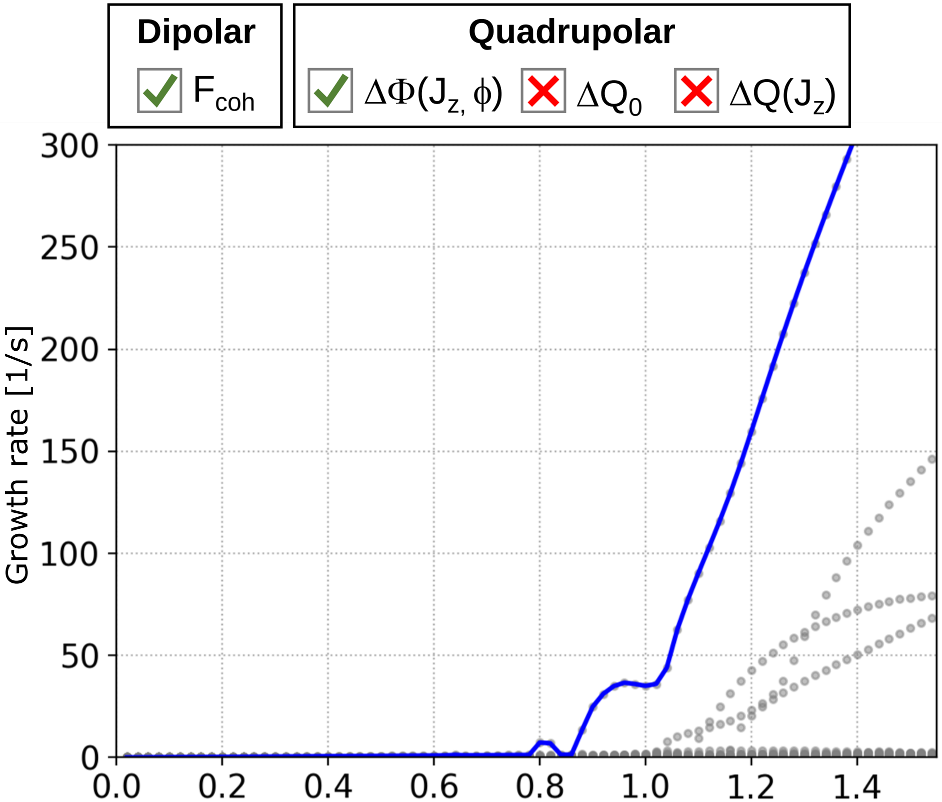

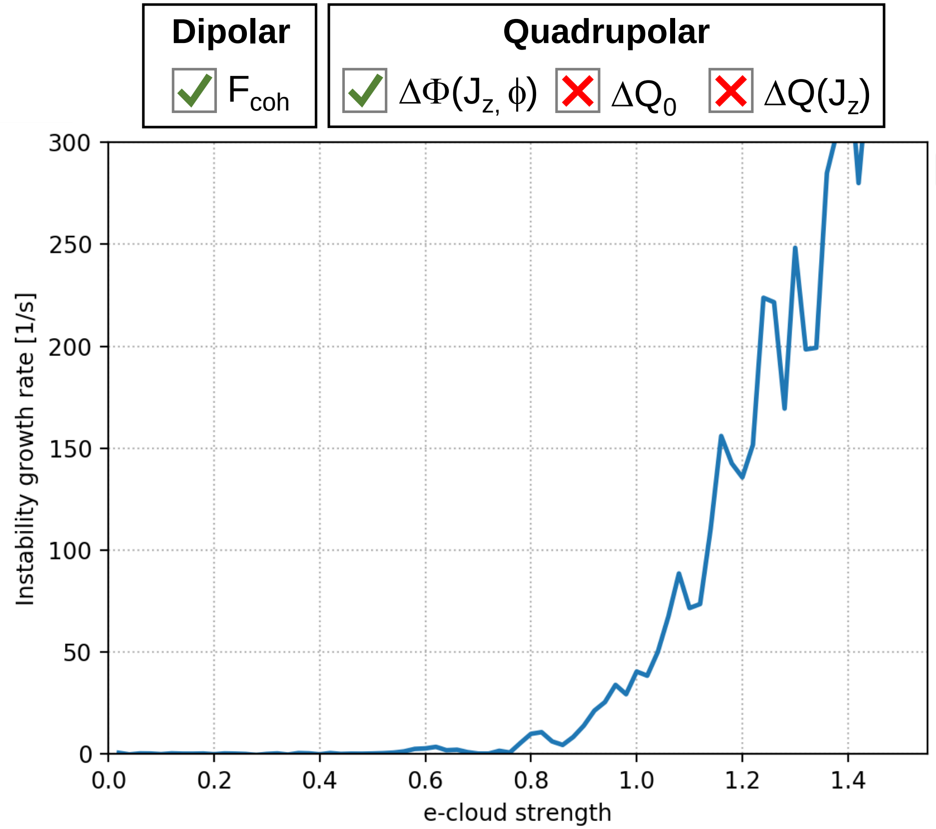

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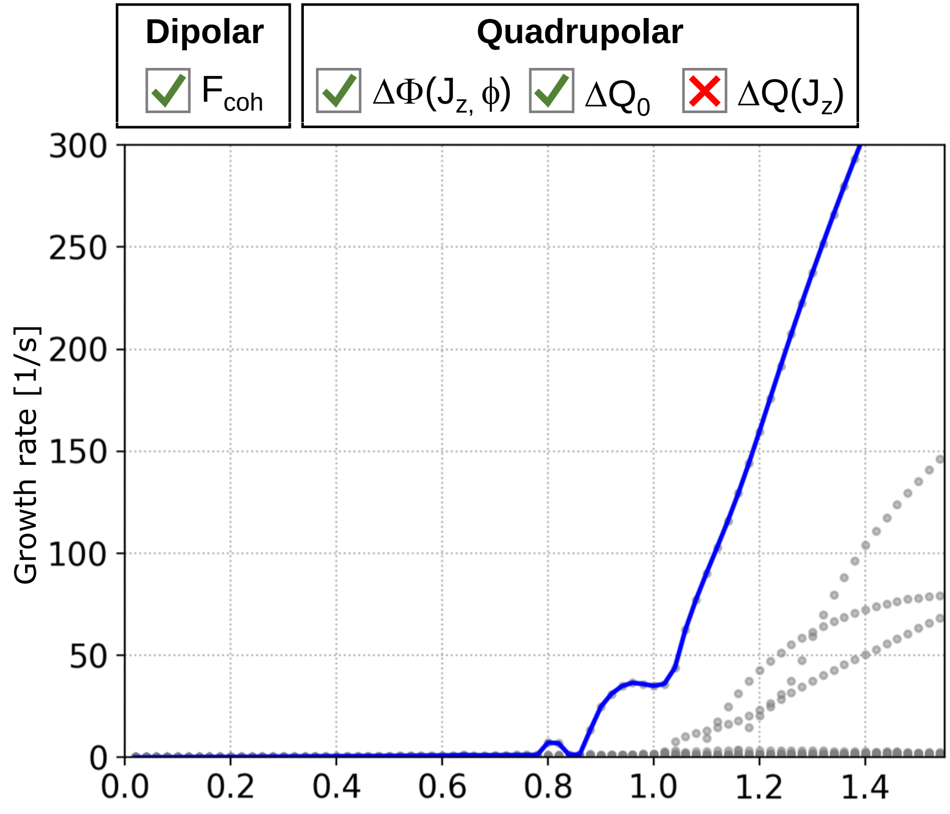

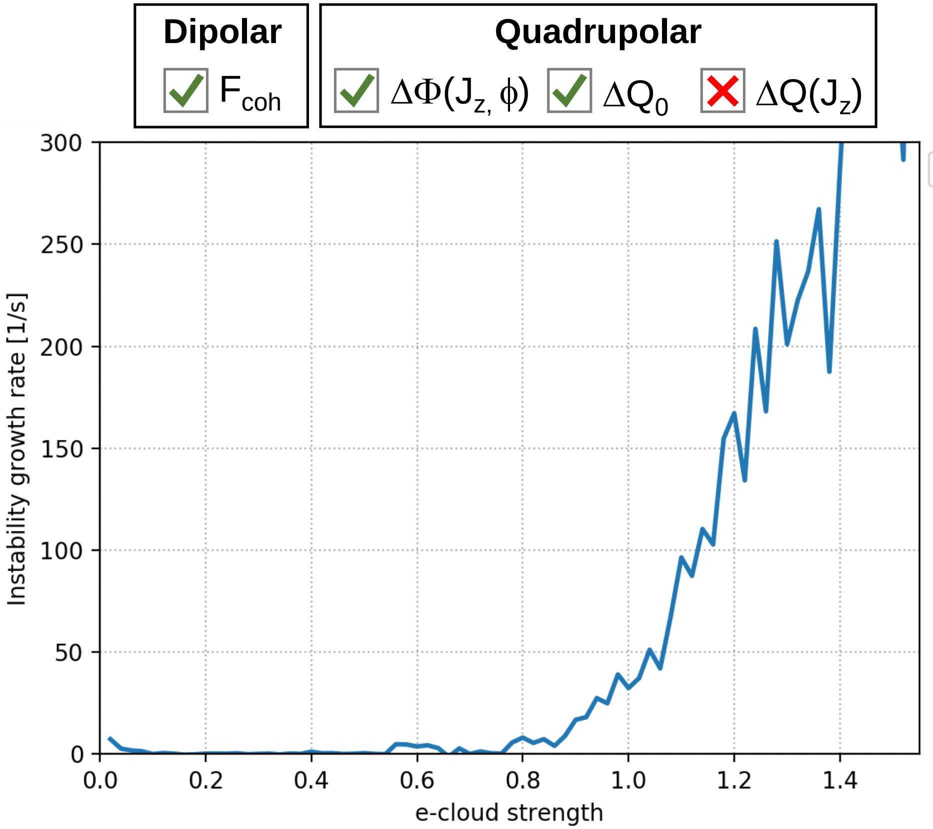

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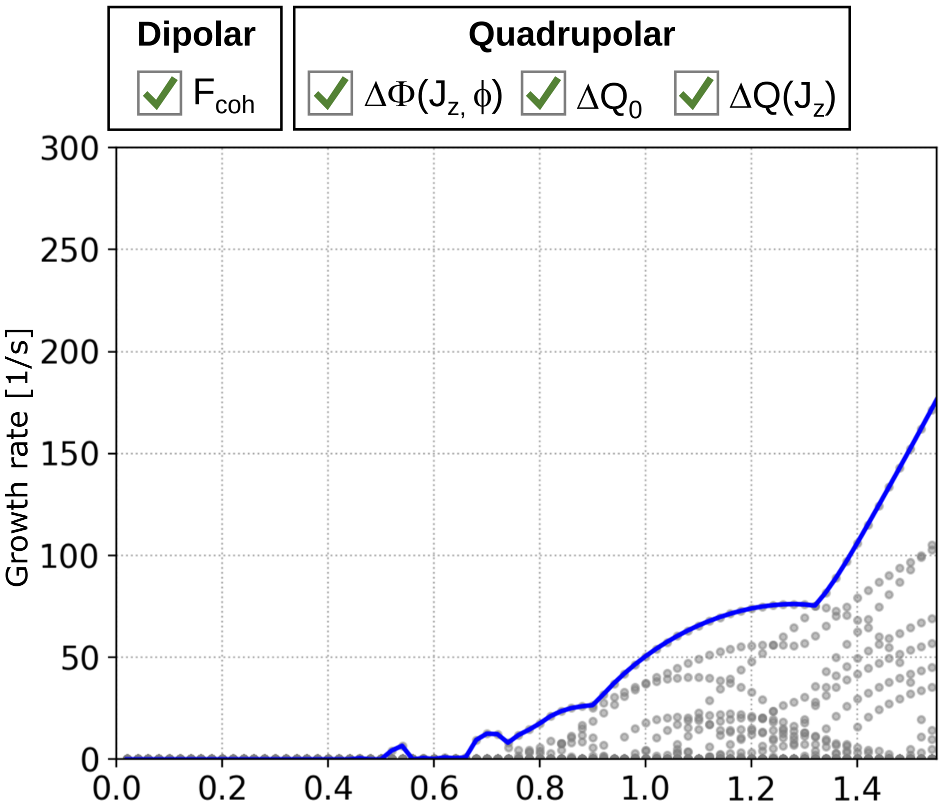
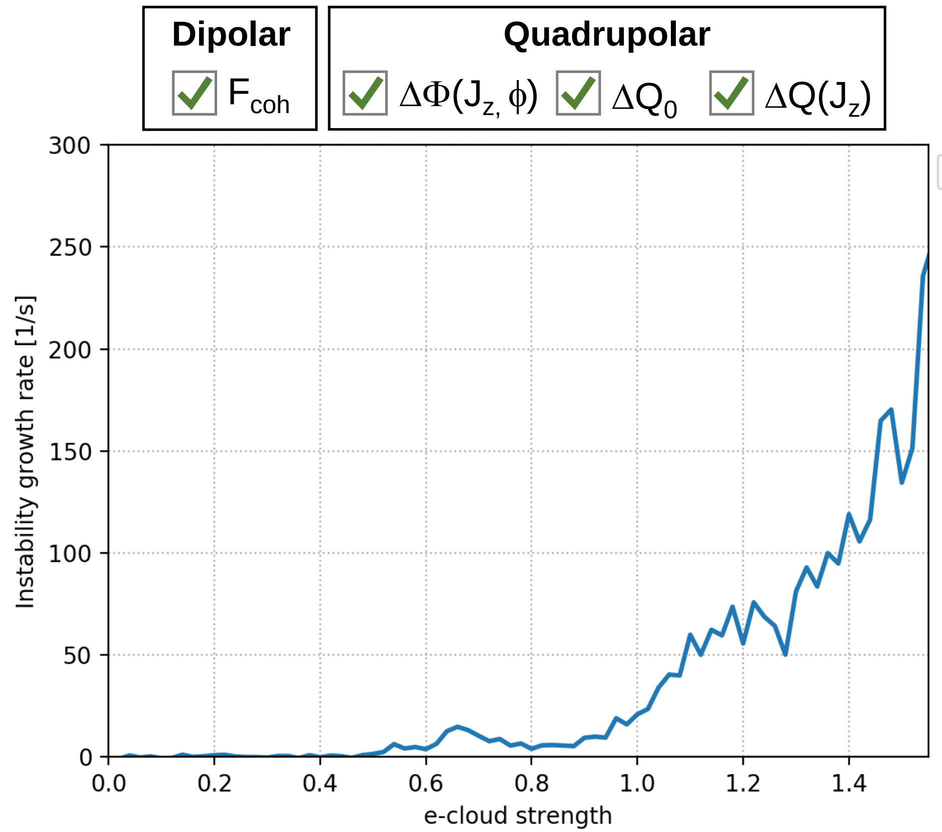
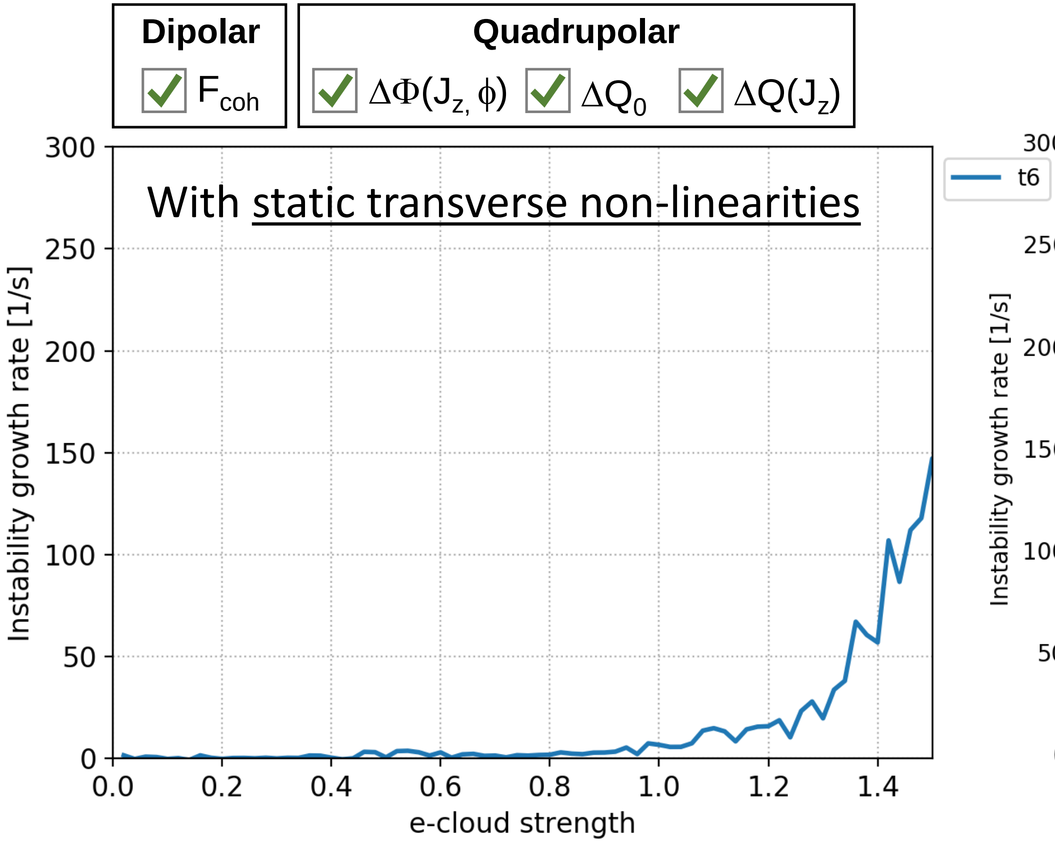


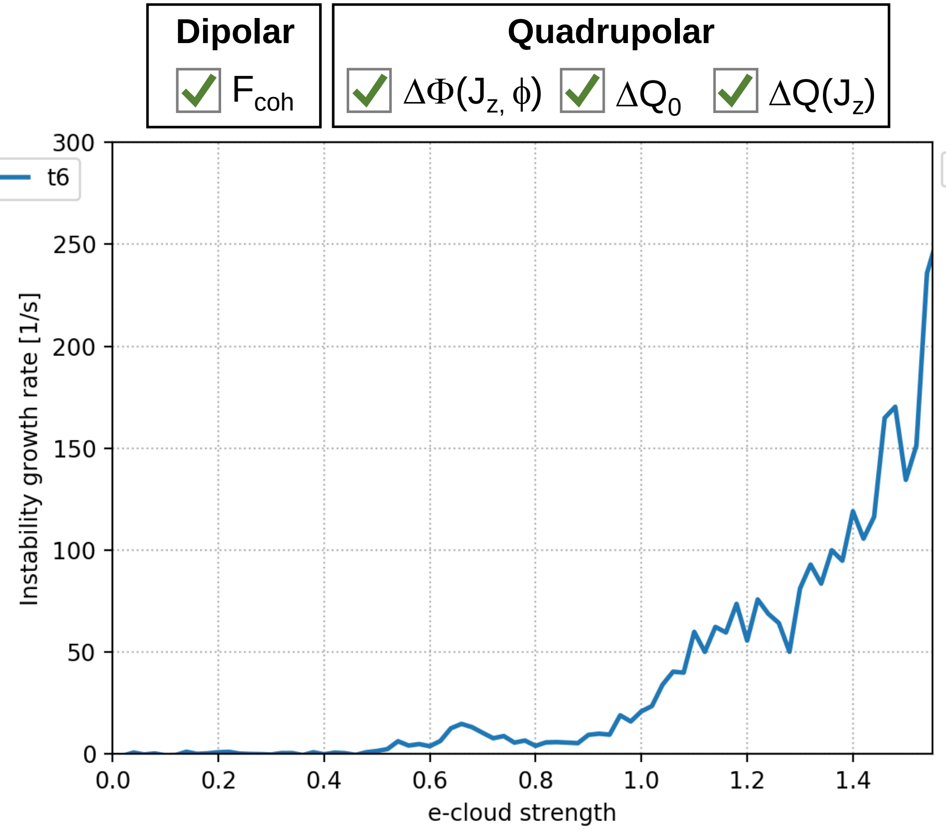


Effect of transverse non-linearities

The only effect that is not included in the Vlasov solver, is the **impact of transverse non-linearities**

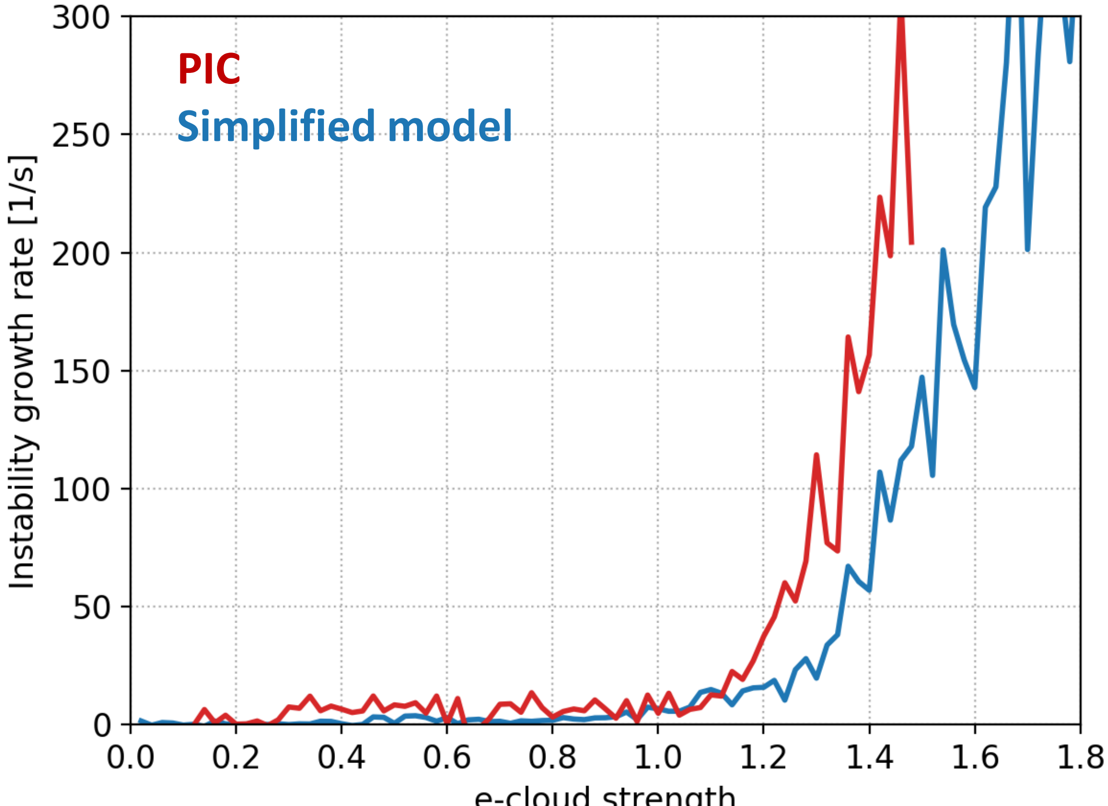
- These are **introduced PyHEADTAIL** in the form of a **static non-linear map** (independent on z)
- We observe a **mitigating effect on the instability** (Landau damping)





Effect of transverse non-linearities

The simulations performed with the simplified model **agree rather well with the full-PIC simulations** when looking at risetime, frequency spectrum and intra-bunch pattern



Effect of transverse non-linearities

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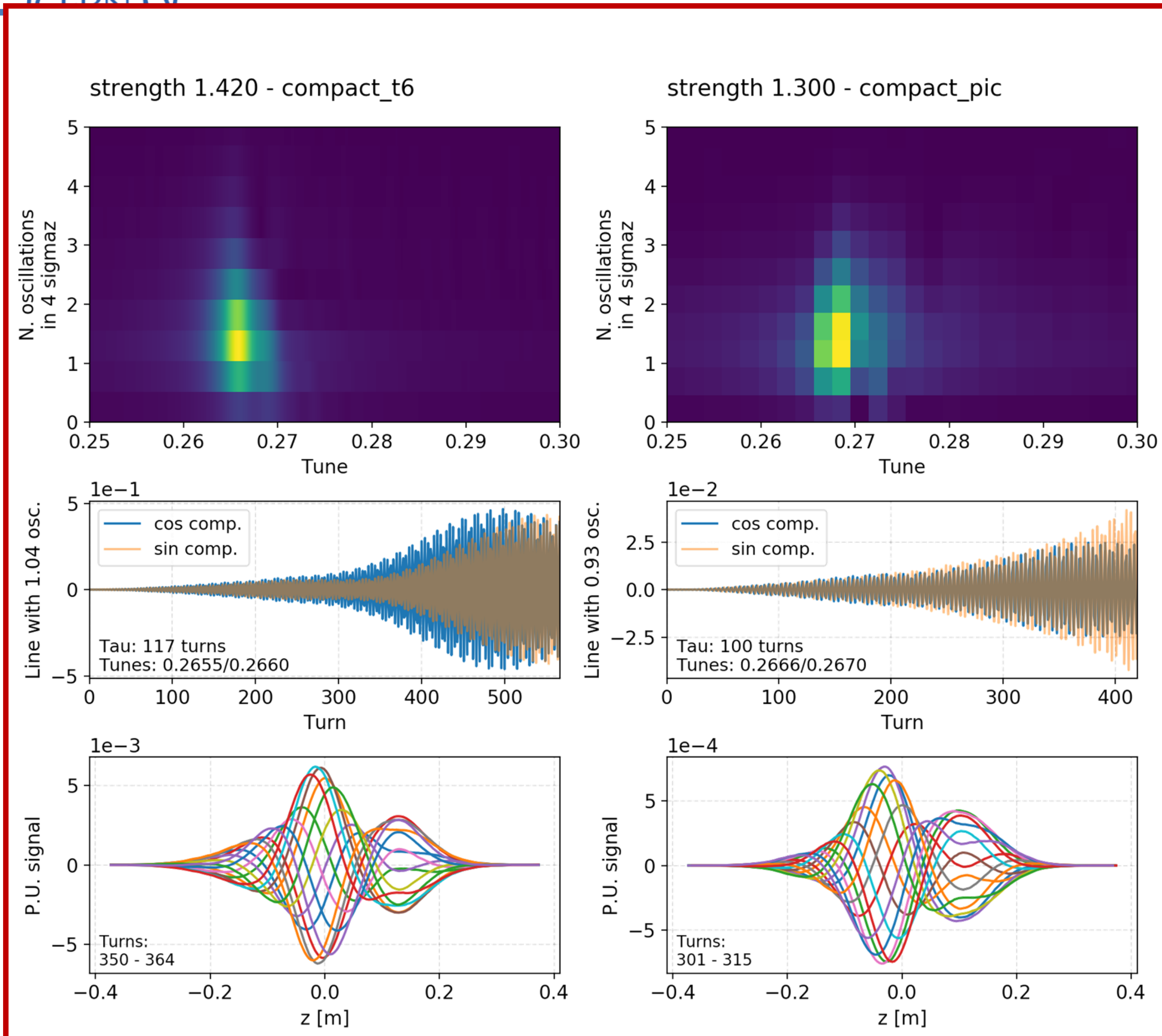

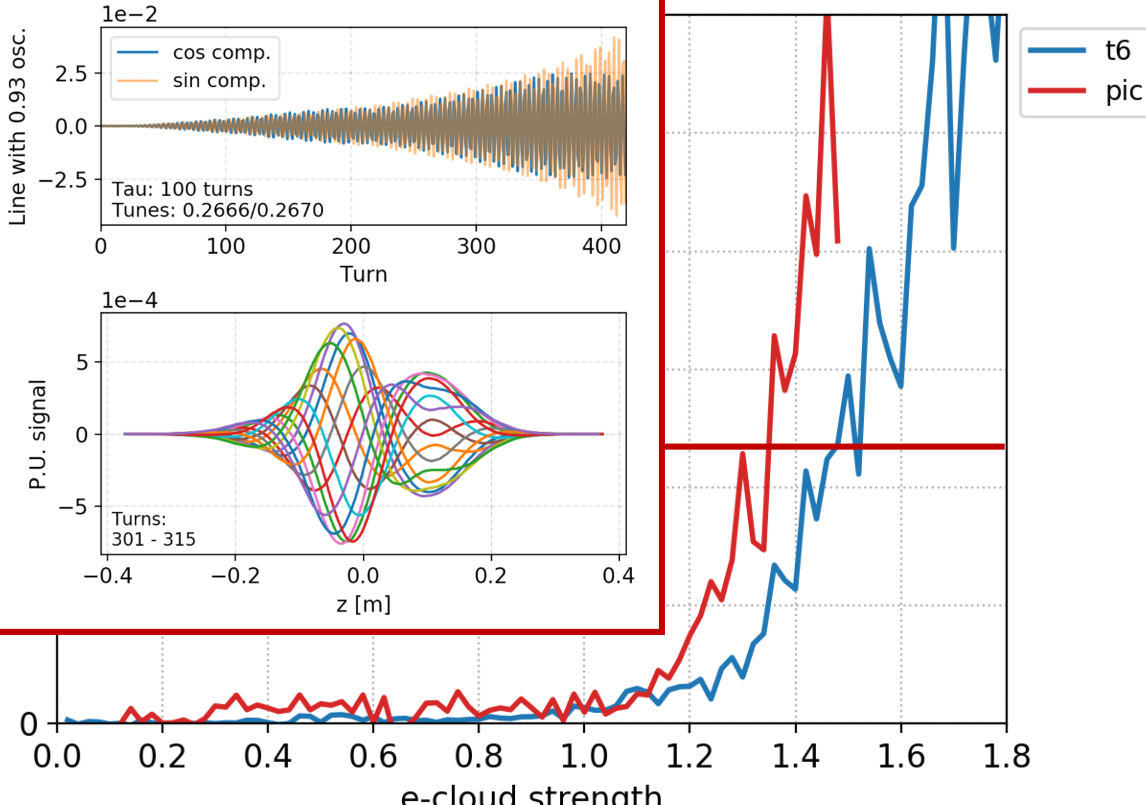

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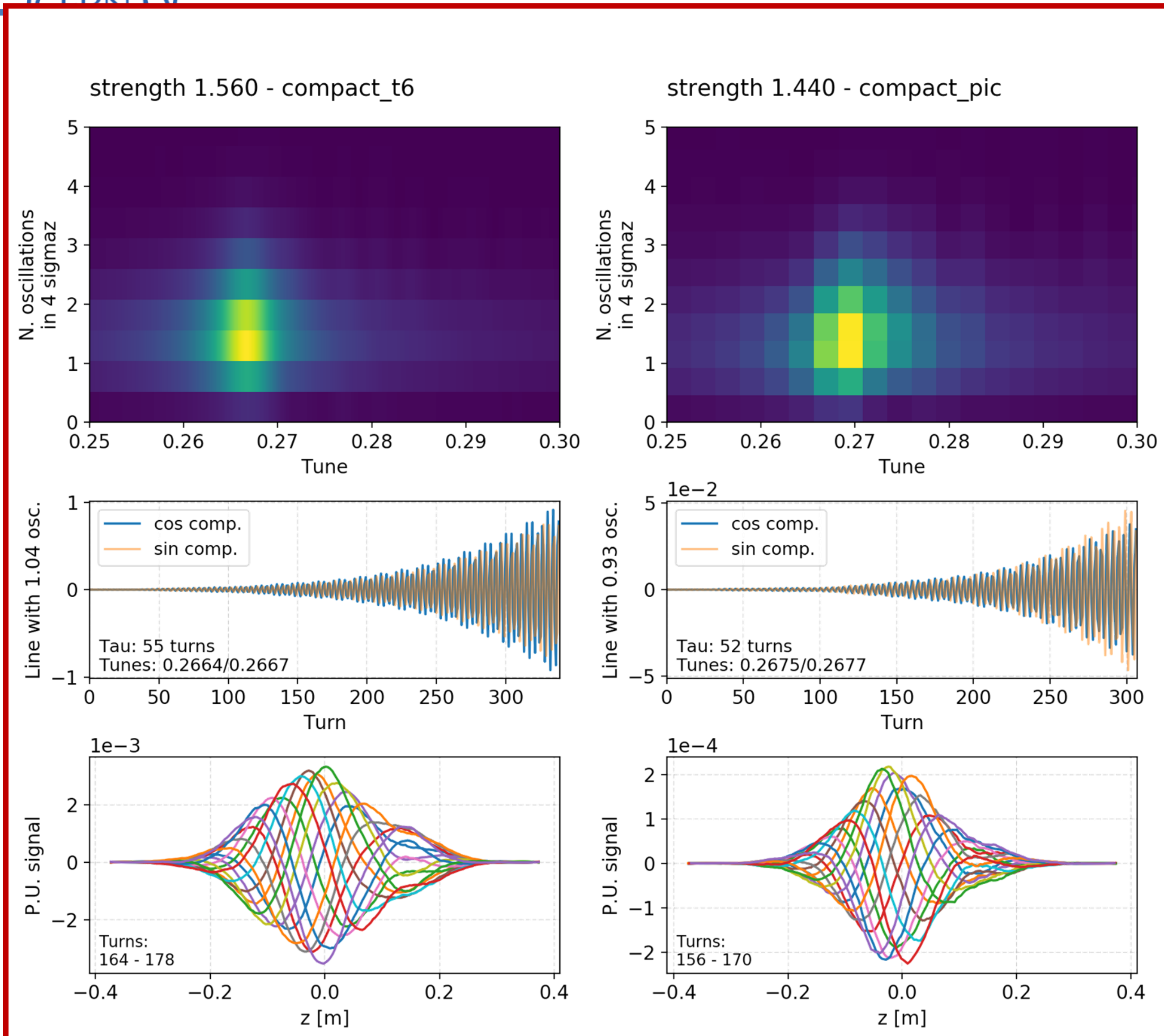

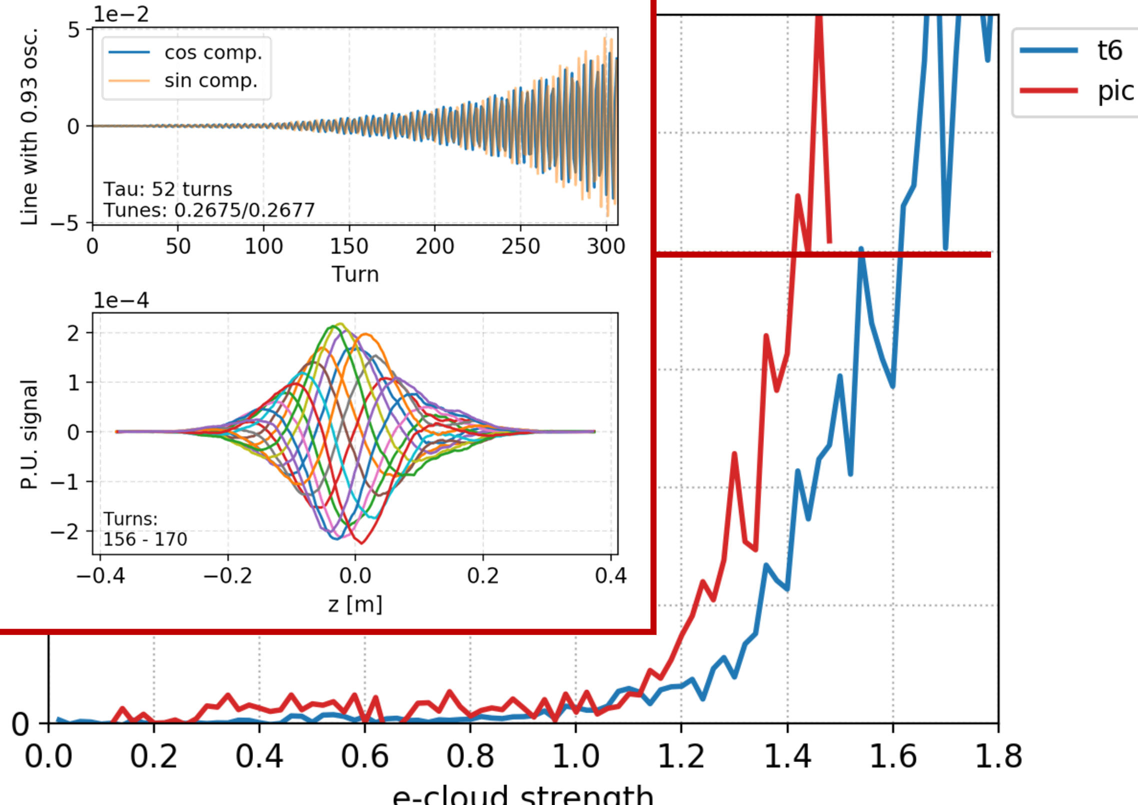

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- **Application to LHC e-cloud instabilities**
  - Comparison against PyHEADTAIL
- **Checks for the case of a dipolar impedance**

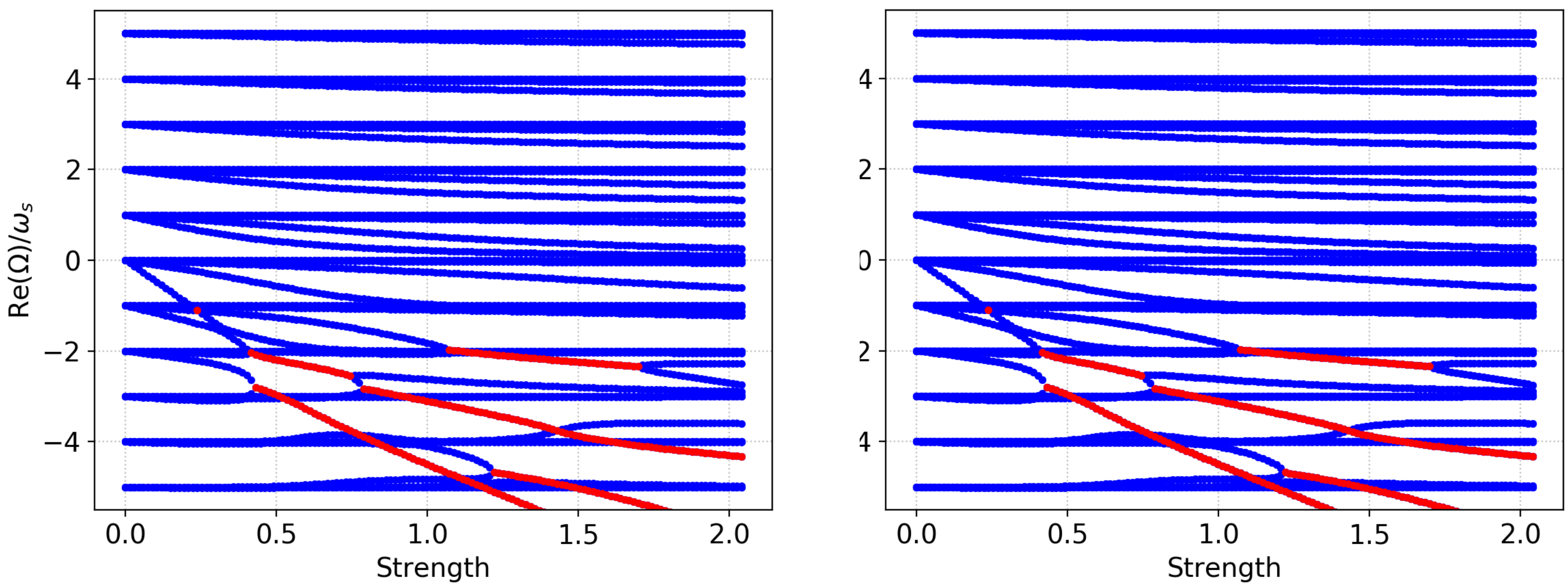
Impedance check 450 GeV

The presented method **can be applied also to the case of a dipolar impedance** (BB resonator) for check purposes:

- The **harmonic response** from the impedance is computed from **short PyHEADTAIL simulations** as done from the e-cloud
- It is possible to prove analytically that the **matrix obtained in this way and the matrix computed by DELPHI should be identical**

  → This was **verified**, also in the presence of chromaticity

New method **Q' = 0** DELPHI



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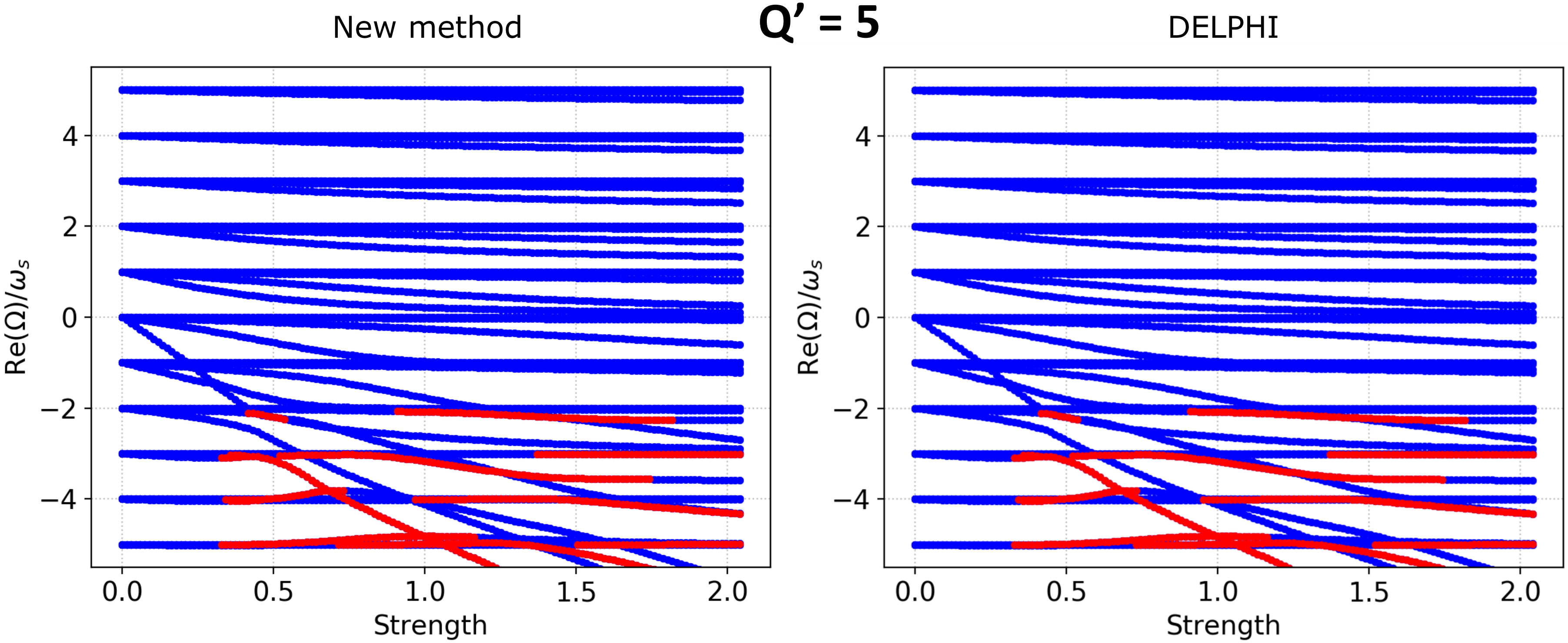
Q' = 5



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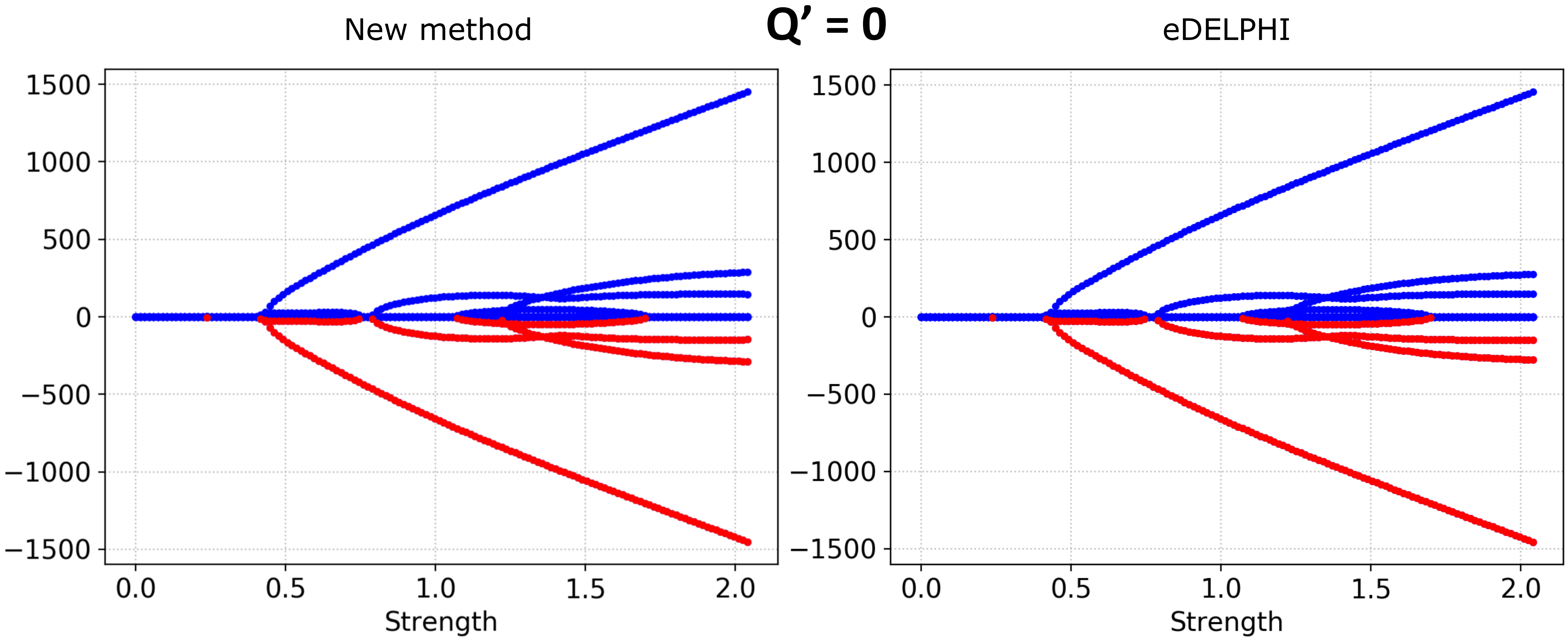

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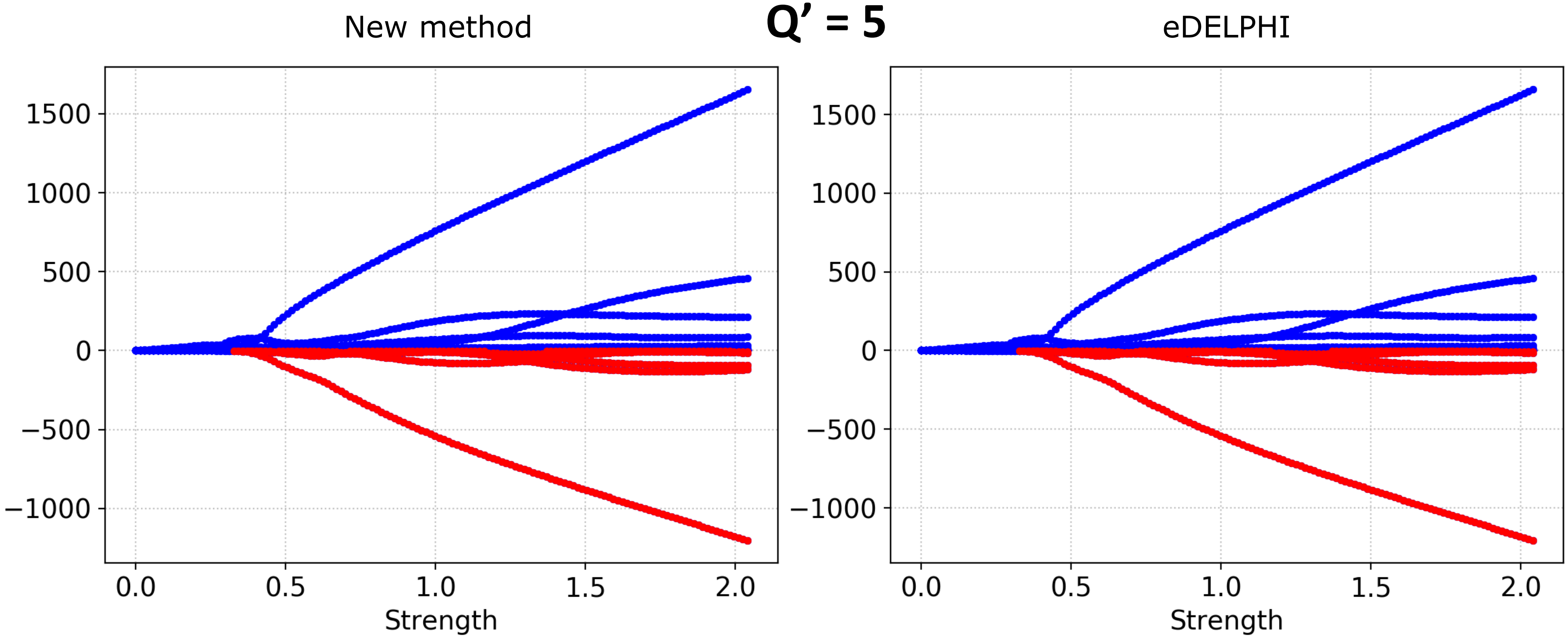

Summary and conclusions

- We have identified a **synthetic way to model the dipolar and quadrupolar forced introduced by an e-cloud** along a bunch

- We have **developed a Vlasov solver which includes this effects**
  - The **dipolar forces** are modelled using a **set of response functions**
  - The **detuning along the bunch** is decomposed into a term responsible for the **detuning with longitudinal ampl**itude and a term responsible for the **headtail phase shift**

- The results have been **successfully compared against PyHEADTAIL** simulations implementing the same model and good agreement is found also against PyECLOUD-PyHEADTAIL PIC simulations when introducing also the effect of transverse non-linearities

Thanks for your attention!

With quadrupolar alone there is no instabilitty