

# The growth of the $\frac{1}{16}$ -BPS index in 4d $N=4$ SYM

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[Based on collaborations with  
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CERN  
May 26 2020

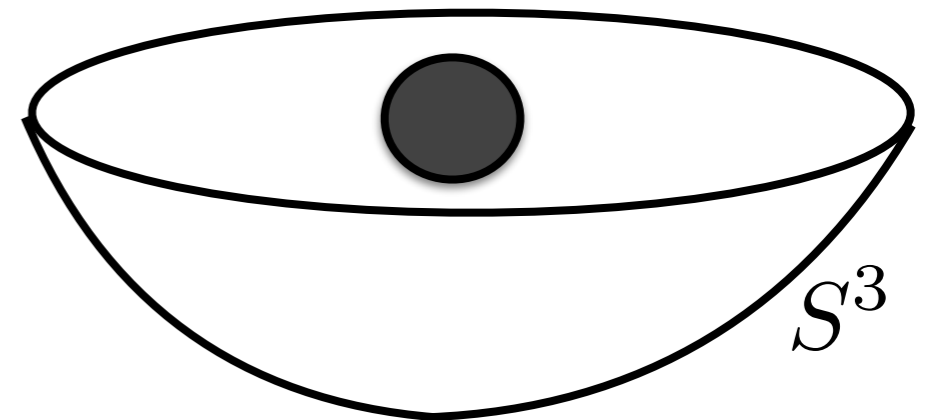


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# Does the most general index in $\mathcal{N} = 4$ SYM capture the susy BH entropy in $\text{AdS}_5 \times S^5$ ?



time ↑



$\mathcal{N} = 4$   $U(N)$  SYM

$\text{AdS}_5 \times S^5$

$\frac{1}{16}$ -BPS states

$\frac{1}{16}$ -BPS BH

$$\mathcal{I}_N(\tau) = \text{Tr}(-1)^F e^{2\pi i \tau_i q_i}$$

$$\log \mathcal{I}_N \stackrel{?}{=} O(N^2)$$

$$S_{\text{BH}} = \frac{A_H}{4G_N} = O(N^2)$$

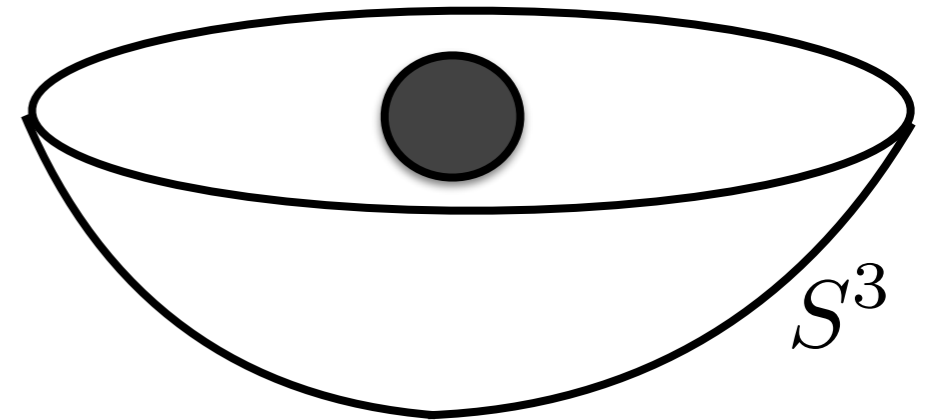
[Sundborg '99; Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk '03; Kinney, Maldacena, Minwalla, Raju '05]

[Gutowski, Reall '04; Chong, Cvetič, Lu, Pope '05; Kunduri, Lucietti, Reall '06]

# Does the most general index in $\mathcal{N} = 1$ SCFT4 capture the susy BH entropy in dual AdS5 ?



time ↑



$\mathcal{N} = 1$  SCFT  
 $\frac{1}{4}$ -BPS states

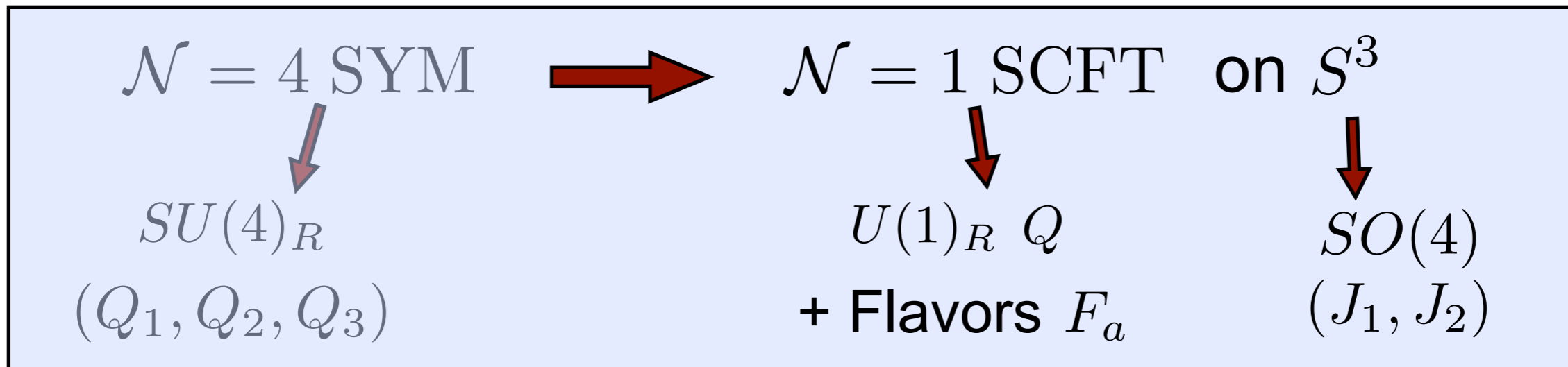
$AdS_5 \times M_5$   
 $\frac{1}{4}$ -BPS BH

$$\mathcal{I}_N(\tau) = \text{Tr}(-1)^F e^{2\pi i \tau q}$$

$$\log \mathcal{I}_N \stackrel{?}{=} O(N^2)$$

$$S_{\text{BH}} = \frac{A_H}{4G_N} = O(N^2)$$

# Index counts BPS states labelled by (R-charge + spin)



**Susy:**  $\{Q, \bar{Q}\} = E - J_1 - J_2 - \frac{3}{2}Q$

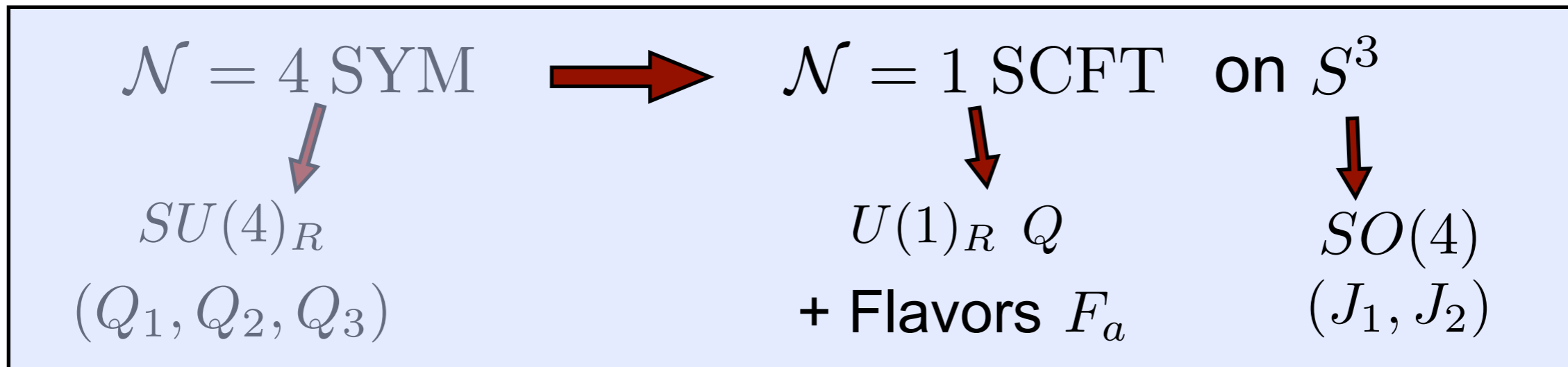
**Commutant:**  $[J_i + \frac{1}{2}Q, Q] = 0$  ,  $[F_a, Q] = 0$

Most general  $\frac{1}{4}$  - BPS index:

$$\text{Tr} (-1)^F e^{-\beta \{Q, \bar{Q}\}} e^{2\pi i \sigma (J_1 + \frac{1}{2}Q) + 2\pi i \tau (J_2 + \frac{1}{2}Q)} e^{2\pi i \mu_a F_a}$$



# Index counts BPS states labelled by (R-charge + spin)



**Susy:**  $\{Q, \bar{Q}\} = E - J_1 - J_2 - \frac{3}{2}Q$

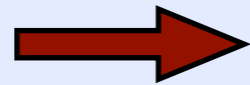
**Commutant:**  $[J_i + \frac{1}{2}Q, Q] = 0$  ,  $[F_a, Q] = 0$

**Simplest  $\frac{1}{4}$ -BPS index**  $(2J = J_1 + J_2)$

$$\mathcal{I}_N(\tau) = \text{Tr} (-1)^F e^{2\pi i \tau (2J + Q)} = \sum_{n \geq 0} d_N(n) e^{2\pi i \tau n/3}$$

# Susy BH has $O(N^2)$ semiclassical entropy

IIB on  $\text{AdS}_5 \times S^5$   
(IIB sugra in 10d)

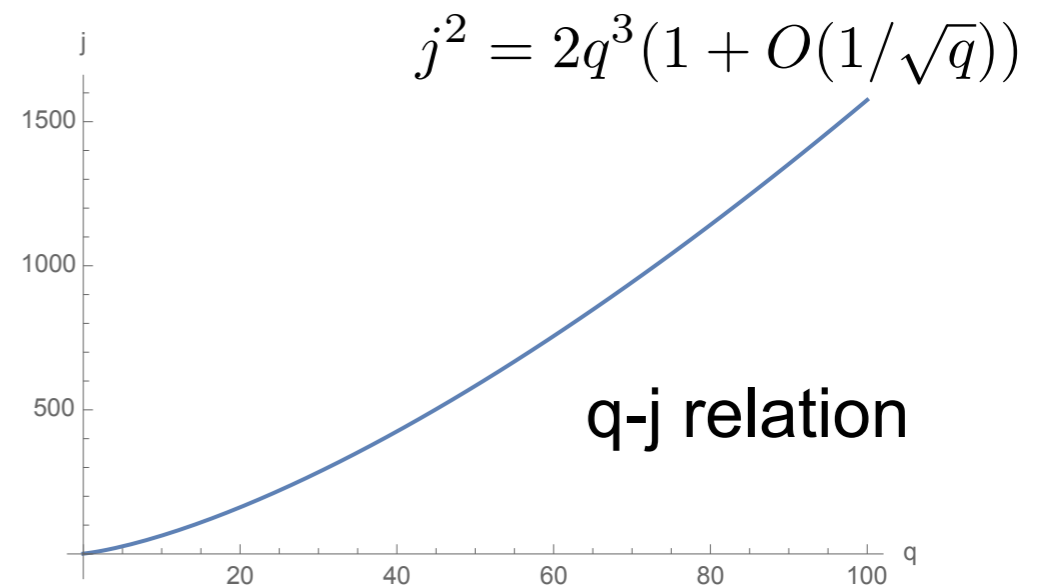


IIB on  $\text{AdS}_5 \times M^5$   
Minimal  $\text{AdS}_5$  sugra  
 $g_{\mu\nu}, A_\mu \quad G_N = 1/N^2$

$\frac{1}{4}$ -BPS BH charges

$$(J_1, J_2, Q) = N^2(j_1, j_2, q) \quad \text{obeying}$$

Simplest:  $j_1 = j_2 = j \neq 0$



$$S_{\text{BH}}(N, j) = N^2 s(j)$$

cf  $d_N(n)$

# What is the meaning of BH/exponential growth of states in the index?

$$\log d_N(n) \xrightarrow{?} N^2 s(n/N^2)$$

Large- $N$   
limit

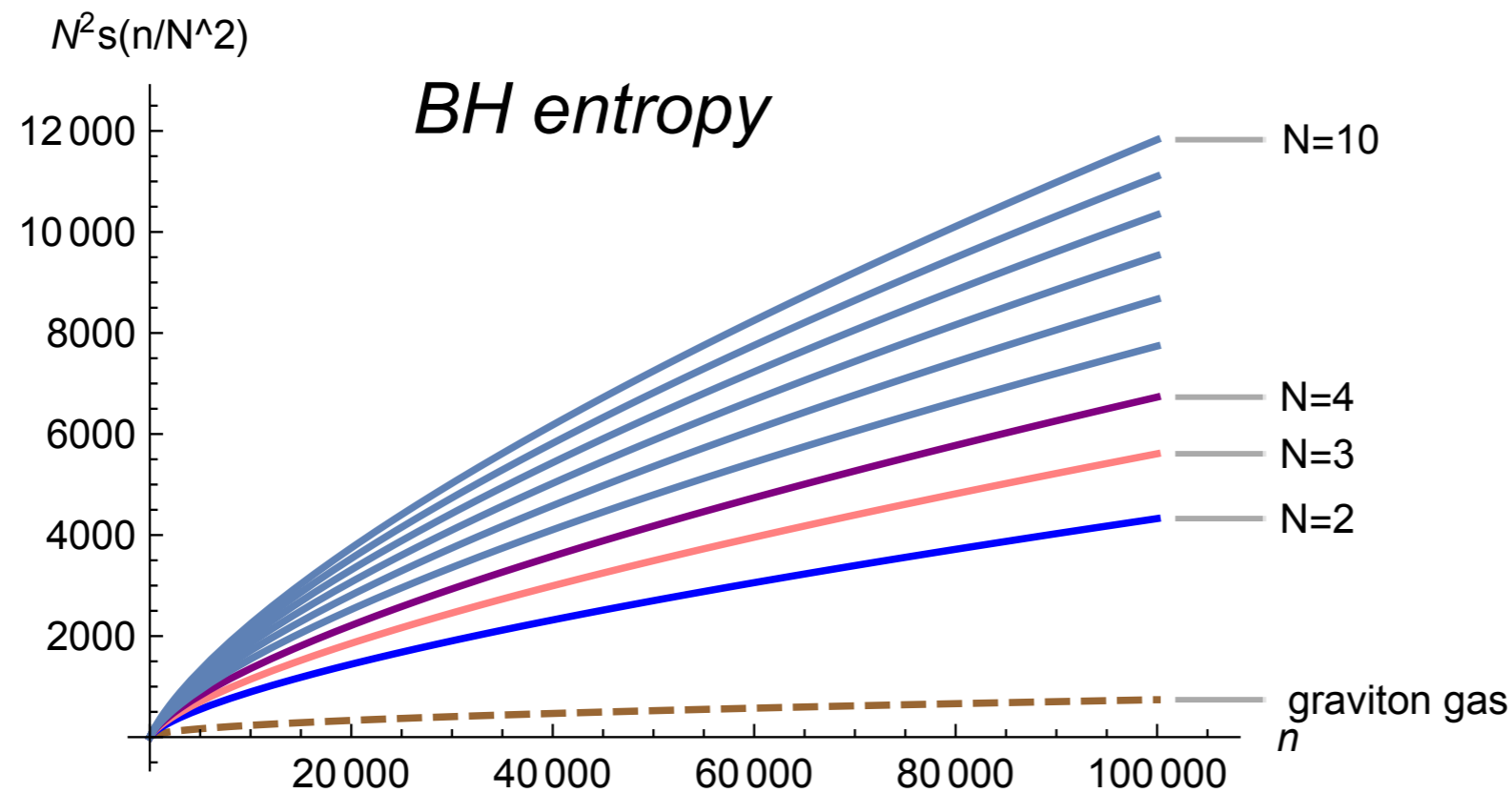
$$n, N \rightarrow \infty$$

$$\text{fixed } j = n/N^2$$

Cardy-  
like limit

$$n/N^2 \rightarrow \infty$$

$$\text{fixed } N$$



$$s(j) = \pi \sqrt{3q^2 - 4j}$$

$$\xrightarrow{j \rightarrow \infty} \sqrt{3} \pi \left( \frac{j^2}{2} \right)^{1/3}$$

Any statement about  $d_n$  can be translated to  $I_N(t)$  and vice versa.

# The N=4 superconformal index is an integral over unitary matrices

[Romelsberger '05;  
Kinney, Maldacena, Minwalla, Raju '05]

$$\mathcal{I}_N(\tau) = \int [DU] \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} i_s(e^{2\pi i n \tau}) \operatorname{tr} U^n \operatorname{tr} (U^\dagger)^n \right)$$

Unitary  
matrix

“Single-letter  
index”

$$i_s(x) = \frac{3x^2 - 3x^4 - 2x^3 + 2x^6}{(1 - x^3)^2}$$

- Purely double trace, i.e. purely two-particle interaction among eigenvalues.

# Recent progress has established BH growth of states in SYM index

- Gravitational thermodynamic analysis  
[Cabo-Bizet, Cassani, Martelli, S.M. 1810]
- Cardy-like limit = infinite charge limit of matrix model  
(BHs much larger than AdS scale) [Choi, Kim, Kim 1810;  
Kim, Kim, Song 1904;  
Cabo-Bizet, Cassani, Martelli, S.M. 1904]
- Bethe-ansatz-type approach at large N.  
(Different contour integral representation of  $\mathcal{I}$ )  
[Benini, Milan 1811, 1812]
- Direct large-N analysis of matrix model. [Cabo-Bizet, S.M. 1909;  
Cabo-Bizet, Cassani, Martelli, S.M. 2005.]  
(= CCMM)
- Numerical results [S.M. 2005.]  
[Agarwal, Choi, Kim, Kim, Nahmgoong 2005.]

# Related recent work on this subject

Cabo-Bizet, Cassani, Martelli, S.M. 1810

Choi, Kim, Kim, Nahmgoong 1810

Choi, Hwang, Kim, Nahmgoong 1811

Benini, Milan 1811, 1812

Suh 1812

Honda 1901

Ardehali 1902

Kim, Kim, Song 1902

Zaffaroni 1902

Cabo-Bizet, Cassani, Martelli, S.M. 1904

Amariti, Garozzo, Lo Monaco 1904

Cassani, Papini 1906

Larsen, Nian, Zeng 1907

Kantor, Papageorgakis, Richmond 1907

Nahmgoong 1907

Lezcano, Pando-Zayas 1907

Lanir, Nedelin, Sela 1908

Cabo-Bizet, S.M. 1909

Goldstein, Jejjala, Lei,  
van Leuven, Li 1910

Ardehali, Hong, Liu 1912

Gadde 2004

Nian, Pando-Zayas 2003

David, Nian, Pando-  
Zayas 2005

S.M. 2005

Agarwal, Choi, Kim, Kim,  
Nahmgoong 2005

# Questions addressed in today's talk

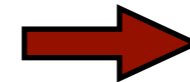
(How) do susy BHs contribute to (gravitational) path integral for index?



Regularized saddles,  
subtle BPS-limit

[CCMM. 1810.]

(How) can BHs be realized as large-N saddle-points of matrix integral? Other phases?



Help from number  
theory: Bloch-Wigner  
elliptic dilogarithm

[Cabo-Bizet, S.M. 1909.]

What about lore: “*gauge-invariant single-trace operators in SYM correspond to gravitons*”?



Representation  
theory of  $U(N)$ ,  
random partitions

[S.M. 2005.]

# Is $S_{\text{BH}} = \log d_N$ a consequence of AdS/CFT?

- BH has thermodynamic entropy.

Boltzmann relation  $\implies$  large degeneracy of states.

Should be reflected in dual SYM.

- Potential holes in argument:

- ❖ Presupposes Hilbert space of quantum gravity.

- ❖ Total number of states not preserved in moduli space.

- ❖  $d_N(n)$  is only a lower bound on the total number of states with charge  $n$ .



# Euclidean functional integral viewpoint

- Start with microscopic SYM index.

$\mathcal{I}(\tau)$  = functional integral on  $S^3 \times S^1$ .

[up to Casimir energy terms, cf CCMM 1810.]

- Index protected. [Not dimension of  $\frac{1}{16}$ -BPS operators cf Beisert '03]

$\Rightarrow \mathcal{I}(\tau)$  = functional integral in AdS gravity.

- Question: does susy BH contribute to the gravitational functional integral with susy boundary conditions?

Subtle: needs regularization of infinite throat.

# The entropy of these BHs can be recast as an extremization problem [\[Hosseini, Hristov, Zaffaroni 1705.\]](#)

$$S_{\text{BH}}(J_1, J_2, Q) = -\text{ext}_{\varphi, \omega_i} (I(\omega_1, \omega_2, \varphi) - \omega_1 J_1 - \omega_2 J_2 - \varphi Q)$$

$$\text{with } I(\omega_1, \omega_2, \varphi) = \frac{4N^2}{27} \frac{\varphi^3}{\omega_1 \omega_2}$$

$$\text{under the constraint } \omega_1 + \omega_2 - 2\varphi = -2\pi i$$

Note similarity to “Quantum statistical relation” [\[Gibbons, Hawking '77\]](#)

$I$  is like the on-shell action of the AdS5 BH (canonical ensemble).

# Does extremization principle follow from semiclassical BH thermodynamics in AdS5?

## BH thermodynamics

Temperature = surface gravity  $\beta$

Chemical potential for angular momentum = angular velocity  $\Omega_{1,2}$

Chemical potential for electric charge = gauge field value  $\Phi$

Susy BH values:  $\beta \rightarrow \infty$  ,  $\Omega_{1,2}^* = 1$  ,  $\Phi^* = \frac{3}{2}$

- Frozen: nothing left to vary!
- Wrong relation!  $\Omega_1^* + \Omega_2^* - 2\Phi^* = -1$

# Treat supersymmetric solutions as a limit of non-susy BHs

[CCMM 1810.]

- Minimal 5d sugra

$$\mathcal{L} = (R + 12g^2) *1 - \frac{2}{3} F \wedge *F + \frac{8}{27} F \wedge F \wedge A$$

- Four-parameter family of BH solutions:  $(a, b, m, q)$

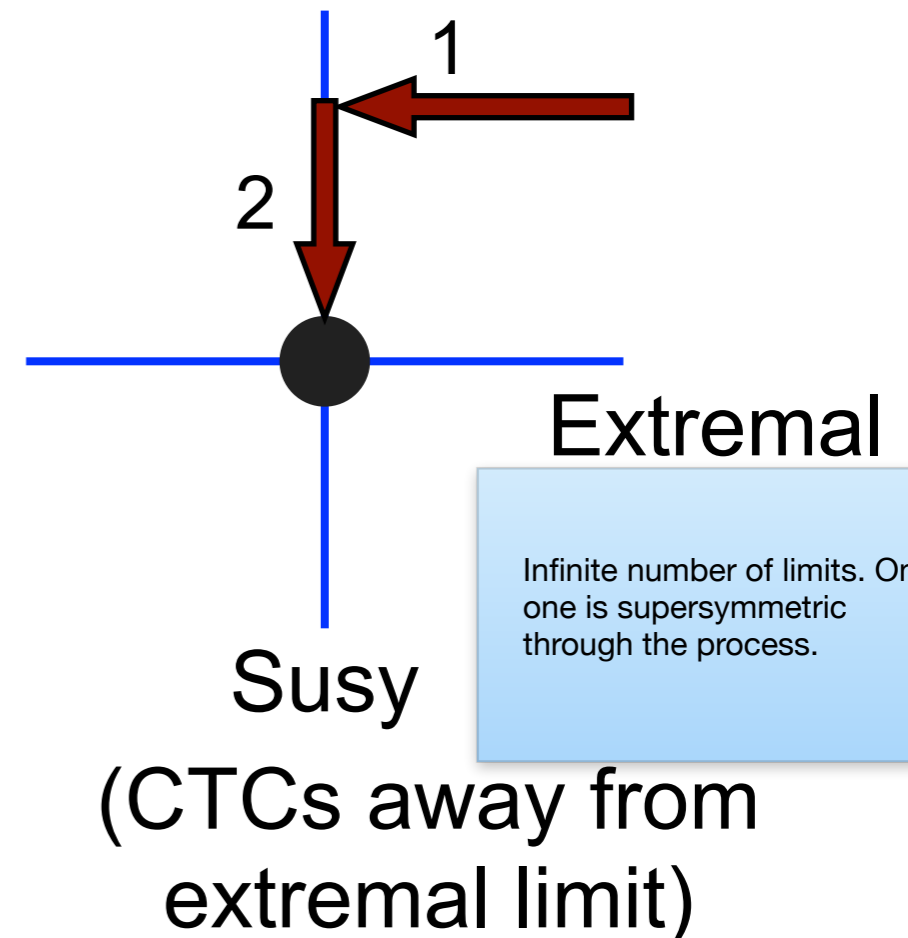
[Gutowski, Reall, '04] [Chong, Cvetič, Lu, Pope, '05; Kunduri, Lucietti, Reall, '06]

- Supersymmetry  $\neq$  extremality

$$q = \frac{m}{1 + a + b}$$

$$r_+ \rightarrow r_* \\ (\beta \rightarrow \infty)$$

[Idea first presented in Silva '06.  
General formalism in CCMM 1810.]  
Extremal non-susy analysis  
in Larsen, Nian, Zeng 1907.]



# We thus reach a new way to describe AdS5 susy BH thermodynamics

[CCMM 1810.]

Extremal limit of susy potentials:  $\Omega_{1,2}(a, b, m) \longrightarrow \Omega_{1,2}^* = 1$   
 $\Phi(a, b, m) \longrightarrow \Phi^* = 3/2$  as  $\beta \rightarrow \infty$

The sub-leading fluctuations of chemical potentials are the interesting variables

Subleading fluctuations:  
 $\omega_{1,2} := \beta(\Omega_{1,2} - \Omega_{1,2}^*)$   
 $\varphi := \beta(\Phi - \Phi^*)$

Constraint  $\Rightarrow \omega_1 + \omega_2 - 2\varphi = -2\pi i$  ✓

On-shell action  $\Rightarrow I = \frac{4N^2}{27} \frac{\varphi^3}{\omega_1 \omega_2}$  ✓

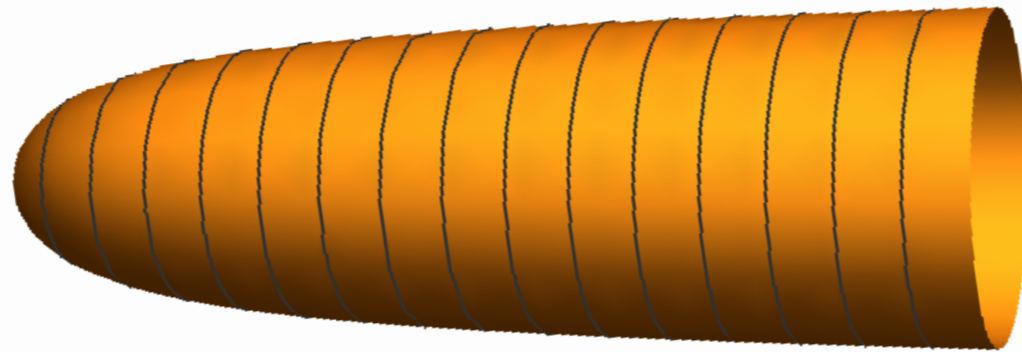
**BH entropy function!**

[Note: here we are discussing IR regulators. There is also the usual UV divergences which is cured by holographic renormalization/background subtraction  
Papadimitrou, Skenderis '05; Chen, Lu, Pope '05; Kunduri, Lucietti '05]

# Extremization principle follows from regularized Euclidean AdS5 path integral

[CCMM 1810.]

- Cigar-like geometry with on-shell action  $I = \frac{4N^2}{27} \frac{\varphi^3}{\omega_1 \omega_2}$



R-symmetry background  
Wilson line

- Global regularity of Killing spinor  $\Rightarrow \omega_1 + \omega_2 - 2\varphi = -2\pi i$

- Solve for  $\varphi$   $\Rightarrow$   
w/  $\omega_1 = \omega_2 = 2\pi i\tau$

$$I = \frac{\pi i N^2 (2\tau + 1)^3}{27 \tau^2}$$

# AdS/CFT prediction for matrix model: large-N free energy = on-shell action of BH

$$\mathcal{I}_N(\tau) \longleftrightarrow d_N(n)$$

Change of ensemble

$$\mathcal{I}_N(\tau) = \sum_{n \geq 0} d_N(n) e^{2\pi i \tau n/3}$$

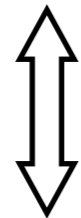
$$\log d_N(n) \xrightarrow{?} N^2 s(n/N^2)$$

$$n \rightarrow \infty$$

fixed  $N$

$$n, N \rightarrow \infty$$

fixed  $j = \frac{n}{N^2}$



$$\log \mathcal{I}_N(\tau) \xrightarrow{?} \frac{\pi i N^2 (2\tau + 1)^2}{27 \tau^2}$$

$$\tau \rightarrow 0$$

fixed  $N$

$$N \rightarrow \infty$$

fixed  $\tau$

Not  $SL_2(\mathbb{Z})$  modular form!

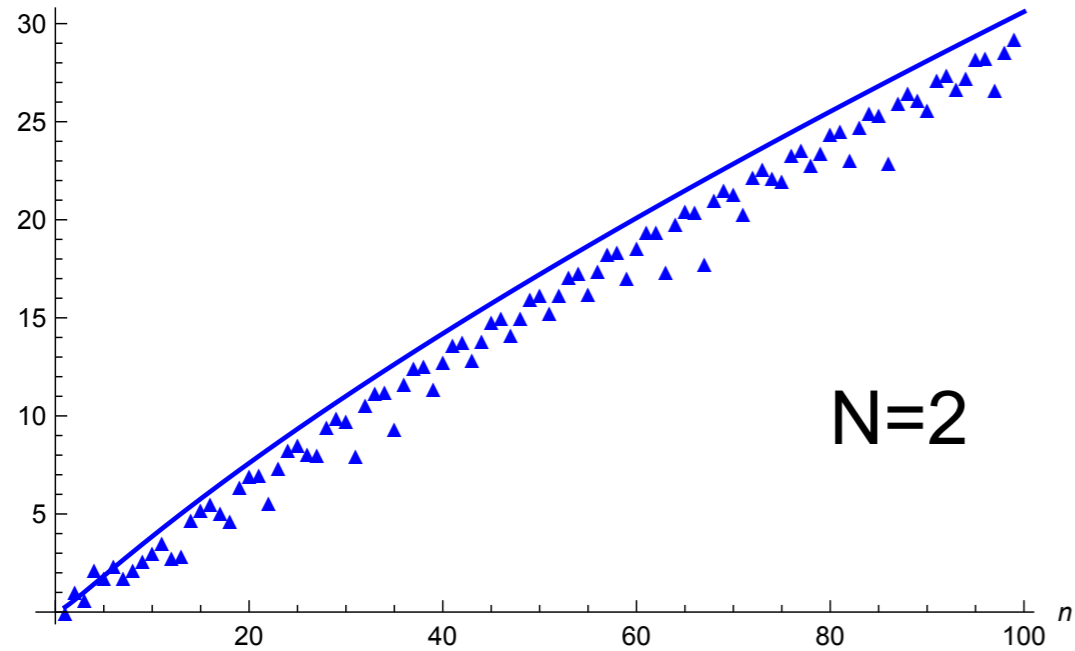
*Cardy-like limit*

*Large-N limit*

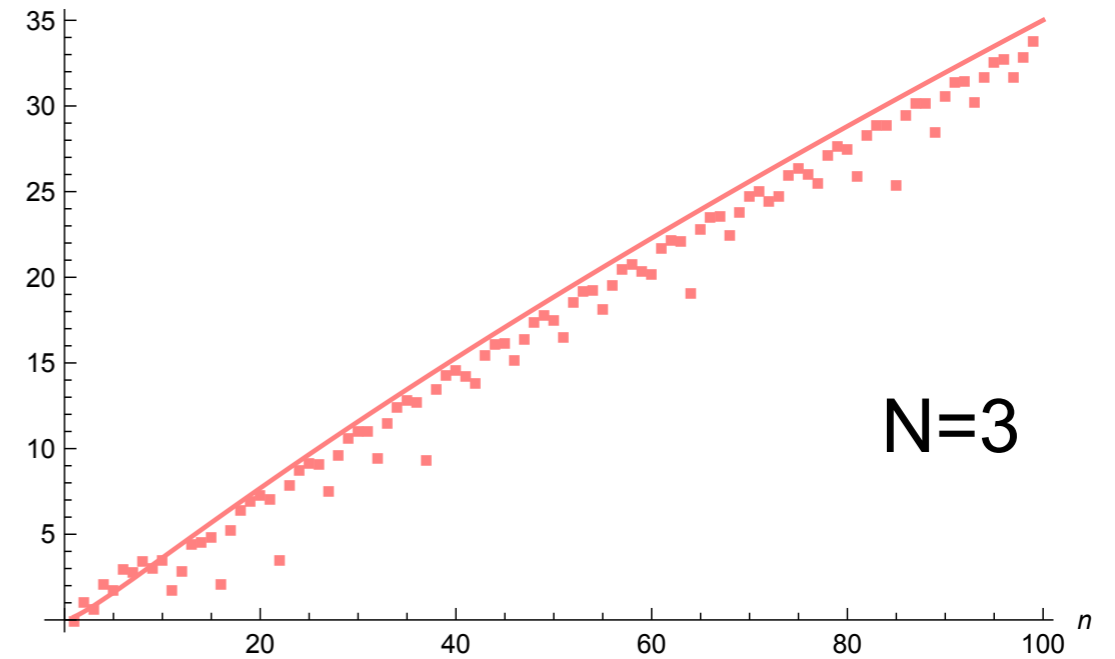
# Comparison for fixed N (Cardy-like limit)

[S.M. 2005]

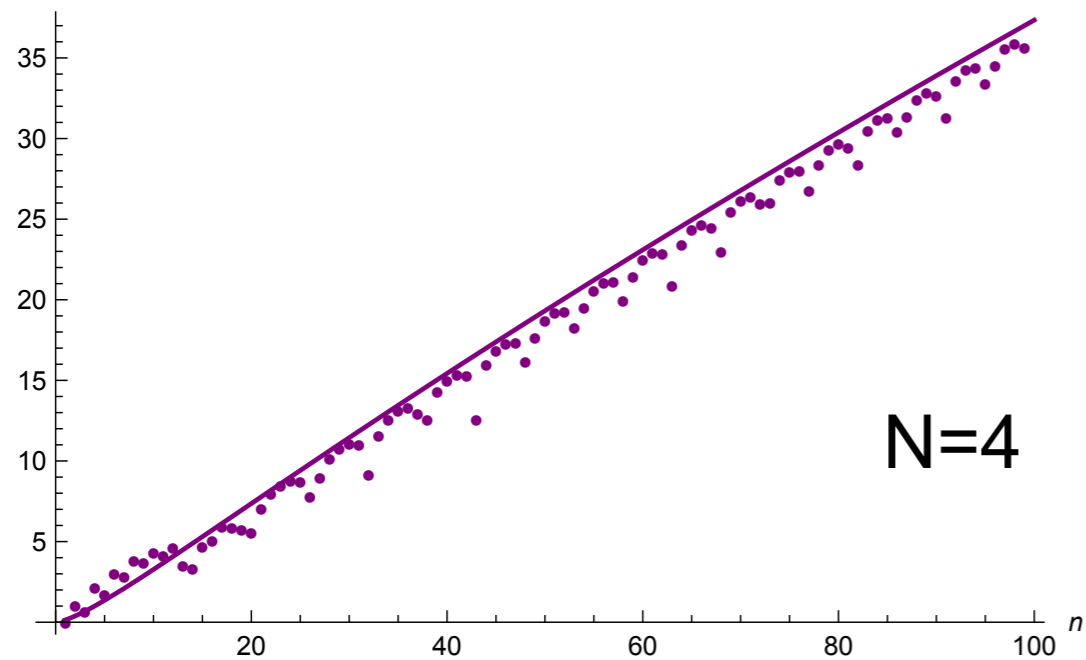
▲  $\log|d_2(n)|$  —  $S_{\text{BH}}(2,n)$



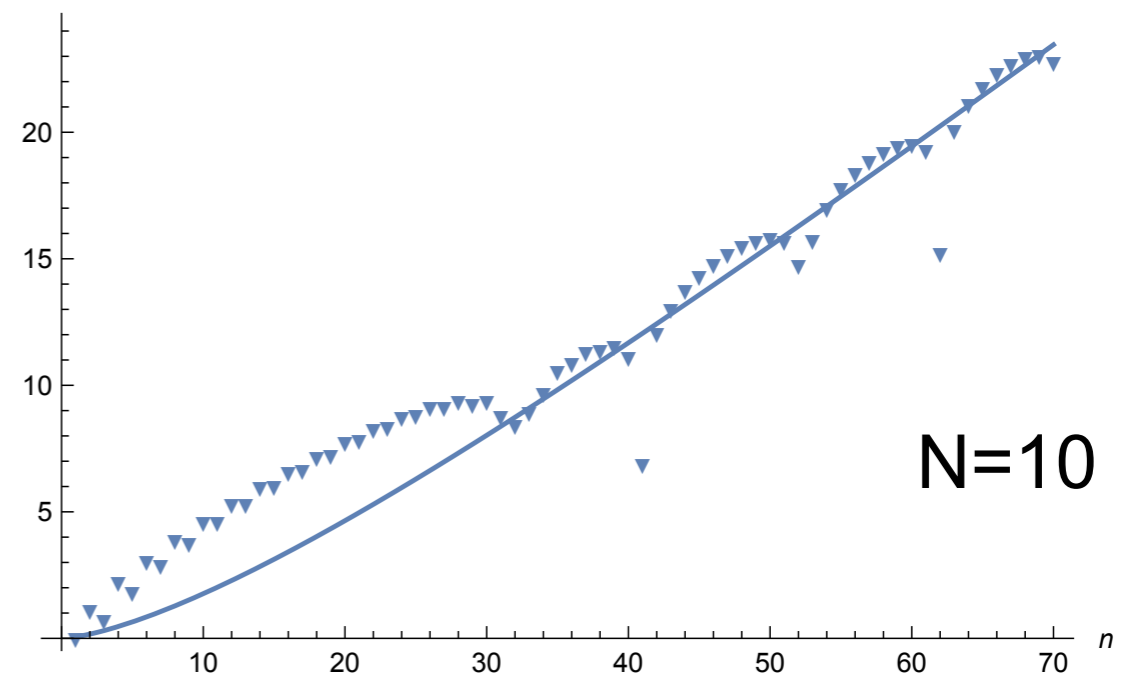
■  $\log|d_3(n)|$  —  $S_{\text{BH}}(3,n)$



●  $\log|d_4(n)|$  —  $S_{\text{BH}}(4,n)$



▼  $\log|d_{10}(n)|$  —  $S_{\text{BH}}(10,n)$



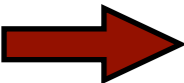


# How to analyze matrix model large-N limit?

$$\mathcal{I}_N(\tau) = \int [DU] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_s(e^{2\pi i n \tau}) \operatorname{tr} U^n \operatorname{tr} (U^\dagger)^n\right)$$

Eigenvalues of  $U$   $\{e^{2\pi i u_i}\}_{i=1,\dots,N}$

$$\underline{u} \equiv \{u_i\}_{i=1,\dots,N} \quad [D\underline{u}] = \frac{1}{N!} \prod_{i=1}^N du_i$$

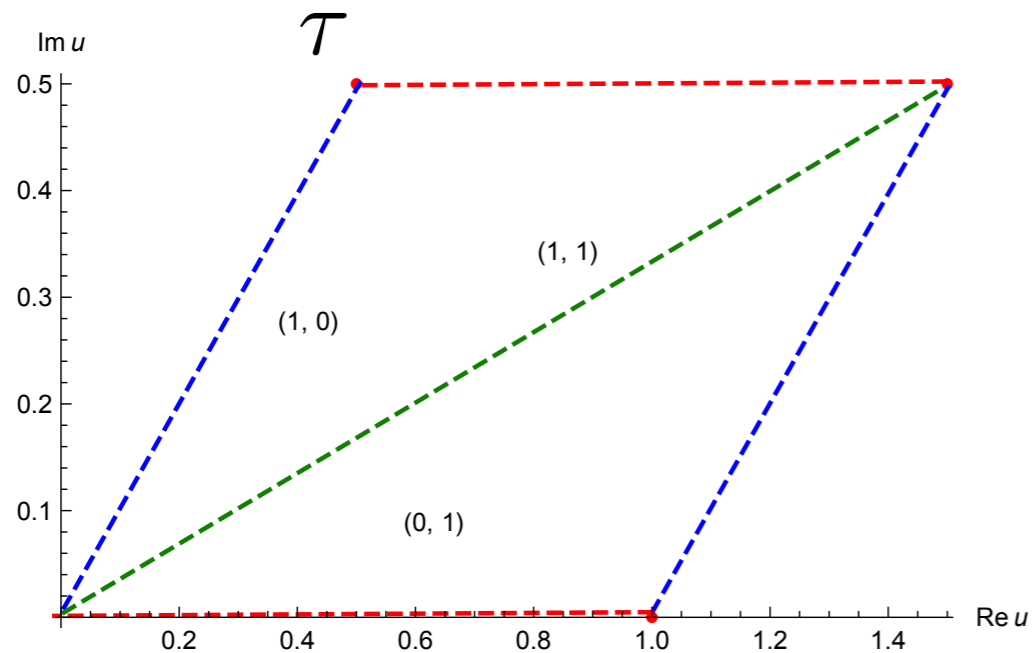

$$\mathcal{I}_N(\tau) = \int [D\underline{u}] \exp(-S(\underline{u})) \quad S(\underline{u}) = \sum_{i,j=1}^N V(u_{ij})$$



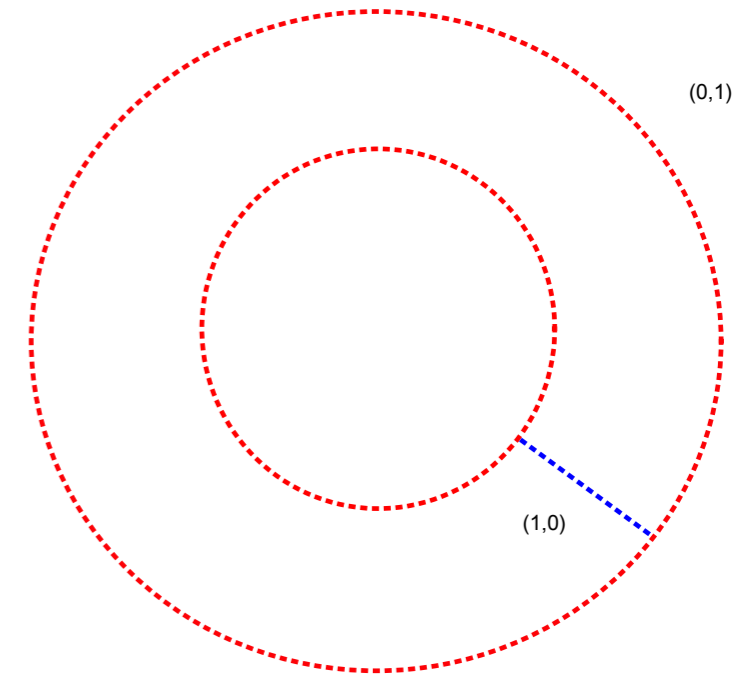
Pure two-particle interaction

# Matrix integral has complex saddles

[Cabo-Bizet, S.M. 1909.]



$$u \mapsto e^{2\pi i u}$$

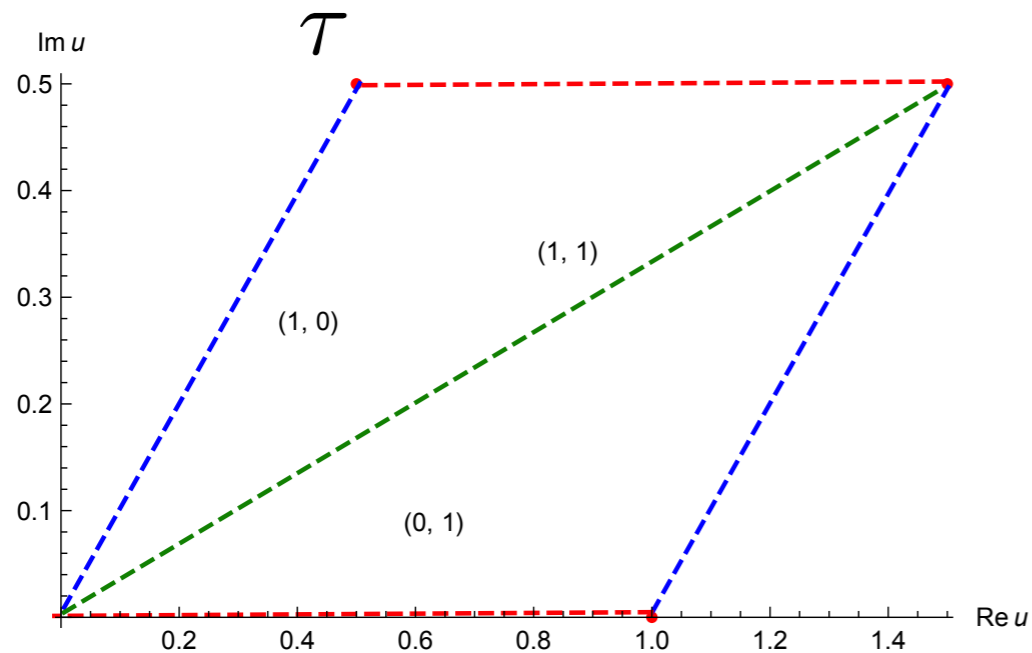


Family of saddles labelled by lattice points  $(m, n)$

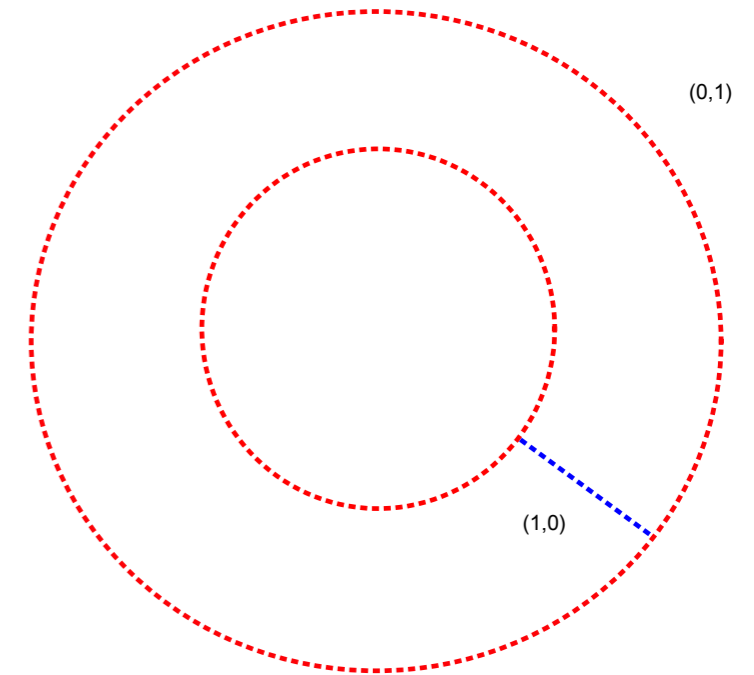
$$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \quad \gcd(m, n) = 1$$

# Matrix integral has complex saddles with macroscopic entropy

[Cabo-Bizet, S.M. 1909.]



$$u \mapsto e^{2\pi i u}$$



Family of saddles labelled by lattice points  $(m, n)$

$$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \quad \gcd(m, n) = 1$$

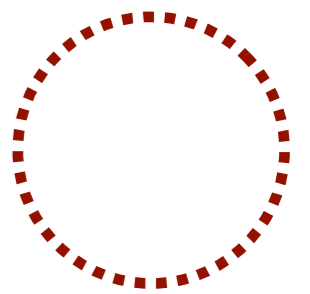
$$\mathcal{S}_{(0,1)} = \text{entropy of pure AdS}_5 = O(1)$$

$$\mathcal{S}_{(1,0)} = \text{entropy of AdS}_5 \text{ BH} = O(N^2)$$

# Lemma: Periodic eigenvalue distributions are saddle points of periodic potentials

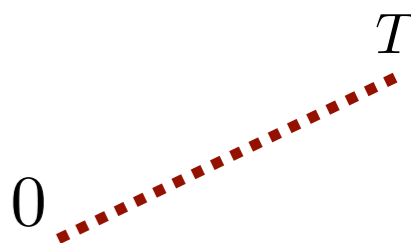
$$\partial_{u_i} S(\underline{u}) = \partial_{\bar{u}_i} S(\underline{u}) = 0, \quad S(\underline{u}) = \sum_{i,j=1}^N V(u_{ij})$$
$$V(-z) = V(z) \quad V(z+T) = V(z)$$

1. Uniform distribution along  $[0, T]$   $u_i = \frac{i}{N} T + u_0$  is a solution of the variational equations.



2. Action of this distribution = average value of  $V$

$$S_{\text{eff}}(T) = \sum_{i,j=1}^N V\left(\frac{i}{N} T\right) = N^2 \int_0^1 dx V(xT)$$



3. Need separate argument for contour deformation.

# The potential of our index is quasi-periodic under lattice translations

$$\mathcal{I}(\tau) = \eta(\tau)^{2N} \int [D\underline{u}] \prod_{i \neq j=1}^N \Gamma_e(u_{ij} + 2\tau; \tau, \tau) \times \\ \times \prod_{i,j=1}^N \Gamma_e(u_{ij} + \frac{1}{3}(2\tau + 1); \tau, \tau)^3$$

[Dolan-Osborn '08 ]

$$\Gamma_e(z; \sigma, \tau) = \prod_{j,k=0}^{\infty} \frac{1 - e^{-2\pi iz} e^{2\pi i\sigma(j+1)} e^{2\pi i\tau(k+1)}}{1 - e^{2\pi iz} e^{2\pi i\sigma j} e^{2\pi i\tau k}}$$

[Felder-Varchenko '99; Spiridonov '10; Spiridonov-Vartanov '12 ]

$$\Gamma_e(z; \tau, \tau) = \Gamma_e(z + 1; \tau, \tau) = \theta_0(z; \tau)^{-1} \Gamma_e(z + \tau; \tau, \tau)$$

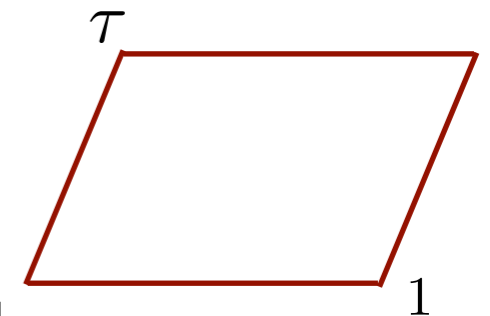
# From quasi-elliptic to elliptic functions

$$\theta_0(z; \tau) = (1 - \zeta) \prod_{n=1}^{\infty} (1 - q^n \zeta) (1 - q^n \zeta^{-1})$$

[Jacobi]

$$\theta_0(z) = \theta_0(z + 1) = -e^{2\pi iz} \theta_0(z + \tau)$$

Lattice  $\mathbb{Z}\tau + \mathbb{Z}$



Construct

$$P(z; \tau) := q^{\frac{1}{2}} B_2(z_2) \theta_0(z; \tau) \quad B_2(x) = x^2 - x + \frac{1}{6}$$

$$q = e^{2\pi i \tau}$$

$$\zeta = e^{2\pi i z}$$

$$z = z_1 + \tau z_2$$

1.  $q^{-\frac{1}{12}} P(z; \tau) = \theta_0(z; \tau)$  when  $z_2 = 0$
2.  $|P(z)|$  is elliptic
3.  $|P(z)|$  is modular!

$$-\log |P(z; \tau)| = \lim_{s \rightarrow 1} \frac{\tau_2^s}{2\pi} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{e^{2\pi i(nz_2 - mz_1)}}{|m\tau + n|^{2s}}$$

[Kronecker's second limit formula]

# The elliptic gamma function can be deformed to an elliptic function

Construct  $Q(z; \tau) = q^{\frac{1}{3}B_3(z_2) - \frac{1}{2}z_2B_2(z_2)} \frac{P(z; \tau)^{z_2}}{\Gamma_e(z + \tau; \tau, \tau)}$

1.  $Q(z; \tau) = \Gamma_e(z + \tau; \tau, \tau)^{-1}$  when  $z_2 = 0$

2.  $|Q(z)|$  is elliptic  $\longrightarrow$  Elliptic dilogarithm [S.Bloch]

Kronecker-Eisenstein series

Period functions

3.  $\log \left| Q\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) \right| - \tau \log |Q(z; \tau)| = F(z; \tau)$

Bloch formula

$$F(z; \tau) = \frac{\tau_2^2}{2\pi^2} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{e^{2\pi i(nz_2 - mz_1)}}{(m\tau + n)(m\bar{\tau} + n)^2}$$

# We can thus extend the SYM action to an elliptic function (the elliptic SYM action)

$$\begin{aligned} S(\underline{u}) = & -2N \log(q^{-1/24} \eta(\tau)) - \frac{1}{6} N \pi i \tau + \frac{8}{27} \pi i \tau N^2 \\ & + \sum_{i \neq j} \log Q(u_{ij} + \tau) + 3 \sum_{i, j} \log Q(u_{ij} - \frac{1}{3} \tau + \frac{1}{3}) \\ & - \sum_{i \neq j} \log P(u_{ij} + \tau) + \sum_{i, j} \log P(u_{ij} - \frac{1}{3} \tau + \frac{1}{3}) \end{aligned}$$

- Well-defined on the torus  $\implies$  saddles labelled by  $(m, n)$   $m\tau + n \in \mathbb{Z}\tau + \mathbb{Z}$   $\gcd(m, n) = 1$
- Action calculated as the integral along a given direction — use double Fourier expansion!



# The action of the (m,n) saddle is a simple rational function of $\tau$

$$S_{\text{eff}}(m, n; \tau) = \frac{N^2 \pi i}{27 m} \frac{(2(m\tau + n) + \chi_1(m + n))^3}{(m\tau + n)^2} \quad m \neq 0$$

$$S_{\text{eff}}(0, 1; \tau) = 0$$

$\ell$	0	1	2
$\chi_1$	0	1	-1

Dirichlet character mod 3.

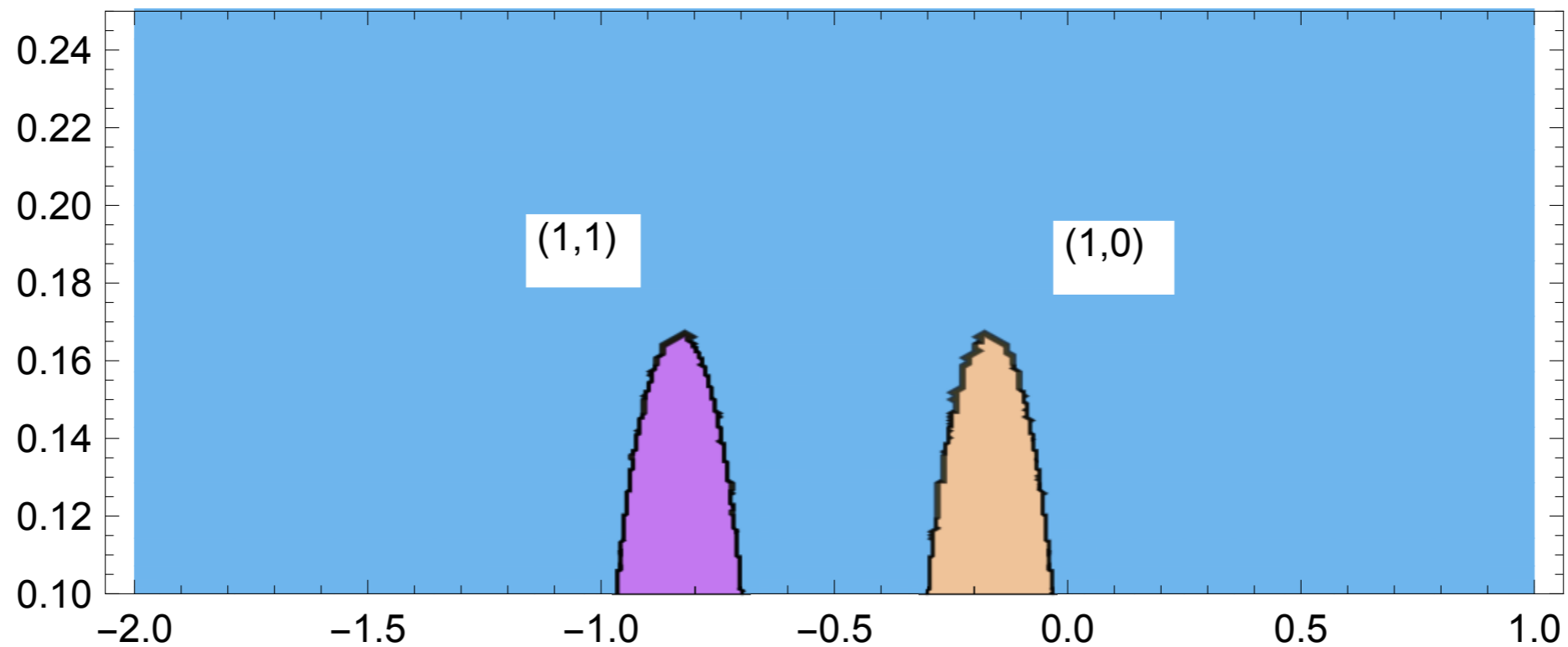
In particular,

$$S_{\text{eff}}(1, 0; \tau) = \frac{N^2 \pi i}{27} \frac{(2\tau + 1)^3}{\tau^2}$$

= the on-shell sugra action of AdS5 BH

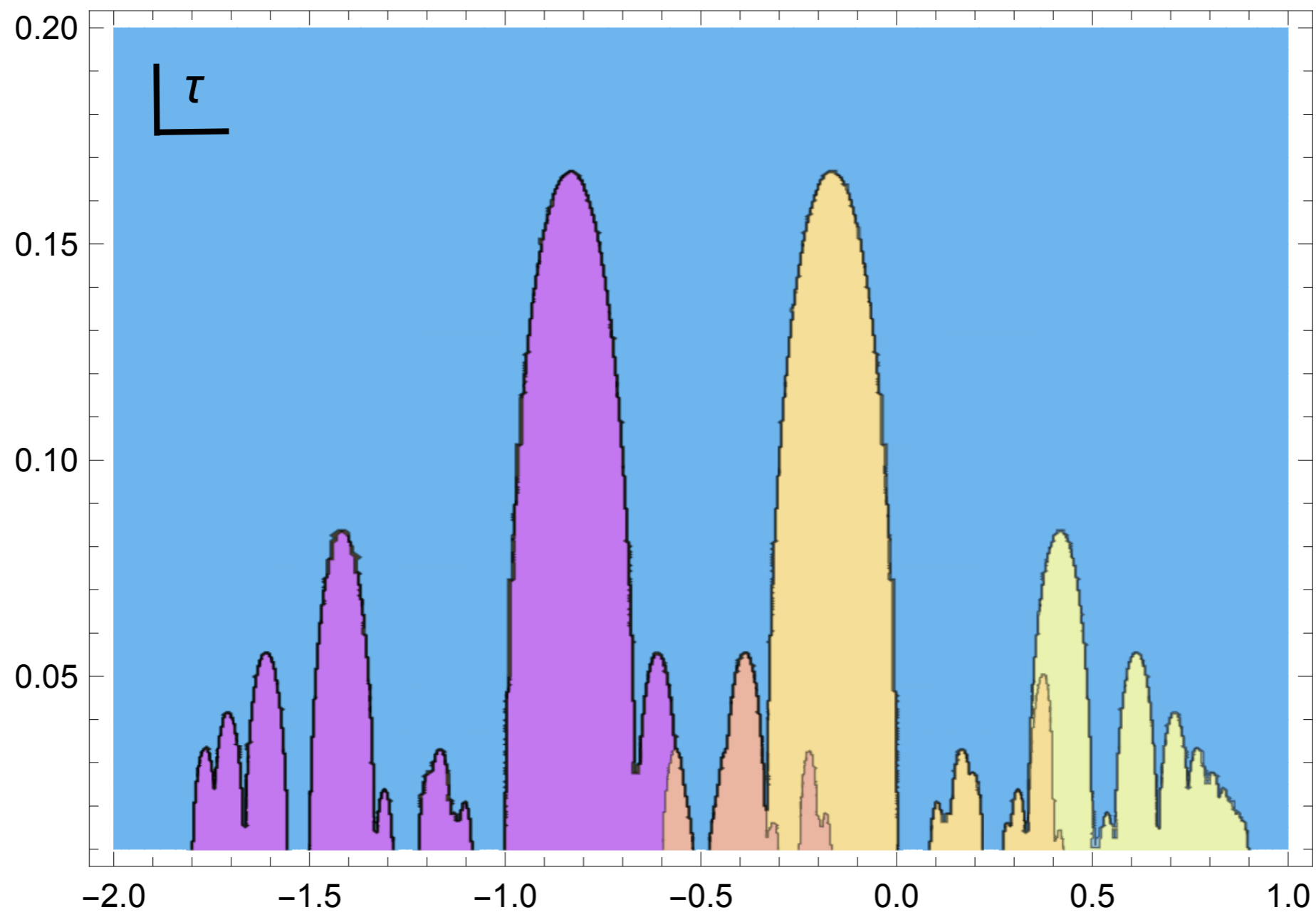
# Phase structure: black hole dominates in a region surrounding $\tau = 0$

[Cabo-Bizet, S.M. 1909.]



# Phase structure: new phases appear near rational points

[Cabo-Bizet, S.M. 1909.]



# Method of elliptic extension is very general

Generic N=1 SCFTs, including flavor chemical potentials

[Cabo-Bizet, Cassani, Martelli, S.M. 2005\]](#)

- ❖ Universal family of saddles with large entropy. Should have gravitational duals.
- ❖ Action controlled by anomalies

$$S_{\text{eff}}(m, n) = \frac{\pi i}{3mT^2} C_{IJK} [\Delta^I]_T^m [\Delta^J]_T^m [\Delta^K]_T^m$$

- ❖ Large family of saddles, classified by finite Abelian groups of order N.

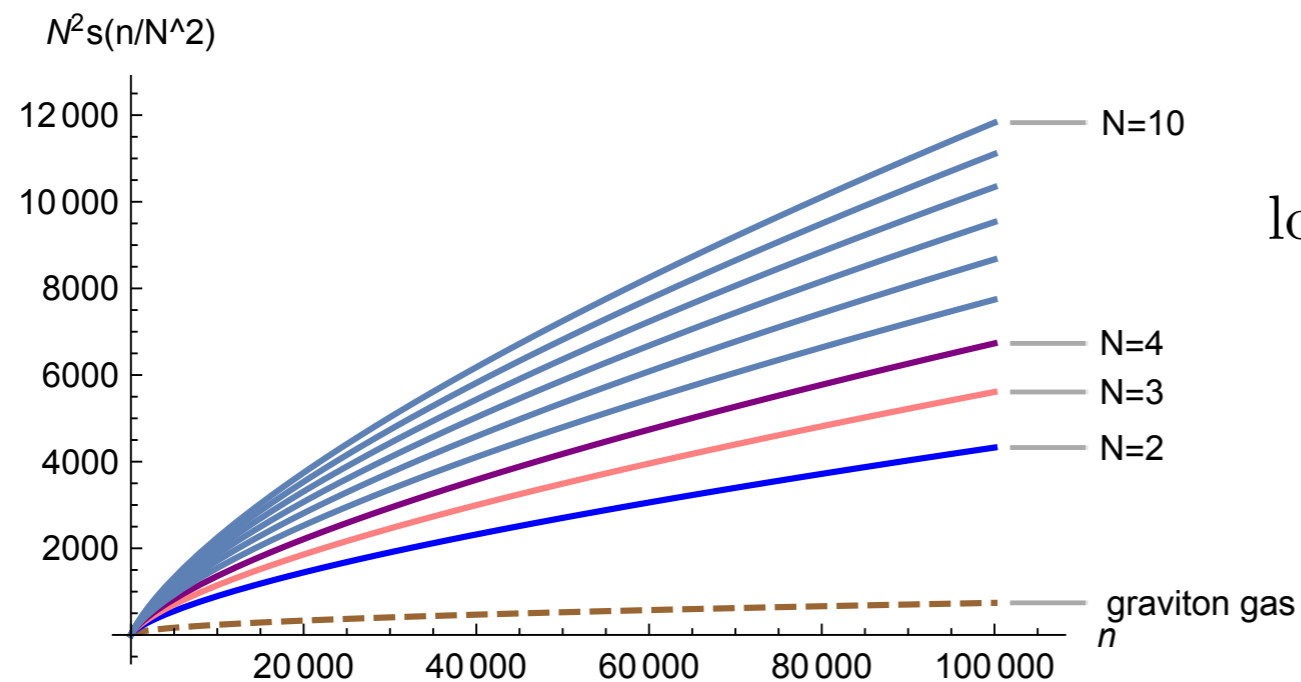
# Are SYM operators BHs or gravitons?

- At  $N = \infty$  single-trace operators = single gravitons.

At  $N = \infty$  multi-trace operators = multi-gravitons.

Made precise in [Chang, Yin '13]

[cf Janik, Trzetrzelewski, '07;  
Grant, Grassi, Kim, Minwalla '08]



$$\log d_N(n) \xrightarrow{n \rightarrow \infty} \frac{\sqrt{3}\pi}{2^{1/3}} N^{2/3} n^{2/3}$$

$$\log d_{\text{grav}}(n) \xrightarrow{n \rightarrow \infty} \frac{\pi}{3} \sqrt{5n}$$

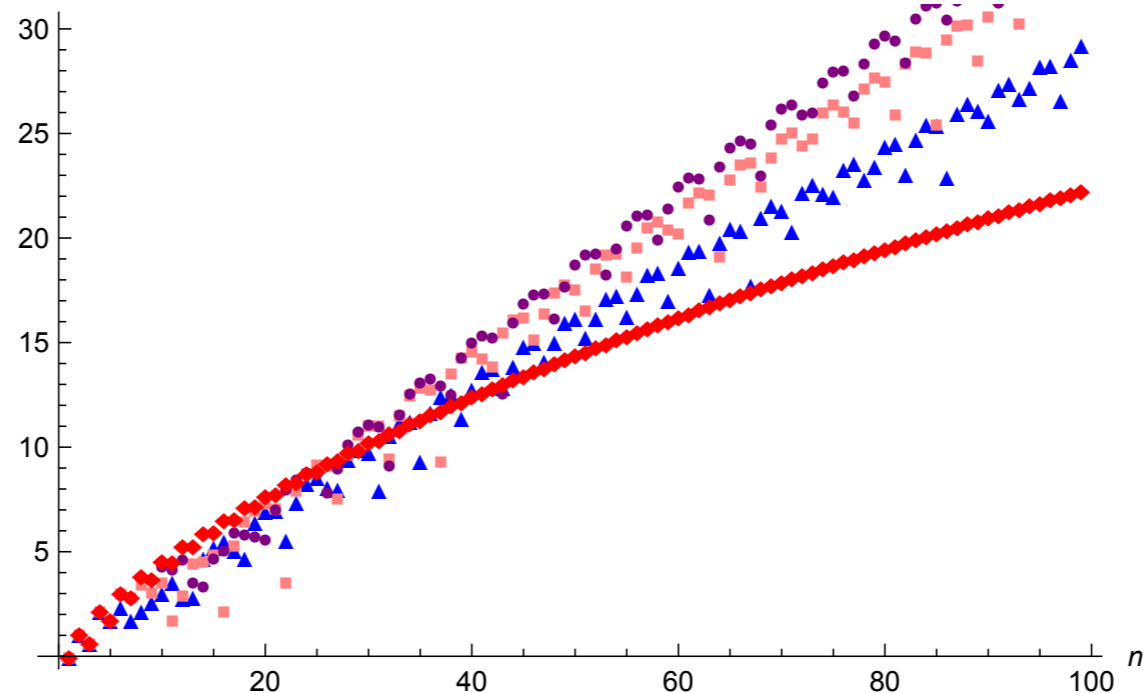
- On the other hand, we clearly see exponential growth of SYM operators controlled by  $N^2$ , as consistent with BH.

# Both! The index *interpolates* between multi-gravitons and BHs

[S.M. 2005]

$n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$d_2$	3	-2	9	-6	11	-6	9	14	-21	36	-17	-18	114	-194	258	-168
$d_3$	3	-2	9	-6	21	-18	33	-22	36	6	-19	90	-99	138	-9	-210
$d_4$	3	-2	9	-6	21	-18	48	-42	78	-66	107	-36	30	114	-165	390
$d_5$	3	-2	9	-6	21	-18	48	-42	99	-96	172	-156	252	-160	195	48
$d_6$	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	345	-340	540	-426
$d_{\text{grav}}$	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	381	-396	711	-750


▲  $\log|d_2(n)|$   
 ■  $\log|d_3(n)|$   
 ●  $\log|d_4(n)|$   
 ◆  $\log|d_{\text{grav}}(n)|$

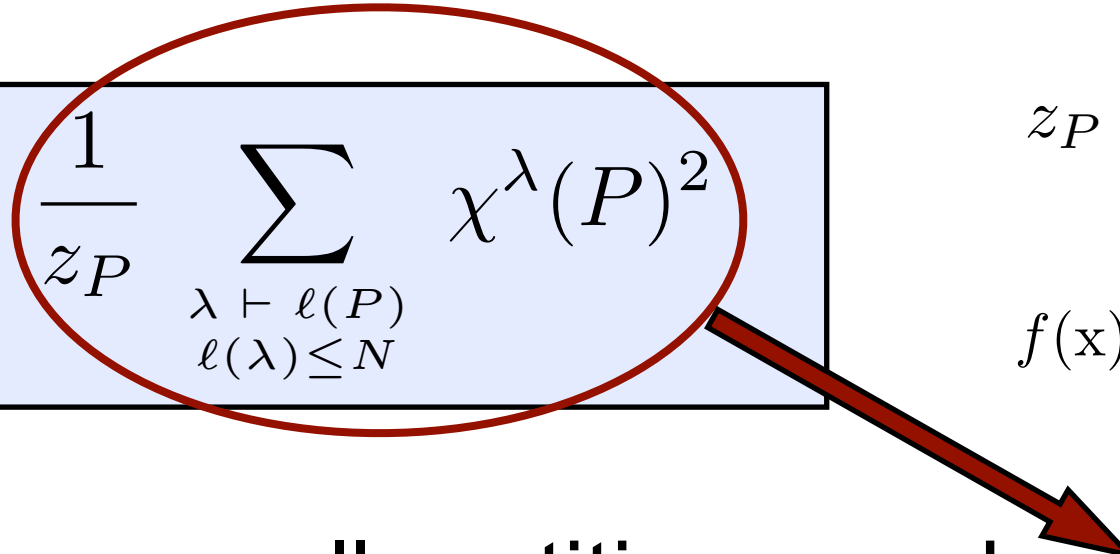


# This can be proved using ideas from representation theory

[S.M. 2005] [cf Dolan '07; Dutta, Gopakumar '07]

- Expand exponential in terms of  $U(N)$  characters.

- Frobenius formula:  $\prod_{j=1}^m (\text{Tr } U^j)^{k_j} = \sum_{\substack{\lambda \vdash \ell(P) \\ \ell(\lambda) \leq N}} \tilde{\chi}_\lambda(U) \chi^\lambda(P)$
- $P \equiv \prod_{j=1}^m (j)^{k_j}$  

$$\mathcal{I}_N(\mathbf{x}) = \sum_P i_s(\mathbf{x})_P \frac{1}{z_P} \sum_{\substack{\lambda \vdash \ell(P) \\ \ell(\lambda) \leq N}} \chi^\lambda(P)^2$$


$$z_P = \prod_{j=1}^m k_j! j^{k_j}$$

$$f(\mathbf{x})_P := \prod_{j=1}^m f(\mathbf{x}^j)^{k_j}$$

- At  $N = \infty$ , we sum over all partitions, and  $\mathcal{I}_N(\mathbf{x}) = 1$

$$\Rightarrow \mathcal{I}_{N=\infty}(\mathbf{x}) = \prod_{k=1}^{\infty} \frac{1}{1 - i_s(\mathbf{x}^k)} = \prod_{n=1}^{\infty} \frac{(1 - \mathbf{x}^{3n})^2}{(1 - \mathbf{x}^{2n})^3} = \sum_n d_{\text{grav}}(n) \mathbf{x}^n$$

# Outlook and Questions

Faith in AdS/CFT reconfirmed (in its simplest form)

- Complex saddles of matrix model important. BHs, many other phases (what are they in gravity?)
- Importance of lattice  $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$  and torus  $\mathbb{C}/\Lambda_\tau$   
Role of modular symmetry:  $\tau^{-1}$  vs  $\tau^{-2}$  period functions
- Elliptic deformation of SYM action — Q-exact term?
- Sum over random partitions (cf 2d gauge theory).
- Deeper role of number theory? Relation to L-functions/ Mahler measure/ Algebraic K-theory.

Thank you very much!



**More details**

# The R-symmetry Wilson line has a subtle effect on the Hamiltonian SCFT index...

[Cabo-Bizet, Cassani, Martelli, M. 1810, 1904.]

$$\mathcal{I}(\sigma, \tau; n_0) = \text{Tr}_{\mathcal{H}_{\text{phys}}} e^{\pi i(n_0+1)F} e^{-\beta\{\mathcal{Q}, \bar{\mathcal{Q}}\} + 2\pi i\sigma J_1 + 2\pi i\tau J_2 + \varphi Q}$$

Use constraint and spin-statistics  $e^{2\pi i n(J_1 + \frac{1}{2}F)} = 1 \rightarrow$

$$\mathcal{I}(\sigma, \tau; n_0) = \mathcal{I}(\sigma - n_0, \tau; 0)$$

$$\mathcal{I}(\sigma, \tau; 0) = \text{Tr}_{\mathcal{H}_{\text{phys}}} (-1)^F e^{-\beta\{\mathcal{Q}, \bar{\mathcal{Q}}\} + 2\pi i\sigma(J_1 + \frac{1}{2}Q) + 2\pi i\tau(J_2 + \frac{1}{2}Q)}$$

# ... with interesting consequences

$$\mathcal{I}(\sigma, \tau; n_0) = \mathcal{I}(\sigma - n_0, \tau; 0)$$

$n_0$  absorbed in shift of potential



1. Field theory index with any value of  $n_0$  is protected.
2.  $n_0$  only affects the phase of the degeneracies of states

$$\mathcal{I}(\sigma, \tau; n_0) = \sum_{n_1, n_2} d(n_1, n_2; n_0) e^{2\pi i(\sigma n_1 + \tau n_2)}$$

$$|d(n_1, n_2; n_0)| = |d(n_1, n_2; 0)|$$

# The method of elliptic extension is general and useful

- Generic N=1 SCFTs, including flavor chemical potentials  
[Cabo-Bizet, Cassani, Martelli, S.M. 2005\]](#)
- ❖ Universal saddles (m,n), should have gravitational dual.
- ❖ Action controlled by anomalies

$$S_{\text{eff}}(m, n) = \frac{\pi i}{3mT^2} C_{IJK} [\Delta^I]_T^m [\Delta^J]_T^m [\Delta^K]_T^m$$

- ❖ Large family of saddles, classified by finite Abelian groups.
- Applies to action at finite N.
    - ❖ Cardy-like limit  $\tau \rightarrow 0$
    - ❖ Possible to calculate 1/N corrections

1 constraint

Charges of  
BPS states/BH

