

# Color Decompositions from Unitarity

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based on [1908.02695](#) with Ben PAGE

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## Aim of this talk

“Proper” color decompositions of the form

$$\begin{aligned} \mathcal{A}(1, X, n) &= \sum_{\sigma \in \mathcal{B}_X^{1,n}} C(1, \sigma, n) A(1, \sigma, n) \\ &= \sum_{\sigma \in \mathcal{B}_X^{1,n}} C \left( \begin{array}{c} \sigma \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{1} \quad \quad \quad n \end{array} \right) A \left( \begin{array}{c} \sigma \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{1} \quad \quad \quad n \end{array} \right) \end{aligned}$$

“stretched” by 2 arb. chosen labels

$$1 \longleftarrow \quad \text{and} \quad \longrightarrow n$$

$\mathcal{B}_X^{1,n}$  — basis of  $\leq (n-2)!$  permutations

indep. under KK relations

Kleiss, Kuijf '88

$$A(1, \alpha, n, \beta) = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup \beta^T} A(1, \sigma, n)$$

$$\begin{aligned}
 \mathcal{A}_n &= \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma(2)} b_1} \tilde{f}^{b_1 a_{\sigma(3)} b_2} \dots \tilde{f}^{b_{n-4} a_{\sigma(n-2)} b_{n-3}} \tilde{f}^{b_{n-3} a_{\sigma(n-1)} a_n} \\
 &\quad \times A(1, \sigma(2), \dots, \sigma(n-1), n) \\
 &= \sum_{\sigma \in S_{n-2}} C \left( \begin{array}{c} \sigma(2) \ \sigma(3) \quad \dots \quad \sigma(n-1) \\ \text{diagram with } n \text{ external legs and } \sigma \text{ internal lines} \end{array} \right) A(1, \sigma(2), \dots, \sigma(n-1), n)
 \end{aligned}$$

I.e. for pure gluons

$$C \left( \begin{array}{c} \sigma \\ \text{diagram with } n \text{ external legs and } \sigma \text{ internal lines} \end{array} \right) = C \left( \begin{array}{c} \sigma(2) \ \sigma(3) \quad \dots \quad \sigma(n-1) \\ \text{diagram with } n \text{ external legs and } \sigma \text{ internal lines} \end{array} \right) \quad \text{non-trivial!}$$

C.f. standard  $SU(N)$  trace decomposition

$$\mathcal{A}_n = \sum_{\sigma \in S_{n-1}} \{ \text{Tr}[T^{a_1} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}] + (-1)^n \text{c.c.} \} A(1, \sigma(2), \dots, \sigma(n))$$

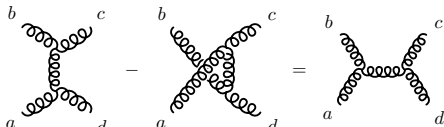
# Color Feynman rules

$C(\dots)$  replaces exposed vertices by

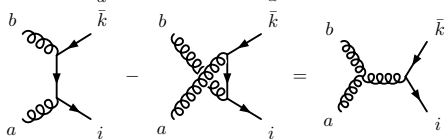
$$\tilde{f}^{abc} = \begin{array}{c} b \\ \diagup \quad \diagdown \\ a \quad c \end{array}, \quad T_{i\bar{j}}^a = \begin{array}{c} a \\ \diagup \quad \diagdown \\ i \quad \bar{j} \end{array}, \quad T_{\bar{j}i}^a = \begin{array}{c} a \\ \diagdown \quad \diagup \\ \bar{j} \quad i \end{array} = -T_{i\bar{j}}^a$$

Basic color algebra:

$$\tilde{f}^{dae} \tilde{f}^{ebc} - \tilde{f}^{dbe} \tilde{f}^{eac} = \tilde{f}^{abe} \tilde{f}^{dec}$$



$$T_{i\bar{j}}^a T_{j\bar{k}}^b - T_{i\bar{j}}^b T_{j\bar{k}}^a = \tilde{f}^{abe} T_{i\bar{k}}^e$$



SU(N) :  $T_{i\bar{j}}^a T_{k\bar{l}}^a = \delta_{i\bar{l}} \delta_{k\bar{j}} - \frac{1}{N_c} \delta_{i\bar{j}} \delta_{k\bar{l}}$  — not used here directly  
underlie color ordering for  $A(\dots)$

# Outline

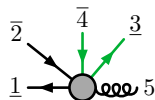
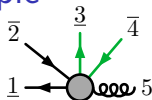
1. Preliminaries
2. Color decompositions with flavored matter
3. Loops at full color
4. Summary & outlook

# Color Decompositions with Flavored Matter\*

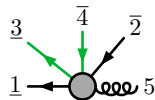
\*NB! Arbitrary matter but in distinct-flavor pairs, fermionic signs separate;  
e.g. 4 quarks at any loop:

$$\mathcal{A} \left( \begin{array}{c} \bar{4} \\ \downarrow \\ \text{3} \\ \swarrow \\ \text{2} \\ \nearrow \\ \bar{1} \end{array} \begin{array}{c} \text{5} \\ \vdots \\ n \end{array} \right) = \mathcal{A} \left( \begin{array}{c} \bar{4} \\ \downarrow \\ \text{3} \\ \swarrow \\ \text{2} \\ \nearrow \\ \bar{1} \end{array} \begin{array}{c} \text{5} \\ \vdots \\ n \end{array} \right) - \mathcal{A} \left( \begin{array}{c} \bar{4} \\ \downarrow \\ \text{3} \\ \swarrow \\ \text{2} \\ \nearrow \\ \bar{1} \end{array} \begin{array}{c} \text{5} \\ \vdots \\ n \end{array} \right)$$

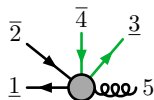
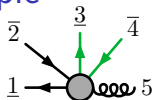
## 5-pt $qg$ -stretch example



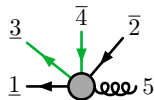
$$\mathcal{A}_{5,2} = C(\underline{1}, \bar{2}, \underline{3}, \bar{4}, 5)A(\underline{1}, \bar{2}, \underline{3}, \bar{4}, 5) + C(\underline{1}, \bar{2}, \bar{4}, \underline{3}, 5)A(\underline{1}, \bar{2}, \bar{4}, \underline{3}, 5) \\ + C(\underline{1}, \underline{3}, \bar{4}, \bar{2}, 5)A(\underline{1}, \underline{3}, \bar{4}, \bar{2}, 5)$$



## 5-pt $qg$ -stretch example



$$\mathcal{A}_{5,2} = C(\underline{1}, \bar{2}, \underline{3}, \bar{4}, 5) A(\underline{1}, \bar{2}, \underline{3}, \bar{4}, 5) + C(\underline{1}, \bar{2}, \bar{4}, \underline{3}, 5) A(\underline{1}, \bar{2}, \bar{4}, \underline{3}, 5) + C(\underline{1}, \underline{3}, \bar{4}, \bar{2}, 5) A(\underline{1}, \underline{3}, \bar{4}, \bar{2}, 5)$$



Factorization limits fix basis above:

$$\mathcal{A}\left(\begin{array}{c} \bar{2} \quad \underline{3} \quad \bar{4} \\ \downarrow \quad \uparrow \quad \uparrow \\ \text{---} \circ \text{---} \\ \uparrow \quad \leftarrow \quad \text{---} \quad \text{---} \quad \rightarrow \quad 5 \end{array}\right) \xrightarrow{s_{45} \rightarrow 0} \mathcal{A}\left(\begin{array}{c} \bar{2} \quad \underline{3} \\ \downarrow \quad \uparrow \\ \text{---} \circ \text{---} \\ \uparrow \quad \leftarrow \quad \text{---} \quad \text{---} \quad \rightarrow \quad \bar{p}_{45} \end{array}\right) \times \frac{i}{s_{45}} \times \mathcal{A}\left(\begin{array}{c} \bar{4} \\ \downarrow \\ \text{---} \circ \text{---} \\ \leftarrow \quad \text{---} \quad \rightarrow \quad 5 \end{array}\right)$$

$$\mathcal{A}\left(\begin{array}{c} \bar{2} \quad \bar{4} \quad \underline{3} \\ \downarrow \quad \downarrow \quad \uparrow \\ \text{---} \circ \text{---} \\ \uparrow \quad \leftarrow \quad \text{---} \quad \text{---} \quad \rightarrow \quad 5 \end{array}\right) \xrightarrow{s_{35} \rightarrow 0} \mathcal{A}\left(\begin{array}{c} \bar{2} \quad \bar{4} \\ \downarrow \quad \uparrow \\ \text{---} \circ \text{---} \\ \uparrow \quad \leftarrow \quad \text{---} \quad \text{---} \quad \rightarrow \quad p_{35} \end{array}\right) \times \frac{i}{s_{35}} \times \mathcal{A}\left(\begin{array}{c} \underline{3} \\ \uparrow \\ \text{---} \circ \text{---} \\ \leftarrow \quad \text{---} \quad \rightarrow \quad 5 \end{array}\right)$$

$$\mathcal{A}\left(\begin{array}{c} \underline{3} \quad \bar{4} \quad \bar{2} \\ \uparrow \quad \downarrow \quad \downarrow \\ \text{---} \circ \text{---} \\ \uparrow \quad \leftarrow \quad \text{---} \quad \text{---} \quad \rightarrow \quad 5 \end{array}\right) \xrightarrow{s_{25} \rightarrow 0} \mathcal{A}\left(\begin{array}{c} \underline{3} \quad \bar{4} \\ \uparrow \quad \uparrow \\ \text{---} \circ \text{---} \\ \uparrow \quad \leftarrow \quad \text{---} \quad \text{---} \quad \rightarrow \quad \bar{p}_{25} \end{array}\right) \times \frac{i}{s_{25}} \times \mathcal{A}\left(\begin{array}{c} \bar{2} \\ \downarrow \\ \text{---} \circ \text{---} \\ \leftarrow \quad \text{---} \quad \rightarrow \quad 5 \end{array}\right)$$



## 5-pt $qg$ -stretch example

Factorization limits fix color factors:

$$\begin{aligned}
 C \left( \begin{array}{c} \bar{2} \quad \bar{3} \quad \bar{4} \\ \swarrow \quad \uparrow \quad \nearrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow 5 \end{array} \right) &= C \left( \begin{array}{c} \bar{2} \quad \bar{3} \\ \swarrow \quad \nearrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow \bar{p}_{45} \end{array} \right) C \left( \begin{array}{c} \bar{4} \\ \downarrow \\ \bullet \\ \leftarrow p_{45} \quad \rightarrow 5 \end{array} \right) = C \left( \begin{array}{c} \bar{2} \quad \bar{3} \quad \bar{4} \\ \downarrow \quad \uparrow \quad \downarrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow 5 \end{array} \right) \\
 C \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad \bar{3} \\ \swarrow \quad \downarrow \quad \nearrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow 5 \end{array} \right) &= C \left( \begin{array}{c} \bar{2} \quad \bar{4} \\ \swarrow \quad \nearrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow p_{35} \end{array} \right) C \left( \begin{array}{c} \bar{3} \\ \uparrow \\ \bullet \\ \leftarrow \bar{p}_{35} \quad \rightarrow 5 \end{array} \right) = C \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad \bar{3} \\ \downarrow \quad \downarrow \quad \uparrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow 5 \end{array} \right) \\
 C \left( \begin{array}{c} \bar{3} \quad \bar{4} \quad \bar{2} \\ \swarrow \quad \downarrow \quad \searrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow 5 \end{array} \right) &= C \left( \begin{array}{c} \bar{3} \quad \bar{4} \\ \swarrow \quad \nearrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow \bar{p}_{25} \end{array} \right) C \left( \begin{array}{c} \bar{2} \\ \downarrow \\ \bullet \\ \leftarrow p_{25} \quad \rightarrow 5 \end{array} \right) = C \left( \begin{array}{c} \bar{3} \quad \bar{4} \quad \bar{2} \\ \leftarrow \quad \rightarrow \quad \downarrow \\ \bullet \\ \leftarrow 1 \quad \rightarrow 5 \end{array} \right)
 \end{aligned}$$

## 5-pt $q\bar{q}$ -stretch example

Factorization limits fix color factors:

$$C \left( \begin{array}{c} \bar{2} \quad \bar{3} \quad \bar{4} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) = C \left( \begin{array}{c} \bar{2} \quad \bar{3} \\ \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) C \left( \begin{array}{c} \bar{4} \\ \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \\ \underline{p_{45}} \quad \underline{5} \end{array} \right) = C \left( \begin{array}{c} \bar{2} \quad \bar{3} \quad \bar{4} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right)$$

$$C \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad \bar{3} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) = C \left( \begin{array}{c} \bar{2} \quad \bar{4} \\ \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) C \left( \begin{array}{c} \bar{3} \\ \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \\ \underline{\bar{p}_{35}} \quad \underline{5} \end{array} \right) = C \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad \bar{3} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right)$$

$$C \left( \begin{array}{c} \bar{3} \quad \bar{4} \quad \bar{2} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) = C \left( \begin{array}{c} \bar{3} \quad \bar{4} \\ \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \\ \underline{1} \quad \underline{\bar{p}_{25}} \end{array} \right) C \left( \begin{array}{c} \bar{2} \\ \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \\ \underline{p_{25}} \quad \underline{5} \end{array} \right) = C \left( \begin{array}{c} \bar{3} \quad \bar{4} \quad \bar{2} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right)$$

$\Rightarrow$  Decomposition:

$$\begin{aligned} \mathcal{A}_{5,2}^{\text{tree}} &= C \left( \begin{array}{c} \bar{2} \quad \bar{3} \quad \bar{4} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) A \left( \begin{array}{c} \bar{2} \quad \bar{3} \quad \bar{4} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) \\ &+ C \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad \bar{3} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) A \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad \bar{3} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) \\ &+ C \left( \begin{array}{c} \bar{3} \quad \bar{4} \quad \bar{2} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) A \left( \begin{array}{c} \bar{3} \quad \bar{4} \quad \bar{2} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} \bullet \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ \underline{1} \quad \underline{5} \end{array} \right) \end{aligned}$$

## Observations from 5-pt example

$$\begin{aligned}
 \mathcal{A}_{5,2} = & C \left( \begin{array}{c} \bar{2} \quad 3 \quad \bar{4} \\ \downarrow \quad \uparrow \quad \downarrow \\ \bar{1} \leftarrow \text{gluon} \leftarrow \text{gluon} \rightarrow 5 \end{array} \right) A \left( \begin{array}{c} \bar{2} \quad 3 \quad \bar{4} \\ \swarrow \quad \uparrow \quad \searrow \\ \bar{1} \leftarrow \text{gluon} \leftarrow \text{gluon} \rightarrow 5 \end{array} \right) \\
 & + C \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad 3 \\ \downarrow \quad \downarrow \quad \uparrow \\ \bar{1} \leftarrow \text{gluon} \leftarrow \text{gluon} \rightarrow 5 \end{array} \right) A \left( \begin{array}{c} \bar{2} \quad \bar{4} \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ \bar{1} \leftarrow \text{gluon} \leftarrow \text{gluon} \rightarrow 5 \end{array} \right) \\
 & + C \left( \begin{array}{c} 3 \quad \bar{4} \quad \bar{2} \\ \leftarrow \quad \leftarrow \quad \downarrow \\ \bar{1} \leftarrow \text{gluon} \leftarrow \text{gluon} \rightarrow 5 \end{array} \right) A \left( \begin{array}{c} 3 \quad \bar{4} \quad \bar{2} \\ \swarrow \quad \downarrow \quad \swarrow \\ \bar{1} \leftarrow \text{gluon} \leftarrow \text{gluon} \rightarrow 5 \end{array} \right)
 \end{aligned}$$

- ▶ Presence of fact. channels in  $\mathcal{A}_{n,k}$  constrain basis of  $A(\dots)$
- ▶ Factorization allows color recursion if lower-pt known
- ▶ Comb-like structures unless stretch by quarks of same flavor
- ▶ Luckily,  $q\bar{q}$  stretch most studied (apart from pure gluons)

## Like-flavor $q\bar{q}$ stretch

### Basis:

Melia '13

Consider  $A(\underline{1}, \bar{2}, \sigma) = A(\bar{2}, \sigma, \underline{1})$ : stretch by  $\bar{2} \leftarrow$  and  $\rightarrow \underline{1}$

$$\mathcal{B}_{n,k}^{2,1} = \left\{ \sigma \in [\text{bracket structures}]_{k-1} \times [\text{gluon insertions}]_{n-2k} \right\}$$

▶ e.g. pure-quark permutation  $A(\bar{2}, \underline{3}, \underline{5}, \bar{6}, \underline{7}, \bar{8}, \bar{4}, \underline{1})$

▶ full Melia basis for  $n = 6$ ,  $k = 3$ :

$$A(\bar{2}, \underline{3}, \bar{4}, \underline{5}, \bar{6}, \underline{1}), A(\bar{2}, \underline{5}, \bar{6}, \underline{3}, \bar{4}, \underline{1}), A(\bar{2}, \underline{3}, \underline{5}, \bar{6}, \bar{4}, \underline{1}), A(\bar{2}, \underline{5}, \underline{3}, \bar{4}, \bar{6}, \underline{1})$$

▶ quark-bracket orientations chosen but fixed

▶ gluon are allowed everywhere (except between  $\underline{1}$  and  $\bar{2}$ )

## Like-flavor $q\bar{q}$ stretch

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$$\mathcal{B}_{n,k}^{2,1} = \left\{ \sigma \in [\text{bracket structures}]_{k-1} \times [\text{gluon insertions}]_{n-2k} \right\}$$

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$$A(\bar{2}, \underline{3}, \bar{4}, \underline{5}, \bar{6}, \underline{1}), A(\bar{2}, \underline{5}, \bar{6}, \underline{3}, \bar{4}, \underline{1}), A(\bar{2}, \underline{3}, \underline{5}, \bar{6}, \bar{4}, \underline{1}), A(\bar{2}, \underline{5}, \underline{3}, \bar{4}, \bar{6}, \underline{1})$$

▶ quark-bracket orientations chosen but fixed

▶ gluon are allowed everywhere (except between  $\underline{1}$  and  $\bar{2}$ )

### Decomposition:

Johansson, AO '15  
proven by Melia '15

$$\mathcal{A}_{n,k} = \sum_{\sigma \in \mathcal{B}_{n,k}^{2,1}} C(\underline{1}, \bar{2}, \sigma) A(\underline{1}, \bar{2}, \sigma)$$



# Like-flavor $q\bar{q}$ stretch

## Basis:

Melia '13

Consider  $A(\underline{1}, \bar{2}, \sigma) = A(\bar{2}, \sigma, \underline{1})$ : stretch by  $\bar{2} \leftarrow$  and  $\rightarrow \underline{1}$

$$\mathcal{B}_{n,k}^{2,1} = \left\{ \sigma \in [\text{bracket structures}]_{k-1} \times [\text{gluon insertions}]_{n-2k} \right\}$$

## Decomposition:

Johansson, AO '15

$$\mathcal{A}_{n,k} = \sum_{\sigma \in \mathcal{B}_{n,k}^{2,1}} C \left( \begin{array}{c} \sigma \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right) A \left( \begin{array}{c} \sigma \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right)$$

5-pt example:

$$\begin{aligned} \mathcal{A}_{5,2} = & C \left( \begin{array}{c} 5 \quad 3 \quad \bar{4} \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right) A \left( \begin{array}{c} 3 \quad \bar{4} \\ 5 \quad \bullet \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right) \\ & + C \left( \begin{array}{c} 3 \quad \bar{4} \quad 5 \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right) A \left( \begin{array}{c} \bar{4} \quad 5 \\ 3 \quad \bullet \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right) \\ & + C \left( \begin{array}{c} 5 \quad 3 \quad \bar{4} \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} + \begin{array}{c} 5 \quad 3 \quad \bar{4} \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right) A \left( \begin{array}{c} 3 \quad \bar{4} \\ 5 \quad \bullet \\ \bar{2} \rightarrow \bullet \rightarrow \underline{1} \end{array} \right) \end{aligned}$$

## Towards arbitrary stretches

**So far:**  $gg$  stretch for pure YM [DDM]

$q\bar{q}$  stretch for QCD [Melia+JO]

**Want generic stretch:**

$$\mathcal{A}(1, X, n) = \sum_{\sigma \in \mathcal{B}_X^{1,n}} C \left( \text{Diagram} \right) A \left( \text{Diagram} \right)$$

The diagram in the equation is a vertex with a grey circle in the center. It has two horizontal lines extending to the left and right, labeled '1' and 'n' respectively. From the top of the vertex, two lines extend upwards and outwards, meeting at a point above the vertex. Above this point is a sigma symbol with three dots below it, representing a multi-line stretch.



## Towards arbitrary stretches

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**Want generic stretch:**

$$\mathcal{A}(1, X, n) = \sum_{\sigma \in \mathcal{B}_X^{1,n}} C \left( \begin{array}{c} \sigma \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} 1 \quad \text{---} n \end{array} \right) A \left( \begin{array}{c} \sigma \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} 1 \quad \text{---} n \end{array} \right)$$


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$$\begin{aligned} \text{Res}_{s_{1P}=0} \mathcal{A} \left( \begin{array}{c} P \cup R \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} 1 \quad \text{---} n \end{array} \right) &= \mathcal{A} \left( \begin{array}{c} P \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} 1 \quad \text{---} \bar{p} \end{array} \right) \times \mathcal{A} \left( \begin{array}{c} R \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} p \quad \text{---} n \end{array} \right) \\ &= \sum_{\pi \in \mathcal{B}_P^{1, \bar{p}}} \sum_{\rho \in \mathcal{B}_R^{p, n}} C \left( \begin{array}{c} \pi \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} 1 \quad \text{---} \bar{p} \end{array} \right) C \left( \begin{array}{c} \rho \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} p \quad \text{---} n \end{array} \right) A \left( \begin{array}{c} \pi \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} 1 \quad \text{---} \bar{p} \end{array} \right) A \left( \begin{array}{c} \rho \\ \vdots \\ \text{---} \bullet \text{---} \\ \text{---} p \quad \text{---} n \end{array} \right) \end{aligned}$$

## Towards arbitrary stretches

**So far:**  $gg$  stretch for pure YM [DDM]

$q\bar{q}$  stretch for QCD [Melia+JO]

**Want generic stretch:**

$$\mathcal{A}(1, X, n) = \sum_{\sigma \in \mathcal{B}_X^{1,n}} C \left( \text{diagram} \right) A \left( \text{diagram} \right)$$

The diagram in the sum is a vertex with two incoming lines from the left (labeled 1 and  $n$ ) and two outgoing lines to the right. The top two lines are labeled with  $\dots$  and  $\sigma$ .

---

$$\begin{aligned} \text{Res}_{s_{1P}=0} \mathcal{A} \left( \text{diagram} \right) &= \mathcal{A} \left( \text{diagram} \right) \times \mathcal{A} \left( \text{diagram} \right) \\ &= \sum_{\pi \in \mathcal{B}_P^{1,\bar{p}}} \sum_{\rho \in \mathcal{B}_R^{p,n}} C \left( \text{diagram} \right) C \left( \text{diagram} \right) A \left( \text{diagram} \right) A \left( \text{diagram} \right) \end{aligned}$$

The first diagram is a vertex with two incoming lines (1 and  $n$ ) and two outgoing lines (labeled  $P \cup R$  and  $\dots$ ). The second diagram is a vertex with two incoming lines (1 and  $\bar{p}$ ) and two outgoing lines (labeled  $P$  and  $\dots$ ). The third diagram is a vertex with two incoming lines ( $p$  and  $n$ ) and two outgoing lines (labeled  $R$  and  $\dots$ ). The fourth diagram is a vertex with two incoming lines (1 and  $\bar{p}$ ) and two outgoing lines (labeled  $\pi$  and  $\dots$ ). The fifth diagram is a vertex with two incoming lines ( $p$  and  $n$ ) and two outgoing lines (labeled  $\rho$  and  $\dots$ ).

---

$$\begin{aligned} \text{Res}_{s_{1P}=0} \mathcal{A} \left( \text{diagram} \right) &= \text{Res}_{s_{1P}=0} \sum_{\sigma \in \mathcal{B}_{P \cup R}^{1,n}} C \left( \text{diagram} \right) A \left( \text{diagram} \right) \\ &= \sum_{(\sigma_1, \sigma_2) \in \mathcal{U}_{P,R}[\mathcal{B}_{P \cup R}^{1,n}]} C \left( \text{diagram} \right) A \left( \text{diagram} \right) A \left( \text{diagram} \right) \end{aligned}$$

The first diagram is a vertex with two incoming lines (1 and  $n$ ) and two outgoing lines (labeled  $P \cup R$  and  $\dots$ ). The second diagram is a vertex with two incoming lines (1 and  $\bar{p}$ ) and two outgoing lines (labeled  $\sigma_1$  and  $\dots$ ). The third diagram is a vertex with two incoming lines ( $p$  and  $n$ ) and two outgoing lines (labeled  $\sigma_2$  and  $\dots$ ).

## Towards arbitrary stretches

$$\begin{aligned}
 & \sum_{\pi \in \mathcal{B}_P^{1, \bar{p}}} \sum_{\rho \in \mathcal{B}_R^{p, n}} C \left( \text{Diagram}(\pi, \bar{p}) \right) C \left( \text{Diagram}(\rho, n) \right) A \left( \text{Diagram}(\pi, \bar{p}) \right) A \left( \text{Diagram}(\rho, n) \right) \\
 &= \sum_{(\sigma_1, \sigma_2) \in \mathcal{U}_{P,R}[\mathcal{B}_{P \cup R}^{1,n}]} C \left( \text{Diagram}(\sigma_1, \sigma_2, n) \right) A \left( \text{Diagram}(\sigma_1, \bar{p}) \right) A \left( \text{Diagram}(\sigma_2, n) \right), \\
 & \mathcal{U}_{P,R}[\mathcal{B}_{P \cup R}^{1,n}] = \left\{ (\pi, \rho) \in \mathcal{S}_P \times \mathcal{S}_R \mid \pi \oplus \rho \in \mathcal{B}_{P \cup R}^{1,n}, \operatorname{Res}_{s_{1P}=0} A(1, \pi, \rho, n) \neq 0 \right\}
 \end{aligned}$$

## Towards arbitrary stretches

$$\sum_{\pi \in \mathcal{B}_P^{1, \bar{p}}} \sum_{\rho \in \mathcal{B}_R^{p, n}} C \left( \begin{array}{c} \pi \\ \vdots \\ \text{---} \bullet \text{---} \\ \bar{p} \end{array} \right) C \left( \begin{array}{c} \rho \\ \vdots \\ \text{---} \bullet \text{---} \\ p \quad n \end{array} \right) A \left( \begin{array}{c} \pi \\ \vdots \\ \text{---} \bullet \text{---} \\ \bar{p} \end{array} \right) A \left( \begin{array}{c} \rho \\ \vdots \\ \text{---} \bullet \text{---} \\ p \quad n \end{array} \right)$$

$$= \sum_{(\sigma_1, \sigma_2) \in \mathcal{U}_{P,R}[\mathcal{B}_{P \cup R}^{1,n}]} C \left( \begin{array}{c} \sigma_1 \quad \sigma_2 \\ \vdots \quad \vdots \\ \text{---} \bullet \text{---} \\ 1 \quad n \end{array} \right) A \left( \begin{array}{c} \sigma_1 \\ \vdots \\ \text{---} \bullet \text{---} \\ \bar{p} \end{array} \right) A \left( \begin{array}{c} \sigma_2 \\ \vdots \\ \text{---} \bullet \text{---} \\ p \quad n \end{array} \right),$$

$$\mathcal{U}_{P,R}[\mathcal{B}_{P \cup R}^{1,n}] = \left\{ (\pi, \rho) \in \mathcal{S}_P \times \mathcal{S}_R \mid \pi \oplus \rho \in \mathcal{B}_{P \cup R}^{1,n}, \text{Res}_{s_{1P}=0} A(1, \pi, \rho, n) \neq 0 \right\}$$

Provided “co-unitarity”  $\mathcal{U}_{P,R}[\mathcal{B}_{P \cup R}^{1,n}] = \mathcal{B}_P^{1, \bar{p}} \times \mathcal{B}_R^{p, n}$

$$\Rightarrow C \left( \begin{array}{c} \pi \quad \rho \\ \vdots \quad \vdots \\ \text{---} \bullet \text{---} \\ 1 \quad n \end{array} \right) = C \left( \begin{array}{c} \pi \\ \vdots \\ \text{---} \bullet \text{---} \\ \bar{p} \end{array} \right) C \left( \begin{array}{c} \rho \\ \vdots \\ \text{---} \bullet \text{---} \\ p \quad n \end{array} \right)$$

Minimal example:

$$\mathcal{A}_{4,2} = C \left( \begin{array}{c} \bar{2} \quad \bar{3} \\ \downarrow \quad \uparrow \\ \text{---} \bullet \text{---} \\ \underline{1} \quad \underline{4} \end{array} \right) A \left( \begin{array}{c} \bar{2} \quad \bar{3} \\ \downarrow \quad \uparrow \\ \text{---} \bullet \text{---} \\ \underline{1} \quad \underline{4} \end{array} \right) = C \left( \begin{array}{c} \bar{3} \quad \bar{4} \\ \leftarrow \quad \rightarrow \\ \text{---} \bullet \text{---} \\ \bar{2} \quad \underline{1} \end{array} \right) A \left( \begin{array}{c} \bar{3} \quad \bar{4} \\ \downarrow \quad \uparrow \\ \text{---} \bullet \text{---} \\ \bar{2} \quad \underline{1} \end{array} \right)$$

OK for  $qQ$  failure for  $q\bar{q}$ : no fact. channel

## Arbitrary stretches

**Observation:** length of all bases' must be Melia's

$$\boxed{\frac{(n-2)!}{k!}}$$

**Skip to result:** allow flips for unclosed brackets

$$\mathcal{Q}_F = \bigcup_{f \in F} \bigcup_{E \in \mathbb{P}(F \setminus f)} \left\{ \begin{array}{c} \{ \\ (f) \oplus \pi \oplus (\bar{f}) \oplus \rho \mid (\pi, \rho) \in \mathcal{Q}_E \times \mathcal{Q}_{(F \setminus f) \setminus E} \end{array} \right\}$$

$$\begin{aligned} \overline{\mathcal{Q}}_F = \bigcup_{f \in F} \bigcup_{E \in \mathbb{P}(F \setminus f)} & \left\{ \begin{array}{c} \{ \\ (f) \oplus \pi \oplus (\bar{f}) \oplus \rho \mid (\pi, \rho) \in \mathcal{Q}_E \times \overline{\mathcal{Q}}_{(F \setminus f) \setminus E} \end{array} \right\} \\ & \cup \left\{ \begin{array}{c} [ \\ (\bar{f}) \oplus \pi \oplus (f) \oplus \rho \mid (\pi, \rho) \in \mathcal{Q}_E \times \overline{\mathcal{Q}}_{(F \setminus f) \setminus E} \end{array} \right\} \end{aligned}$$

$$\mathcal{G}_{n-2k} = \left\{ \sigma \in S_G \mid G = \{g_{2k+1}, \dots, g_n\} \right\}$$

## Arbitrary stretches

**Observation:** length of all bases' must be Melia's

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$$\begin{aligned} \bar{\mathcal{Q}}_F = \bigcup_{f \in F} \bigcup_{E \in \mathbb{P}(F \setminus f)} & \left\{ (f) \oplus \pi \oplus (\bar{f}) \oplus \rho \mid (\pi, \rho) \in \mathcal{Q}_E \times \bar{\mathcal{Q}}_{(F \setminus f) \setminus E} \right\} \\ & \cup \left\{ (\bar{f}) \oplus \pi \oplus (f) \oplus \rho \mid (\pi, \rho) \in \mathcal{Q}_E \times \bar{\mathcal{Q}}_{(F \setminus f) \setminus E} \right\} \end{aligned}$$

$$\mathcal{G}_{n-2k} = \{ \sigma \in S_G \mid G = \{g_{2k+1}, \dots, g_n\} \}$$

$$\left. \begin{aligned} q\bar{q} : \quad \mathcal{B}_{n,k}^{2,1} &= \{ A(\bar{2}, \sigma, \underline{1}) \mid \sigma \in \mathcal{Q}_{2(k-1)} \sqcup \mathcal{G}_{n-2k} \} \\ qQ : \quad \mathcal{B}_{n,k}^{1,4} &= \{ A(\underline{1}, \sigma, \bar{4}) \mid (\underline{1}) \oplus \sigma \oplus (\bar{4}) \in \bar{\mathcal{Q}}_{2k} \sqcup \mathcal{G}_{n-2k} \} \\ qg : \quad \mathcal{B}_{n,k}^{1,n} &= \{ A(\underline{1}, \sigma, n) \mid (\underline{1}) \oplus \sigma \in \bar{\mathcal{Q}}_{2k} \sqcup \mathcal{G}_{n-2k} \} \\ gg : \quad \mathcal{B}_{n,k}^{n-1,n} &= \{ A(n-1, \sigma, n) \mid \sigma \in \bar{\mathcal{Q}}_{2k} \sqcup \mathcal{G}_{n-2k-2} \} \end{aligned} \right\} \text{NEW}$$

Melia '13

Formal construction and proof of co-unitarity in [1908.02695](#)



## Similarly: $qg$ and $gg$ stretches

$$\mathcal{B}_{n,k}^{1,n} = \{A(\underline{1}, \sigma, n) \mid (\underline{1}) \oplus \sigma \in \overline{\mathcal{Q}}_{2k} \sqcup \mathcal{G}_{n-2k}\}$$

$$C\left(\begin{array}{c} \sigma \\ \vdots \\ \swarrow \quad \searrow \\ \text{---} \bullet \text{---} \\ \leftarrow \underline{1} \quad \text{---} n \end{array}\right) = C\left(\begin{array}{c} \pi \quad \bar{q} \quad \rho \\ \vdots \quad \downarrow \quad \swarrow \quad \searrow \\ \swarrow \quad \leftarrow \quad \text{---} \bullet \text{---} \quad \bullet \text{---} \\ \leftarrow \underline{1} \quad \text{---} n \end{array}\right), \quad \text{where } \sigma = (\pi, \bar{q}, \rho)$$

implicitly used @1-loop in [Kälin '17](#)

$$\mathcal{B}_{n,k}^{n-1,n} = \{A(n-1, \sigma, n) \mid \sigma \in \overline{\mathcal{Q}}_{2k} \sqcup \mathcal{G}_{n-2k-2}\}$$

$$C\left(\begin{array}{c} \sigma \\ \vdots \\ \swarrow \quad \searrow \\ \text{---} \bullet \text{---} \\ \text{---} n-1 \quad \text{---} n \end{array}\right) = C\left(\begin{array}{c} \pi \quad q \quad \rho \\ \vdots \quad \uparrow \quad \swarrow \quad \searrow \\ \swarrow \quad \leftarrow \quad \text{---} \bullet \text{---} \quad \bullet \text{---} \\ \text{---} n-1 \quad \text{---} n \end{array}\right), \quad \text{where } \sigma = (\pi, \{q, \rho\})$$



## Similarly: $qg$ and $gg$ stretches

$$\mathcal{B}_{n,k}^{1,n} = \{A(\underline{1}, \sigma, n) \mid (\underline{1}) \oplus \sigma \in \overline{\mathcal{Q}}_{2k} \sqcup \mathcal{G}_{n-2k}\}$$

$$C\left(\begin{array}{c} \sigma \\ \vdots \\ \swarrow \quad \searrow \\ \text{---} \circ \text{---} \\ \leftarrow \underline{1} \quad \text{---} n \end{array}\right) = C\left(\begin{array}{c} \pi \quad \bar{q} \quad \rho \\ \vdots \quad \downarrow \quad \vdots \\ \swarrow \quad \leftarrow \quad \searrow \\ \text{---} \circ \text{---} \quad \text{---} \circ \text{---} \\ \leftarrow \underline{1} \quad \text{---} n \end{array}\right), \quad \text{where } \sigma = (\pi, \bar{q}, \rho)$$

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$$C\left(\begin{array}{c} \sigma \\ \vdots \\ \swarrow \quad \searrow \\ \text{---} \circ \text{---} \\ \text{---} n-1 \quad \text{---} n \end{array}\right) = C\left(\begin{array}{c} \pi \quad q \quad \rho \\ \vdots \quad \uparrow \quad \vdots \\ \swarrow \quad \leftarrow \quad \searrow \\ \text{---} \circ \text{---} \quad \text{---} \circ \text{---} \\ \text{---} n-1 \quad \text{---} n \end{array}\right), \quad \text{where } \sigma = (\pi, \{q, \rho\})$$

NB! Induction via other valid fact. channel possible

### Tree-level summary:

- ▶ All ( $1 \leftarrow$  stretches  $\rightarrow n$ ) intertwined by mutual factorization
- ▶ All decompositions but  $q\bar{q}$  implied by factorization dividing 1 and  $n$   
— for free once amp. bases are chosen co-unitary
- ▶ Fortunately,  $q\bar{q}$  and pure glue previously known [[DDM, Melia + JO](#)]

# Loops at Full Color

# Full-color 2-loop amplitude in pure YM

Badger, Mogull, AO, O'Connell '15

$$\begin{aligned}
 \mathcal{A}_5^{(2)} = & \sum_{\sigma \in S_5} \sigma \circ I \left[ C \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right. \\
 & \left. + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + C \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + \frac{1}{2} C \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + C \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{4} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + C \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{4} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right. \\
 & \left. + \frac{1}{2} \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left( \begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} + \dots \Big]
 \end{aligned}$$

Integrated by Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '19

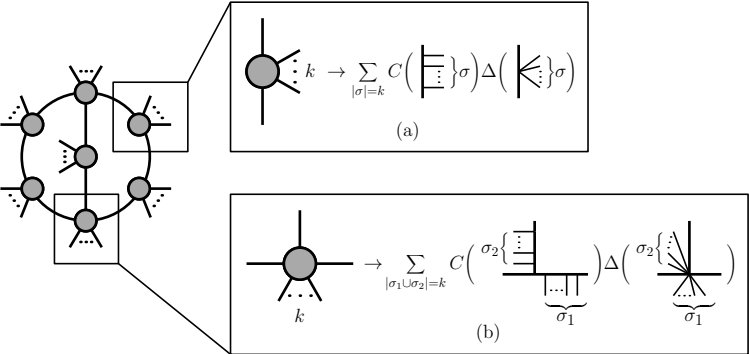


# General color construction

Full color within unitarity/integrand reduction:<sup>\*</sup>

$$\mathcal{A}_n^{(L)} = \sum_{i \in \text{KK-indep. 1PI graphs}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{C_i \Delta_i}{S_i \prod_{l \in i} D_l}$$

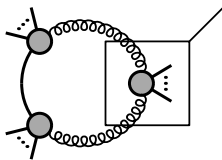
e.g. for pure YM at 2 loops:



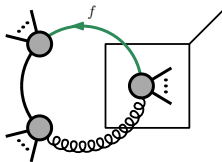
<sup>\*</sup> Bern, Dixon, Dunbar, Kosower '94; Britto, Cachazo, Feng '04; Ossola, Papadopoulos, Pittau '06; Mastroliola, Mirabella, Ossola, Peraro '12; Badger, Frellesvig, Zhang '12; Bourjaily, Herrmann, Trnka '17 [Enrico's talk]<sub>21 / 25</sub>

# 1 loop with matter

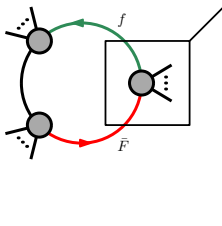
3 types of vertices:



$$\left. \begin{array}{c} \text{wavy line} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} X \rightarrow \sum_{\sigma \in \mathcal{B}_X^{g^*, g^{t^*}}} C \left( \begin{array}{c} g^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{wavy line} \end{array} \right) \sigma \Delta \left( \begin{array}{c} g^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{wavy line} \end{array} \right) \sigma$$



$$\left. \begin{array}{c} \text{wavy line} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{green arrow} \end{array} \right\} X \rightarrow \sum_{\sigma \in \mathcal{B}_X^{f^*, g^*}} C \left( \begin{array}{c} f^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{wavy line} \\ \text{green arrow} \end{array} \right) \sigma \Delta \left( \begin{array}{c} f^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{wavy line} \\ \text{green arrow} \end{array} \right) \sigma$$



$$\left. \begin{array}{c} \text{wavy line} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{green arrow} \\ \text{red arrow} \end{array} \right\} X \rightarrow -\theta[f \simeq \bar{F}] \sum_{\sigma \in \mathcal{B}_X^{f^*, \bar{f}^*}} C \left( \begin{array}{c} f^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{green arrow} \\ \bar{f}^* \end{array} \right) \sigma \Delta \left( \begin{array}{c} f^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{green arrow} \\ \bar{f}^* \end{array} \right) \sigma$$

$$+ \theta[f \simeq q \in X] \sum_{\sigma \in \mathcal{B}_X^{q^*, \bar{F}^*}} C \left( \begin{array}{c} q^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{green arrow} \\ \bar{F}^* \end{array} \right) \sigma \Delta \left( \begin{array}{c} q^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{green arrow} \\ \bar{F}^* \end{array} \right) \sigma$$

fermionic – may be replaced by + for scalars



## Summary & outlook

- ▶ Subject: flexible Kleiss-Kuijf-reduced color representations
- ▶ New bases and decompositions for  $qQ$ ,  $qg$  and  $gg$  stretches
- ▶ Previous results reused via factorization Del Duca, Dixon, Maltoni '99  
Melia '13  
Johansson, AO '15
- ▶ Applicable to graviton-matter amplitudes via color-kinematics Plefka, Wormsbecher '18



# Summary & outlook

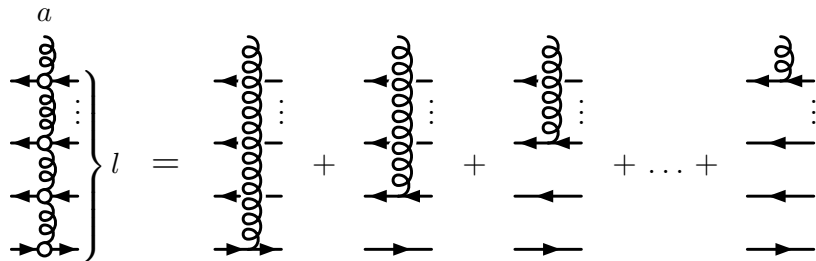
- ▶ Subject: flexible Kleiss-Kuijf-reduced color representations
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Melia '13  
Johansson, AO '15
  - ▶ Applicable to graviton-matter amplitudes via color-kinematics Plefka, Wormsbecher '18
- 
- ▶ Applicable to loops via gen. unitarity via method of Badger, Mogull, AO, O'Connell '15  
AO, Page '16  
also used in Bourjaily, Herrmann, Langer, McLeod, Trnka '19  
[Enrico's talk]
  - ▶ Implemented in numerical unitary framework Abreu, Febres Cordero, Ita, Page, Sotnikov '18  
Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov '19
  - ▶ Orthogonal/complementary to  $SU(N)$  trace methods Bern, Kosower '90  
Bern, Dixon, Kosower '94  
Edison, Naculich '11  
Ita, Ozeren '11  
Reuschle, Weinzierl '13  
Schuster '13
  - ▶ Hopefully helpful for future calculations beyond leading color!

Thank you, and stay safe!

# Backup slides

## Tensor in JO color factors

$$\Xi_l^a = \sum_{s=1}^l \underbrace{1 \otimes \dots \otimes 1 \otimes T^a \otimes 1 \otimes \dots \otimes 1 \otimes \bar{1}}_l^s$$



$$[\Xi_l^a, \Xi_l^b] = \tilde{f}^{abc} \Xi_l^c$$

# Loop-level KK relations

Badger, Mogull, AO, O'Connell '15

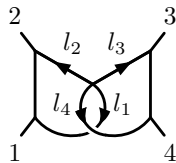
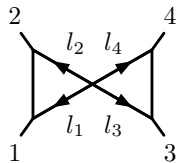
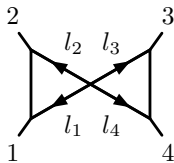
AO, Page '16

Question: do irred. numerators  $\Delta_i$  satisfy extra relations?

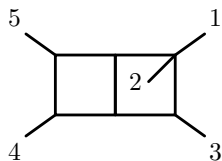
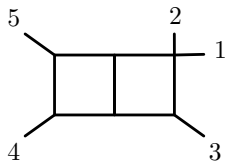
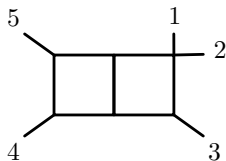
Answer: **yes**, they inherit KK relations from cuts.

$$A(1, 2, 3, 4) + A(1, 2, 4, 3) + A(1, 4, 2, 3) = 0$$

4 points, 2 loops:



5 points, 2 loops:



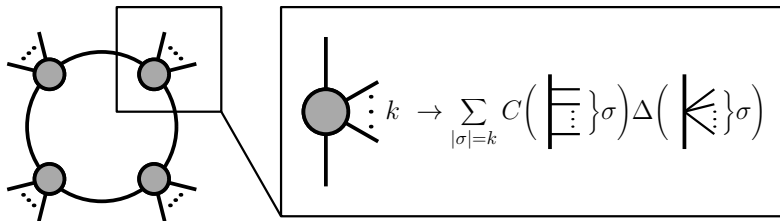
## 2-loop example in detail

$$\begin{aligned}
 \tilde{\Delta} \left( \begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) &= C \left( \begin{array}{c} 4 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left( \begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \\
 &+ C \left( \begin{array}{c} 3 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 4 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left( \begin{array}{c} 3 \\ \ell_1 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 4 \quad \ell_2 \qquad 2 \\ 1 \end{array} \right) \\
 &+ C \left( \begin{array}{c} 4 \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{loop} \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left( \begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{loop} \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \\
 &= \left\{ C \left( \begin{array}{c} 4 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) - C \left( \begin{array}{c} 4 \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{loop} \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \right\} \Delta \left( \begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \\
 &+ \left\{ C \left( \begin{array}{c} 3 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 4 \qquad \qquad 2 \\ 1 \end{array} \right) - C \left( \begin{array}{c} 4 \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{loop} \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \right\} \Delta \left( \begin{array}{c} 3 \\ \ell_1 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 4 \quad \ell_2 \qquad 2 \\ 1 \end{array} \right) \\
 &= C \left( \begin{array}{c} 4 \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{box} \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left( \begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) + C \left( \begin{array}{c} 3 \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{box} \end{array} \\ 4 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left( \begin{array}{c} 3 \\ \ell_1 \\ \begin{array}{cc} \diagdown & \diagup \\ \diagup & \diagdown \end{array} \\ 4 \quad \ell_2 \qquad 2 \\ 1 \end{array} \right)
 \end{aligned}$$

# DDM-based 1-loop decomposition

Del Duca, Dixon, Maltoni '99

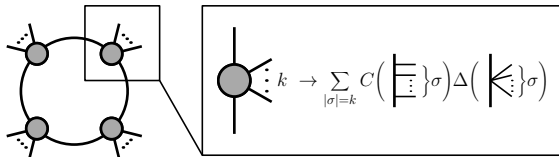
$$\begin{aligned}
 \mathcal{A}_n^{(1)} &= \sum_{\sigma \in S_n/D_n} \tilde{f}^{b_1 a_{\sigma(1)} b_2} \tilde{f}^{b_2 a_{\sigma(2)} b_3} \dots \tilde{f}^{b_n a_{\sigma(n)} b_1} A^{(1)}(\sigma(1), \sigma(2), \dots, \sigma(n)) \\
 &= \sum_{\sigma \in S_n/D_n} C \left( \begin{array}{c} \dots \\ \sigma(2) \text{---} \bigcirc \text{---} \sigma(n-1) \\ \sigma(1) \quad \sigma(n) \end{array} \right) A^{(1)}(\sigma(1), \sigma(2), \dots, \sigma(n))
 \end{aligned}$$



# DDM-based 1-loop decomposition

Del Duca, Dixon, Maltoni '99  
1-loop KK relations by Bern, Kosower '90

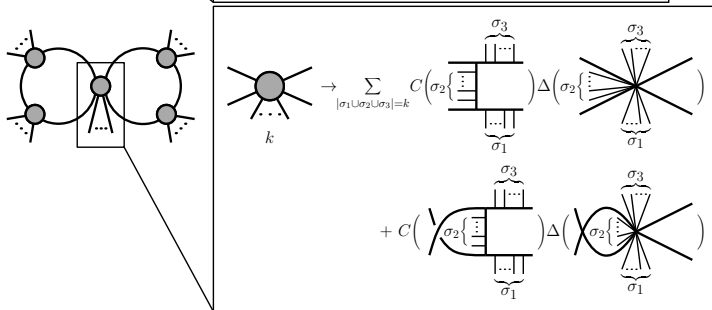
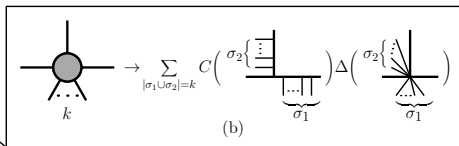
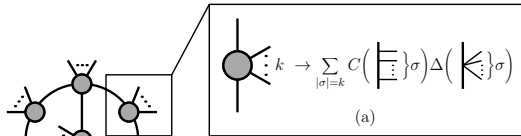
$$\mathcal{A}_n^{(1)} = \sum_{\sigma \in S_n/D_n} C \left( \begin{array}{c} \cdots \\ \sigma(2) \text{ --- } \bigcirc \text{ --- } \sigma(n-1) \\ \sigma(1) \quad \sigma(n) \end{array} \right) A^{(1)}(\sigma(1), \sigma(2), \dots, \sigma(n)),$$



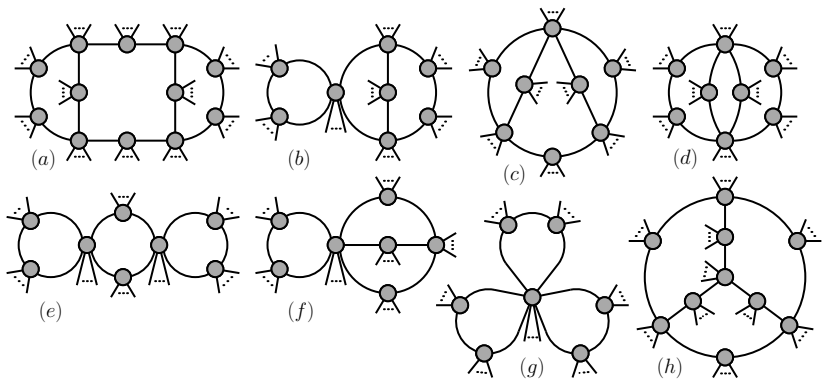
$$A^{(1)}(1, 2, \dots, n) = I \left[ \sum_{\substack{1 \leq i_1 < i_2 < i_3 \\ < i_4 < i_5 \leq n}} \Delta \left( \begin{array}{c} i_5 \quad i_1-1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2-1 \\ \swarrow \quad \searrow \\ i_4-1 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \quad i_3-1 \end{array} \right) + \sum_{\substack{1 \leq i_1 < i_2 \\ < i_3 < i_4 \leq n}} \Delta \left( \begin{array}{c} i_1-1 \quad i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2-1 \\ \swarrow \quad \searrow \\ i_4-1 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \quad i_3-1 \end{array} \right) \right. \\ \left. + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \Delta \left( \begin{array}{c} i_3 \quad i_1-1 \\ \swarrow \quad \searrow \\ i_3-1 \quad i_1 \\ \swarrow \quad \searrow \\ i_2 \quad i_2-1 \end{array} \right) + \sum_{1 \leq i_1 < i_2 \leq n} \Delta \left( \begin{array}{c} i_1-1 \quad i_1 \\ \swarrow \quad \searrow \\ i_2 \quad i_2-1 \\ \swarrow \quad \searrow \end{array} \right) + \sum_{1 \leq i_1 \leq n} \Delta \left( \begin{array}{c} i_1-1 \quad i_1 \\ \swarrow \quad \searrow \\ i_1-1 \quad i_1-1 \end{array} \right) \right]$$



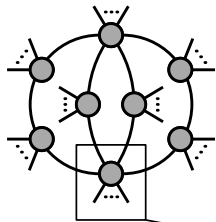
# DDM stretches at 2 loops



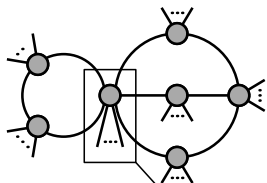
## 3-loop topologies



### 3-loop topologies (*d*) and (*f*)

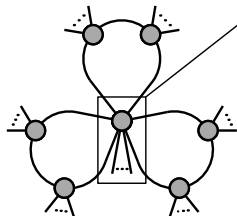


$$\begin{aligned}
 & \text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2 \cup \sigma_3|=k} C \left( \begin{array}{c} \sigma_3 \{ \dots \} \sigma_1 \\ \ell_2 \quad \underbrace{\dots}_{\sigma_2} \quad \ell_1 \end{array} \right) \Delta \left( \begin{array}{c} \sigma_3 \{ \dots \} \sigma_1 \\ \ell_2 \quad \underbrace{\dots}_{\sigma_2} \quad \ell_1 \end{array} \right) \\
 & + C \left( \begin{array}{c} \sigma_3 \{ \dots \} \sigma_1 \\ \underbrace{\dots}_{\sigma_2} \\ \ell_2 \quad \ell_1 \end{array} \right) \Delta \left( \begin{array}{c} \sigma_3 \{ \dots \} \sigma_1 \\ \underbrace{\dots}_{\sigma_2} \\ \ell_2 \quad \ell_1 \end{array} \right)
 \end{aligned}$$

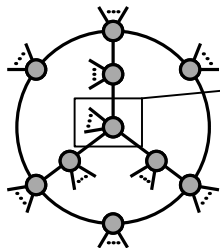


$$\begin{aligned}
 & \text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4|=k} C \left( \begin{array}{c} \sigma_1 \\ \dots \\ \sigma_2 \\ \ell_2 \\ \sigma_3 \\ \dots \\ \sigma_4 \end{array} \right) \Delta \left( \begin{array}{c} \sigma_1 \\ \dots \\ \sigma_2 \\ \ell_2 \\ \sigma_3 \\ \dots \\ \sigma_4 \end{array} \right) \\
 & + C \left( \begin{array}{c} \sigma_1 \\ \dots \\ \sigma_2 \\ \underbrace{\dots}_{\sigma_4} \\ \sigma_3 \\ \ell_3 \end{array} \right) \Delta \left( \begin{array}{c} \sigma_1 \\ \dots \\ \sigma_2 \\ \underbrace{\dots}_{\sigma_4} \\ \sigma_3 \\ \ell_3 \end{array} \right) + \dots
 \end{aligned}$$

### 3-loop topologies ( $g$ ) and ( $h$ )



$$\begin{aligned}
 & \text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4 \cup \sigma_5|=k} C \left( \begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \vdots \\ \sigma_1 \{ \vdots \} \end{array} \right) \Delta \left( \begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \vdots \\ \sigma_1 \{ \vdots \} \end{array} \right) \\
 & + C \left( \begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \vdots \\ \sigma_1 \{ \vdots \} \end{array} \right) \Delta \left( \begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \vdots \\ \sigma_1 \{ \vdots \} \end{array} \right) + \dots
 \end{aligned}$$



$$\text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2|=k} C \left( \begin{array}{c} \sigma_2 \{ \vdots \} \\ \vdots \\ \sigma_1 \{ \vdots \} \end{array} \right) \Delta \left( \begin{array}{c} \sigma_2 \{ \vdots \} \\ \vdots \\ \sigma_1 \{ \vdots \} \end{array} \right)$$

# 3-loop $\mathcal{N} = 4$ example

AO, Page '16

integrand by Bern, Carrasco, Dixon, Johansson, Kosower, Roiban '07

integrated by Henn, Mistlberger '16

$$\begin{aligned}
 \mathcal{A}_{\mathcal{N}=4}^{(3)} = & \sum_{\sigma \in S_4} \sigma \circ I \left[ \frac{1}{8} C \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) \Delta \left( \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) \right. \\
 & + \frac{1}{4} C \left( \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right) \Delta \left( \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right) + \frac{1}{4} C \left( \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right) \Delta \left( \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right) \\
 & + \frac{1}{8} C \left( \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} \right) \Delta \left( \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} \right) + \frac{1}{16} C \left( \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \right) \Delta \left( \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} \right) \\
 & + \frac{1}{2} C \left( \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array} \right) \left\{ \Delta \left( \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} \right) + \Delta \left( \begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \end{array} \right) \right\} \\
 & + \frac{1}{2} C \left( \begin{array}{c} \text{Diagram 27} \\ \text{Diagram 28} \end{array} \right) \left\{ \Delta \left( \begin{array}{c} \text{Diagram 29} \\ \text{Diagram 30} \end{array} \right) + \Delta \left( \begin{array}{c} \text{Diagram 31} \\ \text{Diagram 32} \end{array} \right) \right\} \\
 & + C \left( \begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array} \right) \left\{ \Delta \left( \begin{array}{c} \text{Diagram 35} \\ \text{Diagram 36} \end{array} \right) + \Delta \left( \begin{array}{c} \text{Diagram 37} \\ \text{Diagram 38} \end{array} \right) \right\} \\
 & \left. + C \left( \begin{array}{c} \text{Diagram 39} \\ \text{Diagram 40} \end{array} \right) \left\{ \frac{1}{2} \Delta \left( \begin{array}{c} \text{Diagram 41} \\ \text{Diagram 42} \end{array} \right) + \Delta \left( \begin{array}{c} \text{Diagram 43} \\ \text{Diagram 44} \end{array} \right) + \frac{1}{3} \Delta \left( \begin{array}{c} \text{Diagram 45} \\ \text{Diagram 46} \end{array} \right) \right\} \right]
 \end{aligned}$$