THE TWO-LOOP FOUR-GRAVITON SCATTERING AMPLITUDES

ZOOMPLITUDES 2020

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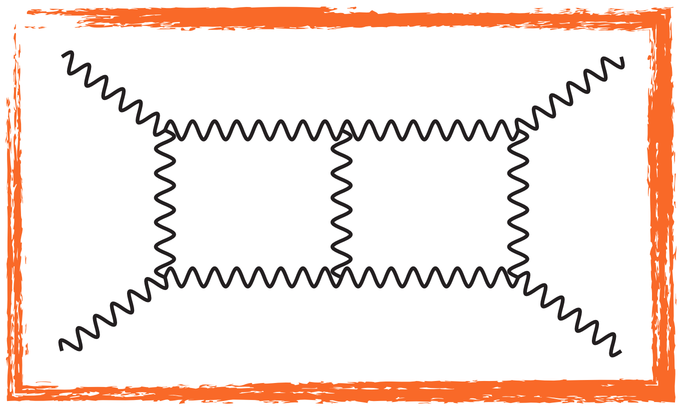
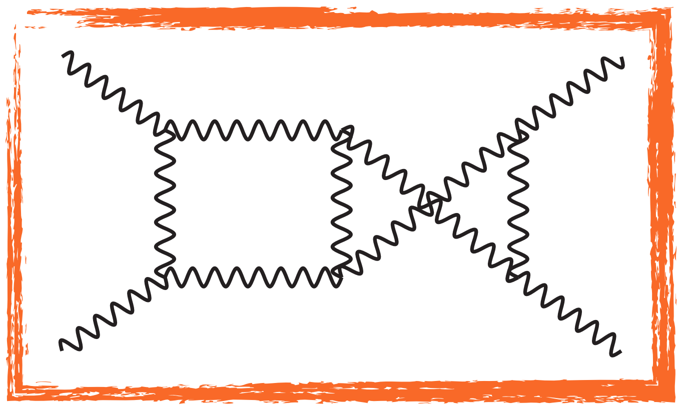
arXiv:2002.12374, to appear in PRL

with: F. Febres Cordero, H. Ita, M. Jaquier, B. Page, M. Ruf, V. Sotnikov

MOTIVATION





- Compute an important amplitude in pure GR:
  - ✓ All-plus known, compute remaining helicities [Bern et al.], [Dunbar et al.]
  - ✓ UV well studied, compute finite pieces [Weinberg], ['t Hooft and Veltman], [Goroff and Sagnotti], [Donoghue], [Bern et al]…
- Insights into relation between multi-loop amplitudes/classical gravitational dynamics

Julio's talk

- Develop tools to compute GR amplitudes
- Explore two-loop numerical unitarity framework beyond planar QCD

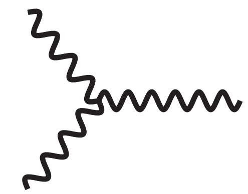
EFT Lagrangian (at 2-loops)

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{GB}} + \mathcal{L}_{\mathrm{R}^3}$$

Emil's talk

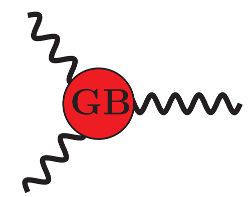
[Weinberg], ['t Hooft and Veltman], [Goroff and Sagnotti], [Donoghue], ...

$$\mathcal{L}_{\mathrm{EH}} = -\frac{2}{\kappa^2}\sqrt{|g|}R$$



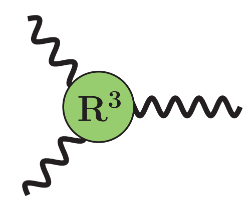
$\mathcal{O}(\kappa)$

$$\mathcal{L}_{\mathrm{GB}} = \frac{\mathcal{C}_{\mathrm{GB}}}{(4\pi)^2}\sqrt{|g|}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$$

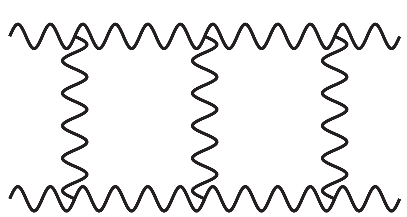
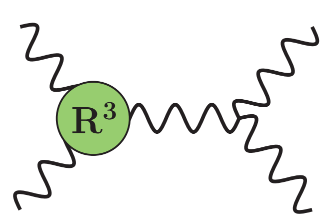


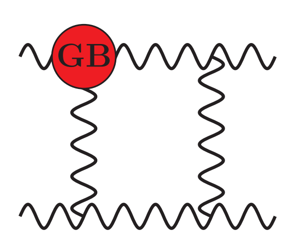
$\mathcal{O}(\kappa^3)$

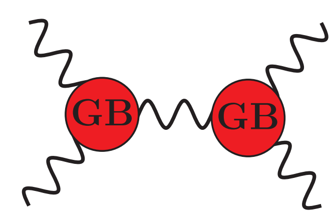
$$\mathcal{L}_{\mathrm{R}^3} = \frac{\mathcal{C}_{\mathrm{R}^3}}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^2\sqrt{|g|}R_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}$$



$\mathcal{O}(\kappa^5)$

$\mathcal{A}^{(2)} =$  $+$ 

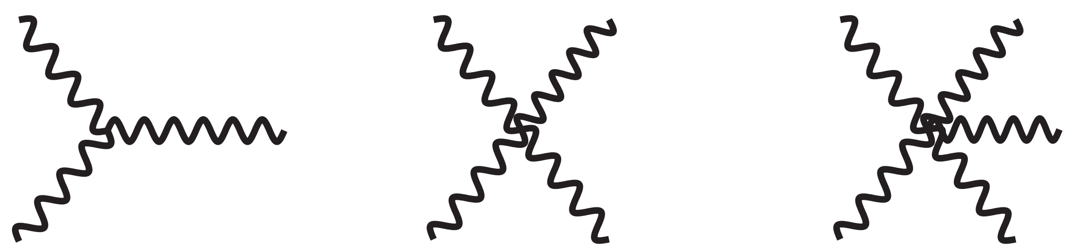
$+$ 

$+$ 

$\mathcal{O}(\kappa^6)$

Main Challenges

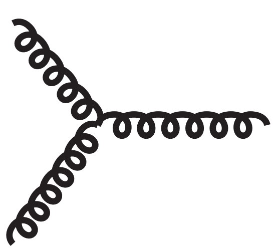
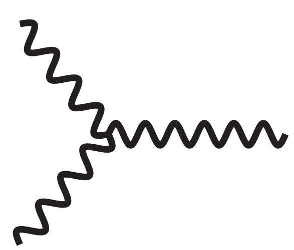
- EH Feynman rules are complicated



Terms: $\mathcal{O}(100)$ $\mathcal{O}(1000)$ $\mathcal{O}(10000)$

Feynman-diagram based calculation out of question
$\Rightarrow$ (Generalised) Unitarity

- EH interactions have high power-counting (QCD²)

 $\sim \mathcal{O}(\ell)$ vs  $\sim \mathcal{O}(\ell^2)$

Complicated integrand
$\Rightarrow$ Analytics from numerics

Issues addressed by
two-loop numerical unitarity

Caravel

Two-loop (numerical) unitarity

[Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng, 17,18,19]

- Computing the amplitude, $D = 4 - 2\epsilon$

$$\mathcal{A} = \sum_{\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(D)\, I_{\Gamma,i}(D)$$

- All integrals (4-point 2-loop massless) are known

[Anastasiou, Smirnov, Tausk, Tejeda-Yeomans, Veretin, 99,00]

- Decomposition of integrand

$$\mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D) \frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j \in P_\Gamma} \rho_j}$$

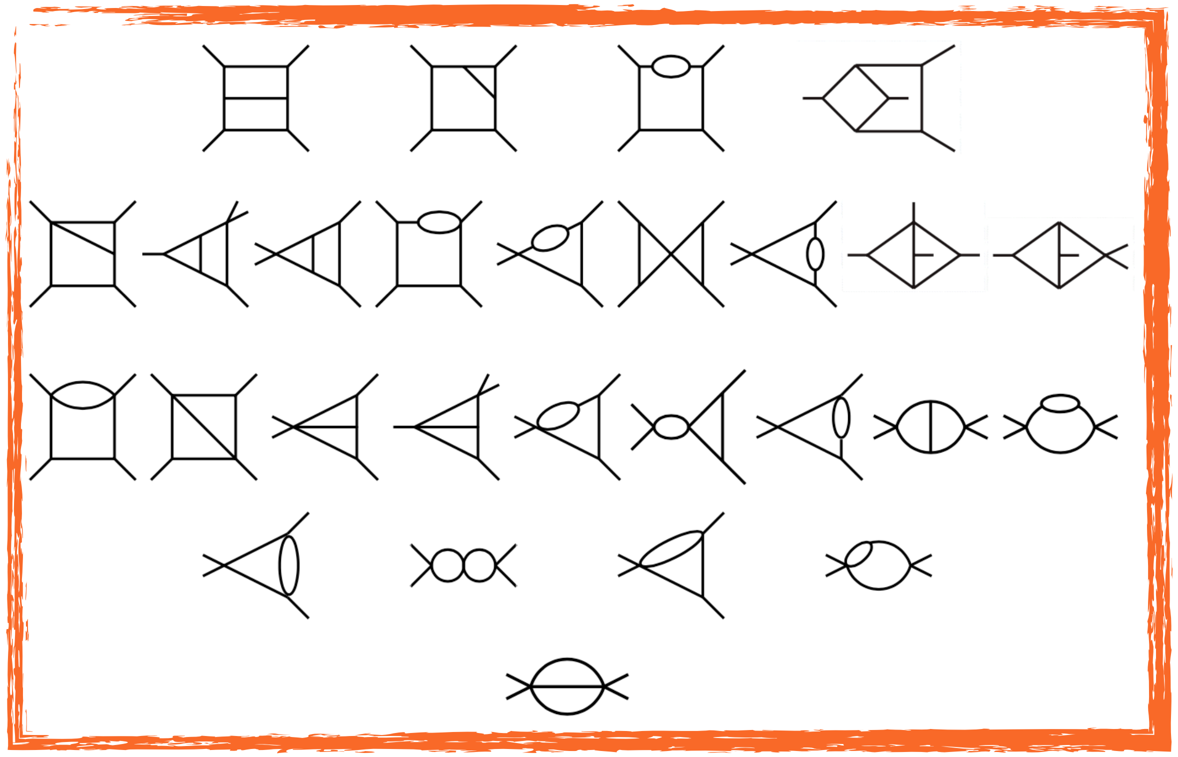
  - ✓ Dimension $|M_\Gamma \cup S_\Gamma|$ simple to determine
  - ✓ Construct surface S_Γ terms that integrate to zero
  - ✓ Compute coefficients $c_{\Gamma,i}$ from generalised unitarity

Two-loop (numerical) unitarity — surface terms I

[Ita 15; Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng 17]

- For each Γ, construct numerators such that

$$\int [d^D \ell_l] \frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j \in P_\Gamma} \rho_j} = 0$$



- "Integration-by-parts" IBP relations …

$$\int [d^D \ell_l] \frac{\partial}{\partial \ell_i^\nu} \left[\frac{u_i^\nu}{\prod_{k \in P_\Gamma} \rho_k} \right] = 0$$

- … that don't increase propagator power

$$u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j$$

[Gluza, Kajda, Kosower 10; Schabinger 11]

  - ✓ Compute the u_i^ν by solving syzygy equations

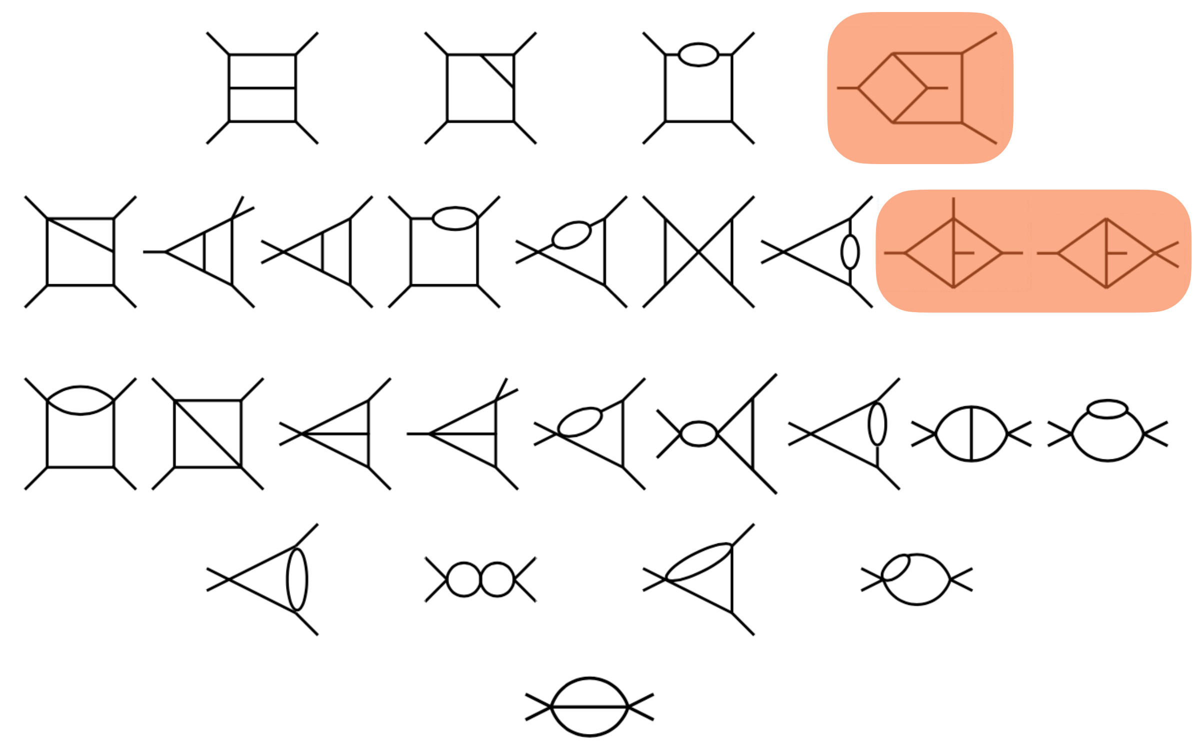
see also [Boehm, Georgoudis, Larsen, Schulze, Zhang, … 16 - 19] and [Agarwal, von Manteuffel 19]

Two-loop (numerical) unitarity — surface terms II

[Ita 15; Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng 17]

- Surface terms with u_i^ν and $t(\ell_l)$: $m_{\Gamma,k} = u_i^\nu \frac{\partial t(\ell_l)}{\partial \ell_i^\nu} + t(\ell_l)\left(\frac{\partial u_i^\nu}{\partial \ell_i^\nu} - \sum_{k \in P_\Gamma} f_k\right)$
- Linear algebra: independent surface terms, determine $|S_\Gamma|$, infer $|M_\Gamma|$
- u_i^ν are power-counting/theory independent

Analytic decomposition of integrand into master integrands and surface terms



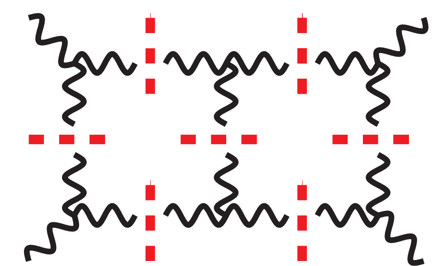
Two-loop numerical unitarity — coefficients

$$\mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D) \frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j \in P_\Gamma} \rho_j}$$

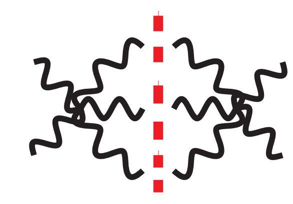
[Bern, Dixon, Dunbar, Kosower 94,95]

✦ In on-shell configuration of ℓ_l, integrand factorises

$$\sum_{\text{states}} \prod_{k \in T_\Gamma} \mathcal{A}_k^{\text{tree}}(\ell_l^\Gamma) = \sum_{\Gamma' \geq \Gamma} \frac{c_{\Gamma'}\, m_{\Gamma'}(\ell_l^\Gamma)}{\prod_{j \in (P_{\Gamma'} \setminus P_\Gamma)} \rho_j(\ell_l^\Gamma)}$$



✦ Need efficient computation of tree amplitudes (numerical)



✓ Berends-Giele recursion [Berends, Giele, 88]

✓ D_s-dimensional state sum, $D_s = 6,\ldots,10$

✦ Solve cut equations numerically: never construct analytic integrand

Compute amplitude at a numerical phase-space point

Two-loop numerical unitarity — analytics I

$$\mathcal{A} = \sum_{\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(\epsilon, D_s, s, x = t/s)\, I_{\Gamma,i}$$

[Peraro, 16]
[Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov 18, 19]

- Rational dependence on all variables: compute in finite field
  - ✓ No loss of precision in numerical manipulations

[von Manteuffel, Schabinger, 15], [Peraro, 16]

- Analytic ϵ dependence
  - ✓ Compute at enough values to determine rational function

[Thiele's formula]

- Analytic D_s dependence
  - ✓ EH: Quartic polynomial
  - ✓ GB: Cubic polynomial / $(D_s - 2)$, vanishes at $D_s = 4$

Analytic ϵ and D_s dependence at numerical phase-space point amplitude from numerics

Two-loop numerical unitarity — analytics II

[Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov 18, 19]

$$\mathcal{A} = \sum_{\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(\epsilon, D_s, s, x = t/s)\, I_{\Gamma,i}$$

- Analytic dependence in x is rational but complicated
  - ✓ Target genuine two-loop contribution

- Reconstruct analytic form of finite remainder ($D_s = 4 - 2\epsilon$)
  - ✓ Remove soft $\mathcal{R}^{(2)} = \mathcal{A}^{(2)} - \mathcal{S}\mathcal{A}^{(1)} + \frac{\mathcal{S}^2}{2}\mathcal{A}^{(0)}$ $\quad \mathcal{S} = \sum_{i<j} \frac{1}{\epsilon^2}(-(p_i + p_j)^2)^{1-\epsilon}$
  - ✓ UV finite by construction

$$\mathcal{R}^{(2)}_{\vec{h}} = \sum_i d_i(x)\, h_i(x)$$

Compute analytic amplitude from numerics

Two-loop numerical unitarity

- Full framework automated in Caravel
  [Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, Sotnikov 20]
- Well tested in QCD calculations
  - ✓ Analytic planar 5-parton amplitudes
    [Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov 18, 19]
- New developments for four-graviton amplitudes
  - ✓ Non-planar topologies
  - ✓ GB, R^3 Feynman rules
  - ✓ For EH: cubic-interaction reformulation [Cheung, Remmen 17]
- 20 numerical evaluations for each helicity
  [Abreu, Febres Cordero, Ita, Jaquier, Page, Ruf, Sotnikov 20]

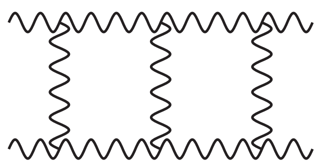
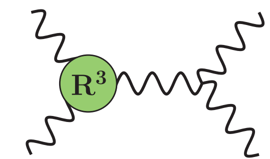
Caravel

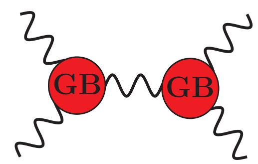
Soon to be released!

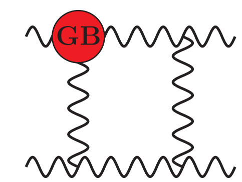
Analytic 4-graviton 2-loop amplitudes:

$$\mathscr{A}(++++),\ \mathscr{A}(-+++),\ \mathscr{A}(--++)$$

Results

- Computed the three independent helicities for
  - ✓ EH gravity 
  - ✓ Tree-level R^3 

  - ✓ Tree-level GB-GB 

  - ✓ One-loop GB 

- Checks
  - ✓ Remainders are finite, correct symmetry, no spurious poles
  - ✓ 1-loop amplitudes [Dunbar, Norridge 95], [Bern, Cheung, Chi, Davies, Dixon, Nohle, unpub.]
  - ✓ GB tree and 1-loop: $++++$ and $--++$ [Bern, Cheung, Chi, Davies, Dixon, Nohle, 15, unpub.]
  - ✓ R^3 tree [Bern, Cheung, Chi, Davies, Dixon, Nohle, 15], [Dunbar, Jehu, Perkins, 17]
  - ✓ 2-loop: $++++$ [Bern, Cheung, Chi, Davies, Dixon, Nohle, unpub.], [Dunbar, Jehu, Perkins, 17]

Results — remainders

- Remainders
  - ✓ $++++$: up to weight 1, symmetric on s, t, u
  - ✓ $-+++$: up to weight 2, only logs, symmetric on s, t, u
  - ✓ $--++$: up to weight 4, classical polylogs

$$\mathcal{R}^{(2)}_{++++} = f(s,t,u) + f(t,u,s) + f(u,s,t) + \frac{117617}{21600} s\,t\,u + 30\,s\,t\,u\,(c_{\mathrm{GB}}(\mu) - 2c_{\mathrm{R}^3}(\mu))$$

$$f(s,t,u) = \frac{s}{60}\left(14s^2 - 9t\,u\right)\log(-s)$$

[Bern, Cheung, Chi, Davies, Dixon, Nohle, unpub.]
[Dunbar, Jehu, Perkins, 17]

s-channel Regge limit ($s \gg -t > 0$):

$$\mathcal{R}^{(2)}_{\{-,-,+,+\}} = s^3 \Bigg\{ 2\frac{s}{t}\pi^2\left(\frac{i\pi}{2} - L\right)^2 - 3\pi^2 L^2 + \frac{107}{10}\pi^2 L + \frac{14191}{1350}\pi^2 - \frac{158}{45}\pi^4 - \frac{13049}{2160} + i\pi\left[-\frac{14}{3}L^3 + \frac{87}{10}L^2 - \left(8\pi^2 - \frac{17749}{450}\right)L - 20\zeta_3 + \frac{2621}{210}\pi^2 - \frac{11221}{375}\right] + \mathcal{O}(-t/s)\Bigg\}$$

$$L = \log(-s/t)$$

Julio's talk

Results — couplings and μ-dependence

$$\mathcal{C}_{\mathrm{GB}} = \left(\frac{53}{90}\frac{1}{\epsilon} + c_{\mathrm{GB}}(\mu)\right)\mu^{-2\epsilon} \qquad \mathcal{C}_{\mathrm{R}^3} = \left(\frac{209}{1440}\frac{1}{\epsilon} + c_{\mathrm{R}^3}(\mu)\right)\mu^{-4\epsilon}$$

- Remainders depend on a single coupling $c(\mu) = c_{\mathrm{GB}}(\mu) - 2\, c_{\mathrm{R}^3}(\mu)$

$$\mathcal{R}^{(2)}_{\{+,+,+,+\}} = 30\, s\, t\, u\, c(\mu) + \ldots \qquad \mathcal{R}^{(2)}_{\{-,+,+,+\}} = 3\, s\, t\, u\, c(\mu) + \ldots$$

$$\mathcal{R}^{(2)}_{\{-,-,+,+\}} = 0\, c(\mu) + \ldots$$

  - ✓ same combination in 5-point 2-loop all-plus [Dunbar, Jehu, Perkins, 17]

- μ-dependence of remainder

$$\mu\frac{\partial}{\partial\mu}\mathcal{R}^{(2)}_{\vec{h}} = \left[\frac{1}{120} + \mu\frac{\partial}{\partial\mu}c(\mu)\right]\frac{\mathcal{A}^{\mathrm{tree},\mathrm{R}^3}_{\vec{h}}}{\mathcal{C}_{\mathrm{R}^3}}$$

$$\mathcal{A}^{\mathrm{tree},\mathrm{R}^3}_{+,+,+,+} = -\mathcal{C}_{\mathrm{R}^3}\frac{209}{24\epsilon}s\,t\,u \qquad \mathcal{A}^{\mathrm{tree},\mathrm{R}^3}_{-,+,+,+} = -\mathcal{C}_{\mathrm{R}^3}\frac{209}{240\epsilon}s\,t\,u \qquad \mathcal{A}^{\mathrm{tree},\mathrm{R}^3}_{-,-,+,+} = 0$$

  - ✓ GB counter-term is evanescent

[Bern, Cheung, Chi, Davies, Dixon, Nohle, 15]
[Bern, Chi, Dixon, Edison, 17]
[Dunbar, Godwin, Jehu, Perkins, 17]

Summary

- ✦ Two-loop numerical unitarity is a mature framework
  - ✓ Efficient approach to obtain analytic results
  - ✓ Generic: state of the art results in QCD & GR

- ✦ Computed the full set of four-graviton helicity amplitudes
  - ✓ Confirm known $++++$
  - ✓ New results for $-+++$ and $--++$
  - ✓ Led to new insights into relation between multi-loop amplitudes and classical gravitational dynamics [Bern, Ita, Parra-Martinez, Ruf, 20]
  - ✓ Interesting coupling and μ dependence in remainder

THANK YOU!