A Coaction for Feynman Integrals

Part I

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Feynman integrals

- Feynman integrals are the cornerstone of perturbative QFT.
  - ➡ We want to understand them as well as we can!
- Possible questions:
  - ➡ Which class of functions?
  - ➡ Is there some 'hidden' algebraic structure?
- In many cases we get multiple polylogarithms (MPLs):

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

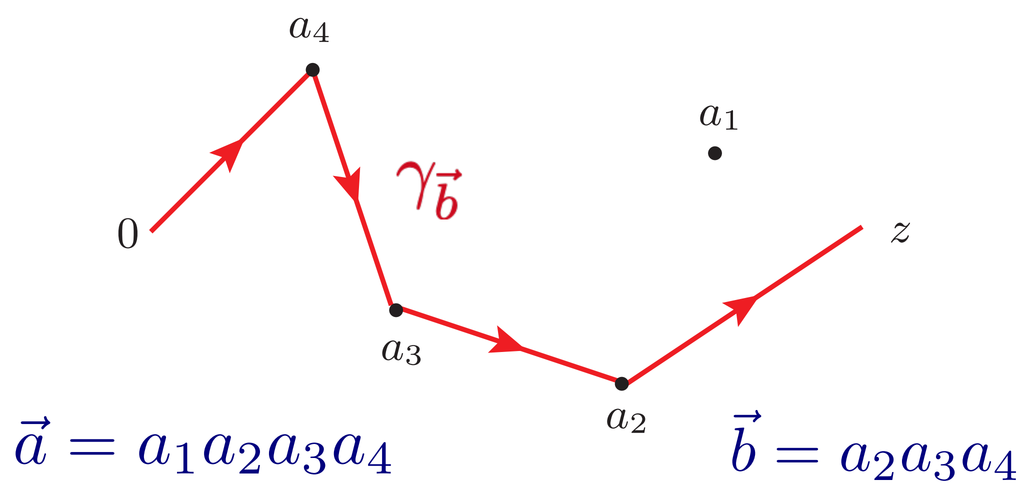
$$G(a_1;z) = \log\left(1-\frac{z}{a_1}\right) \qquad G(0,1;z) = -\text{Li}_2(z) \qquad G(0,1;1) = -\text{Li}_2(1) = -\zeta_2$$

- Beyond one loop: Also other functions may appear (e.g., elliptic)

Polylogarithms

- (Motivic) periods can be equipped with a coaction. [Brown]
- For MPLs it takes the form: [Goncharov, Brown]

$$\Delta(G(\vec{a};z)) = \sum_{\vec{b}\subseteq\vec{a}} G(\vec{b};z)\otimes G_{\vec{b}}(\vec{a};z)$$



- Examples:

$$\Delta(G(1;z)) = G(1;z)\otimes 1 + 1\otimes G(1;z) \qquad G(1;z) = \log(1-z)$$

$$\Delta(G(0,1;z)) = G(0,1;z)\otimes 1 + G(1;z)\otimes G(0;z) + 1\otimes G(0,1;z)$$

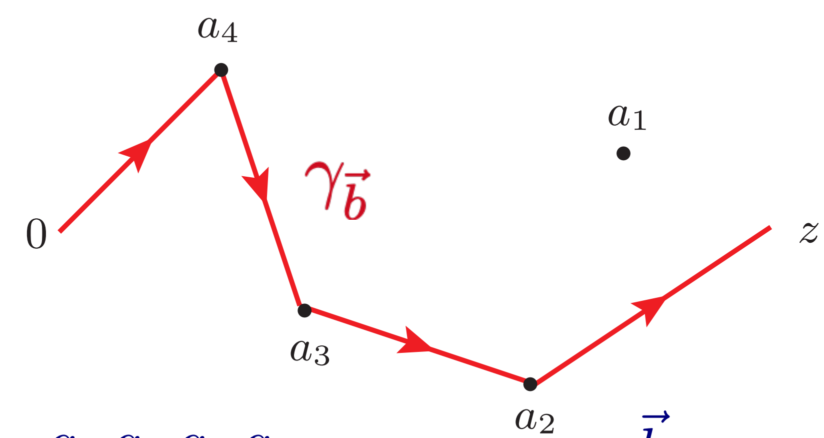
$$\Delta(2\pi i) = 2\pi i\otimes 1$$

Polylogarithms

- (Motivic) periods can be equipped with a coaction. [Brown]
- For MPLs it takes the form: [Goncharov, Brown]

$$\Delta(G(\vec{a};z)) = \sum_{\vec{b}\subseteq\vec{a}} G(\vec{b};z) \otimes G_{\vec{b}}(\vec{a};z)$$

$$\Delta\left(\int_0^z \omega_{\vec{a}}\right) = \sum_{\vec{b}\subseteq\vec{a}} \int_0^z \omega_{\vec{b}} \otimes \int_{\gamma_{\vec{b}}} \omega_{\vec{a}}$$



$\vec{a} = a_1a_2a_3a_4$ $\vec{b} = a_2a_3a_4$

sum over master integrals integral over contour 'dual' to $\omega_{\vec{b}}$

- To which other class of integrals can this be applied?
- Can we apply it directly to Feynman integrals/amplitudes?
  - ➡ Motivic coaction naturally acts on Feynman integral. [Brown]

The class of integrals

- Polylogarithms are integrals of dlog-forms integrated over a polytope.
  - ➡ Special case of integrals defined via positive geometries:

- Positive geometry ~ stratified space Y s.t.: [Arkani-Hamed, Bai, Lam]
  - ➡ Unique normalised differential form $\Omega(Y)$ with logarithmic singularities on the boundary ∂Y.
  - ➡ Each boundary component is itself a positive geometry.

- Canonical integrals: $\int_{Y_1} \Omega(Y_2)$ where Y_1 and Y_2 are positive geometries.

The coaction

- Example: Straight-line $Y = [0,1]$ is a positive geometry with canonical form

$$\Omega([0,1]) = d\log\frac{u}{u-1} = \frac{du}{u} + \frac{du}{1-u}$$

We will consider integrals like $I_Y = \int_0^1 u^\epsilon (1-u)^\epsilon \left(\frac{du}{u} + \frac{du}{1-u}\right).$

'Dimensional regularisation' of singularities at $u = 0, 1$.

- Proposal for the coaction [see Ruth's talk for details]:

$$\Delta\left(\int_\gamma \omega\right) = \int_\gamma \omega_i \otimes C^{-1}_{ij} \int_{\gamma_j} \omega$$

[Abreu, Britto, CD, Gardi, Matthew]

$\{\gamma_i\}$: basis of contours.

$$\int_{\gamma_i} \omega_j = C_{ij}(1+\mathcal{O}(\epsilon))$$

$\{\omega_i\}$: basis of integrands.

[C_{ij} = intersection matrix]

The coaction

- Example: $$I_Y = \int_0^1 u^\epsilon (1-u)^\epsilon \left(\frac{du}{u} + \frac{du}{1-u}\right)$$
$$= \frac{2}{\epsilon} - \frac{\pi^2}{3}\epsilon + 4\zeta_3\,\epsilon^2 - \frac{\pi^4}{20}\epsilon^3 + \ldots \qquad \zeta_n = \mathrm{Li}_n(1)$$
$$\Delta(I_Y) = \Delta\left(\int_\gamma \omega\right) = \int_\gamma \omega \otimes C_{11}^{-1} \int_\gamma \omega = I_Y \otimes \frac{\epsilon}{2} I_Y$$

- One would have obtained the same answer by acting with coaction on MPLs on the zeta values in ϵ - expansion.

$$\Delta(\zeta_{2n+1}) = \zeta_{2n+1} \otimes 1 + 1 \otimes \zeta_{2n+1} \qquad \Delta(\zeta_{2n}) = \zeta_{2n} \otimes 1$$

- Highly non-trivial conjecture [Abreu, Britto, CD, Gardi, Matthew]:

> The coaction is consistent with the expansion in DimReg.

[See also work by Brown & Dupont, Talk by Brown at Amplitudes 2019.]

The coaction

- How much evidence do we have for this conjecture?

- What does it have to do with Feynman integrals?

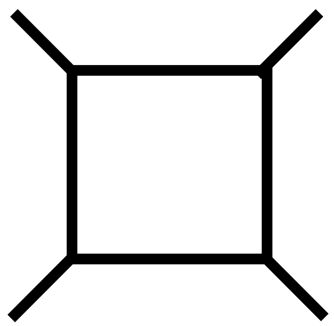
  ➡ This talk (Part I): The coaction on all one-loop integrals.

  ➡ Next talk (Part II): Extension to hypergeometric functions and to some two-loop cases.

One-loop integrals

- One-loop integrals provide examples of positive geometries.

- Example: one-loop box integral ($p_i^2 = 0$):



$$\sim \int_0^\infty d^4x \, \delta(1 - \sum_i x_i) \frac{\mathcal{U}^{4-D}}{\mathcal{F}^{\nu - D/2}}$$

$$\mathcal{U} = \sum_i x_i \qquad \mathcal{F} = (-s)x_1x_3 + (-t)x_2x_4 + \mathcal{U}\sum_i m_i^2 x_i$$

Singular surfaces: $\mathcal{U} = 0$ $\qquad$ $\mathcal{F} = 0$ Quadratic

Linear

Integration boundaries: $x_i = 0$

➡ Geometry for n-point 1-loop: 1 quadric + $(n+1)$ hyperplanes.

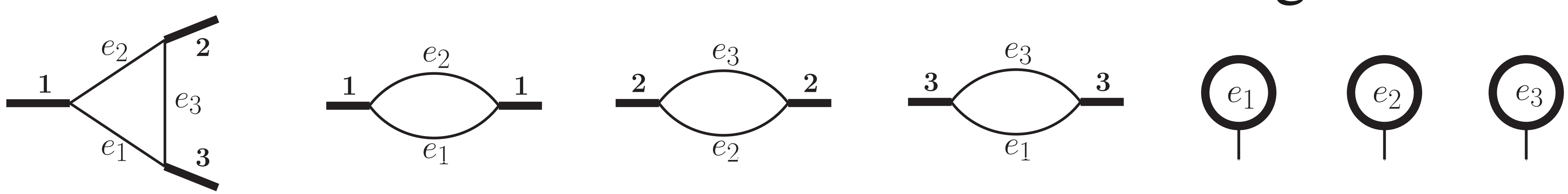
➡ Defines positive geometry. [Arkani-Hamed, Bai, Lam; Arkani-Hamed, Yuan]

One-loop master integrals

- **Need:** a basis of integrands (‘master integrals’) and a basis of contours.
- At one-loop we know a basis of integrands.
- **Example:** Every integral of the type

$$\int d^D k \, \frac{N(k,p_1,p_2)}{[k^2+m_1^2]\,[(k+p_1)^2+m_2^2]\,[(k+p_1+p_2)^2+m_3^2]}$$

can be written as a linear combination of the integrals



$D = 4 - 2\epsilon$ $\qquad$ $D = 2 - 2\epsilon$ $\qquad$ $D = 2 - 2\epsilon$

One-loop contours

- Contours associated to 1-loop integrals were studied in the 60's

[Fotiadi, Pham; Teplitz, Hwa; Federbusch; Landshof, Polkinghorne, ...]

1. $\Gamma_{\emptyset}$: computes Feynman integral
2. $\Gamma_1, \Gamma_{13}, \Gamma_{134}, \ldots$: computes residues where subset of propagators were put on shell.
   ➡ Cut integral, cf. $\dfrac{1}{p^2 - m^2 + i\varepsilon} \longrightarrow 2\pi i\, \delta(p^2 - m^2)\, \theta(p^0)$
3. $\Gamma_{\infty 1}, \Gamma_{\infty 13}, \ldots$: computes residues at a subset of propagators and at singularity at infinity.

- There are more of these contours than master integrals.

  ➡ There must be relations among these contours.

Relations among contours

- Relations involving singularity at infinity (C = subset of props.)

$$\Gamma_{\infty C} = -2x_C\,\Gamma_C + \sum_{C\subset X} (-1)^{\lceil |C|/2\rceil + \lceil |X|/2\rceil}\Gamma_X \qquad x_C = \begin{cases} 1\,, & \text{if } |C| \text{ odd}\,,\\ 0\,, & \text{if } |C| \text{ even}\,,\end{cases}$$

$$C \subseteq \{1\dots n\}$$

➡ 'Cuts of singularities at infinity' are not independent basis elements.

- Relations involving uncut integral:

$$\sum_{i\in[n]} \mathcal{C}_i I_n + \sum_{\substack{i,j\in[n]\\ i<j}} \mathcal{C}_{ij} I_n = -\epsilon\, I_n \mod i\pi$$

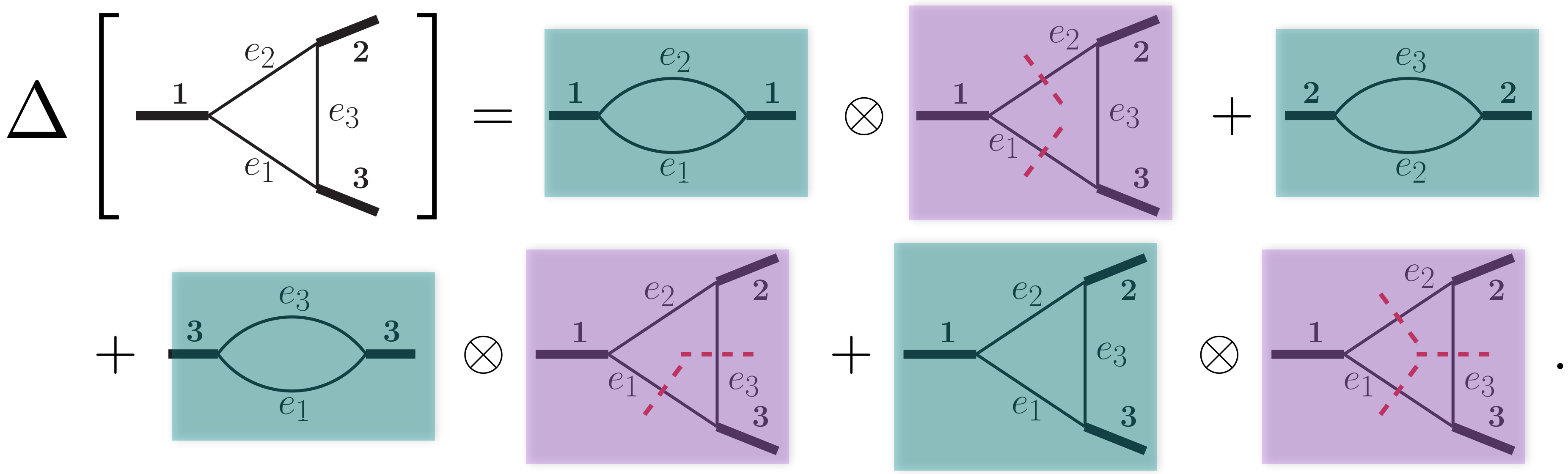
➡ Sum over single and double cuts reproduces original integral.

- Only contours where a subset of propagators are cut remain.

➡ Matches precisely the number of master integrals.

The diagrammatic coaction

- Example: Triangle, $m_i^2 = 0\,, p_i^2 \neq 0\,.$



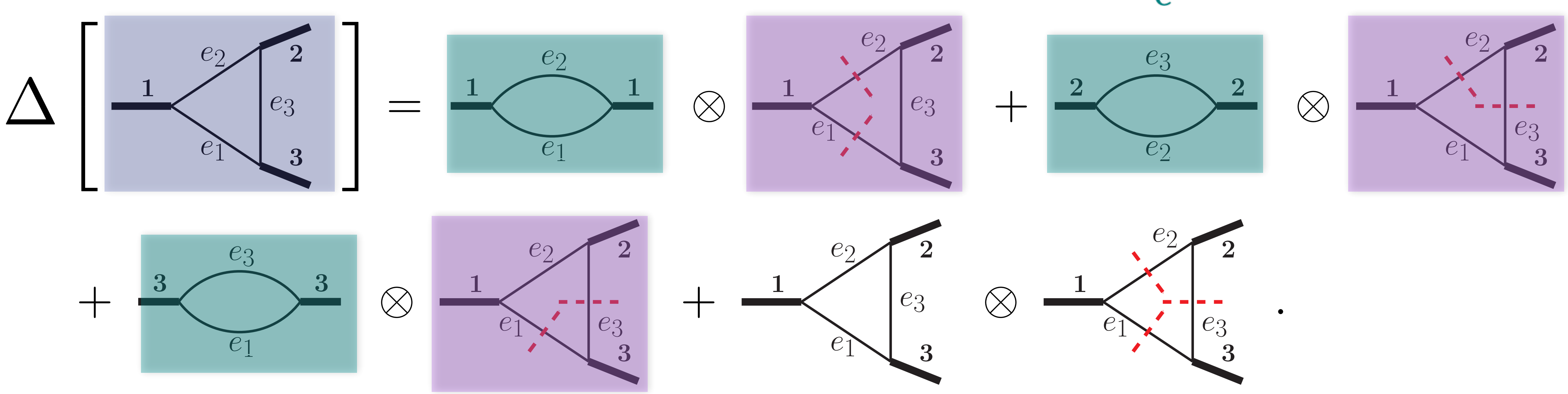
➡ Each graph represents a Laurent series in dimensional regularisation.

➡ Checked consistency of Laurent expansion and coaction up to terms of weight 4.

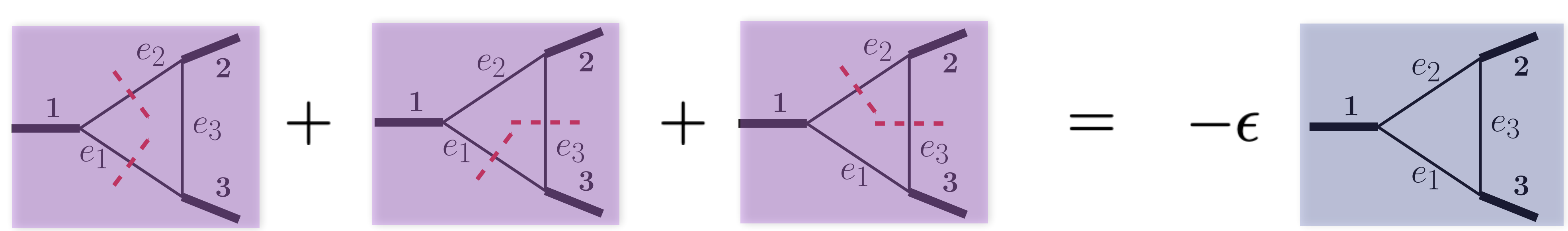
The diagrammatic coaction

- Example: Triangle, $m_i^2 = 0\,,\, p_i^2 \neq 0\,.$

Finite   Bubble integrals diverge: $-\frac{1}{\epsilon} + \ldots$

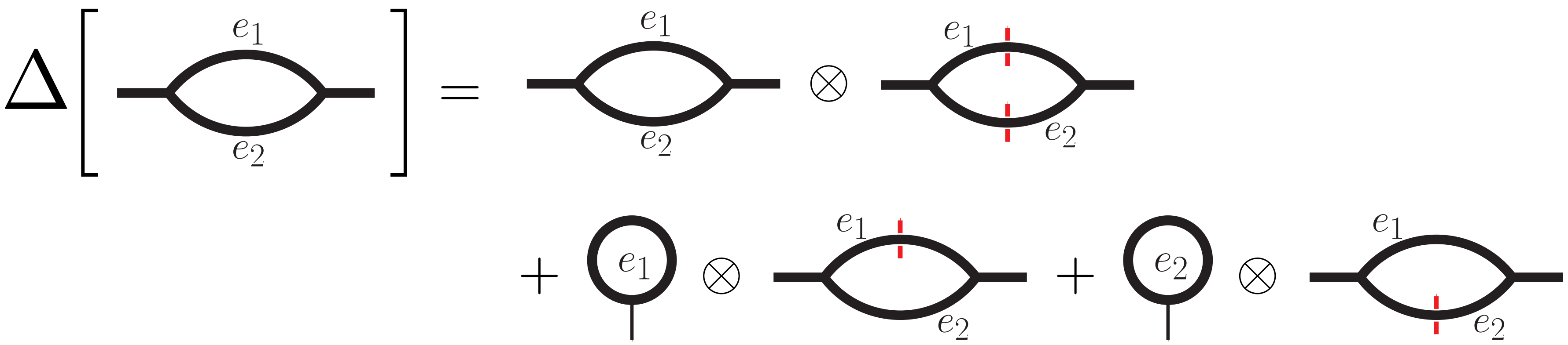


➡ Pole cancels due to relation among cut and uncut integrals:



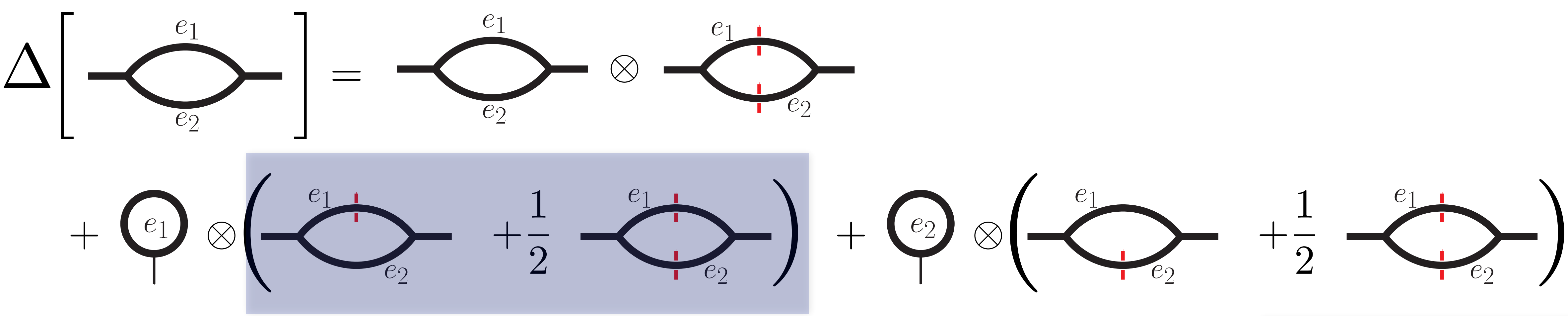
The diagrammatic coaction

- Bubble with massive propagators:



➡ This relation is incorrect…

- … but the following relation holds!



➡ Additional terms from $\Gamma_{\infty 1} = -2\Gamma_1 - \Gamma_{12} = -2\left(\Gamma_1 + \frac{1}{2}\Gamma_{12}\right)$.

The one-loop conjecture

- Conjecture: The coaction on one-loop integrals in DimReg can be represented entirely in terms of graphs: [Abreu, Britto, CD, Gardi]

$$(G,C) = \text{ one-loop graph } G \text{ with subset } C \text{ of propagators cut.}$$

$$\Delta(G,C) = \sum_{C\subseteq X} (G_X, C) \otimes \left[(G,X) + a_X \sum_{e\in E_G\setminus C} (G, X\cup e)\right]$$

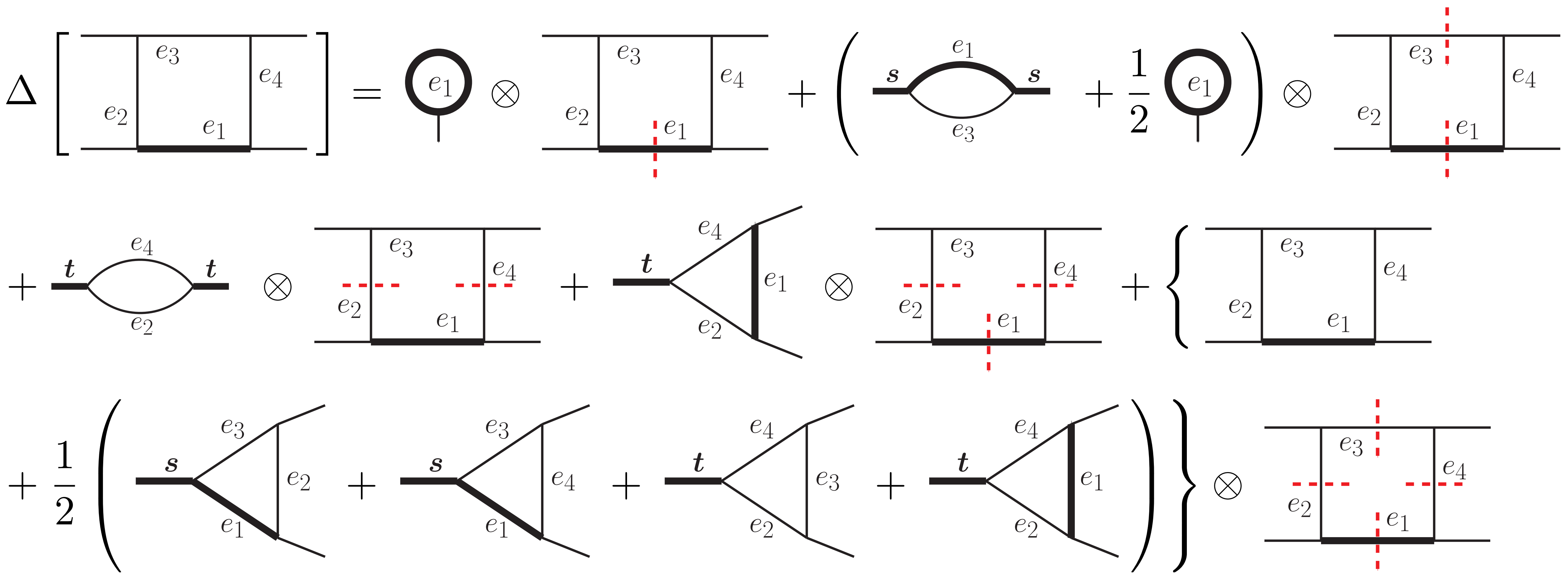
Graph with all edges pinched but those in X

$$a_X = \begin{cases} 1/2\,, & |X| \text{ odd}\,, \\ 0\,, & |X| \text{ even}\,. \end{cases}$$

➡ Recently proven to hold for (motivic) finite triangle and box integrals in D=4. [Tapuskovic]

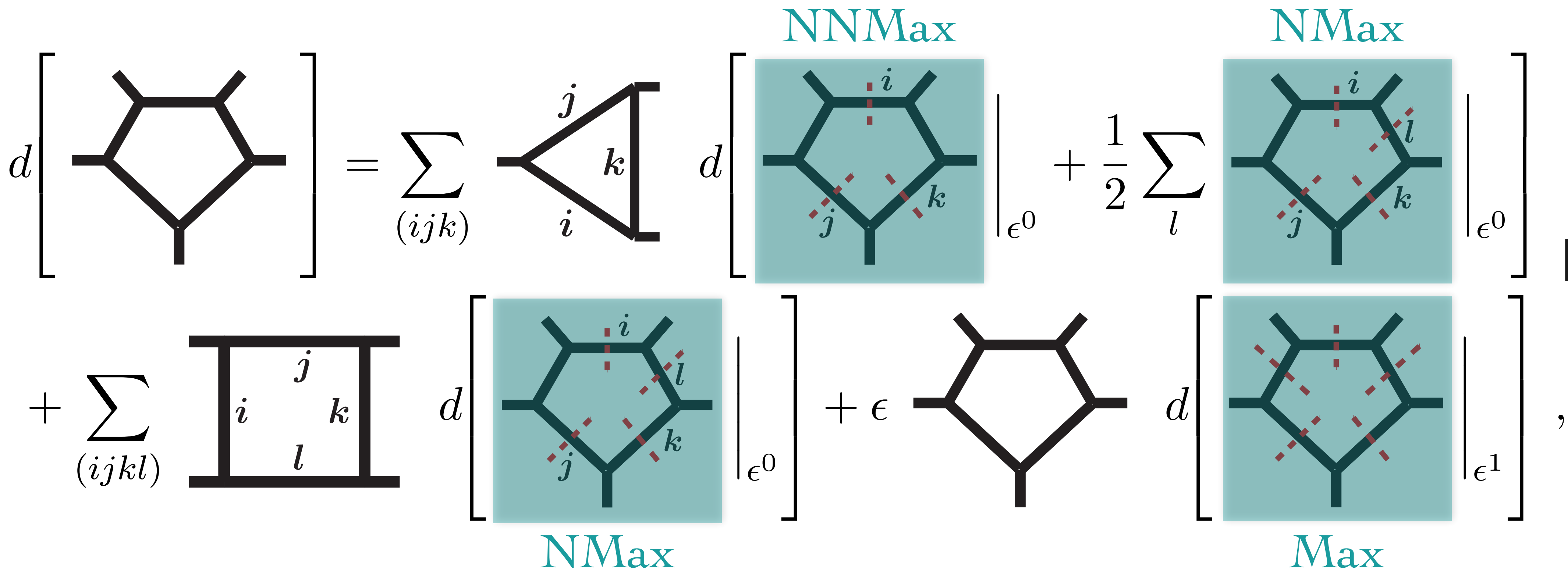
The one-loop conjecture

- Example:



Differential equations

- We can obtain differential equations [Henn] from the coaction, e.g.:



[Abreu, Britto, CD, Gardi]

➡ $d\log$-forms are related to maximal, next-to-maximal (NMax) and NNMax cuts.

➡ The relevant cuts can be computed for an arbitrary number of external and propagator masses!

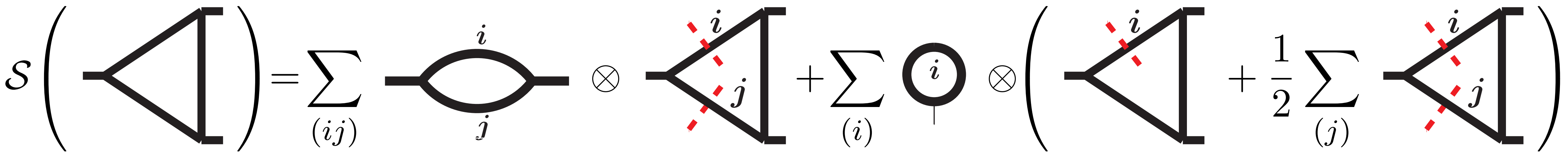
➡ We obtain explicit canonical differential equations for ALL one-loop integrals!

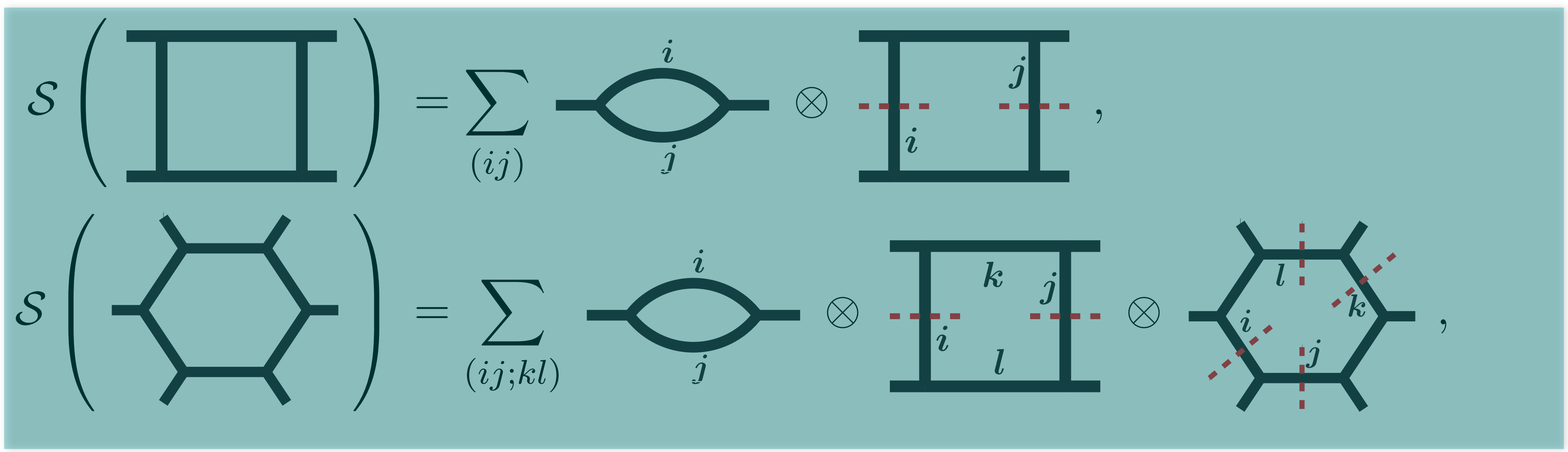
[Relation to Caron-Huot's talk?]

One-loop symbols

- From canonical differential equations one can obtain symbols.
- Example: ϵ^0 (higher orders are similar)

[Abreu, Britto, CD, Gardi]



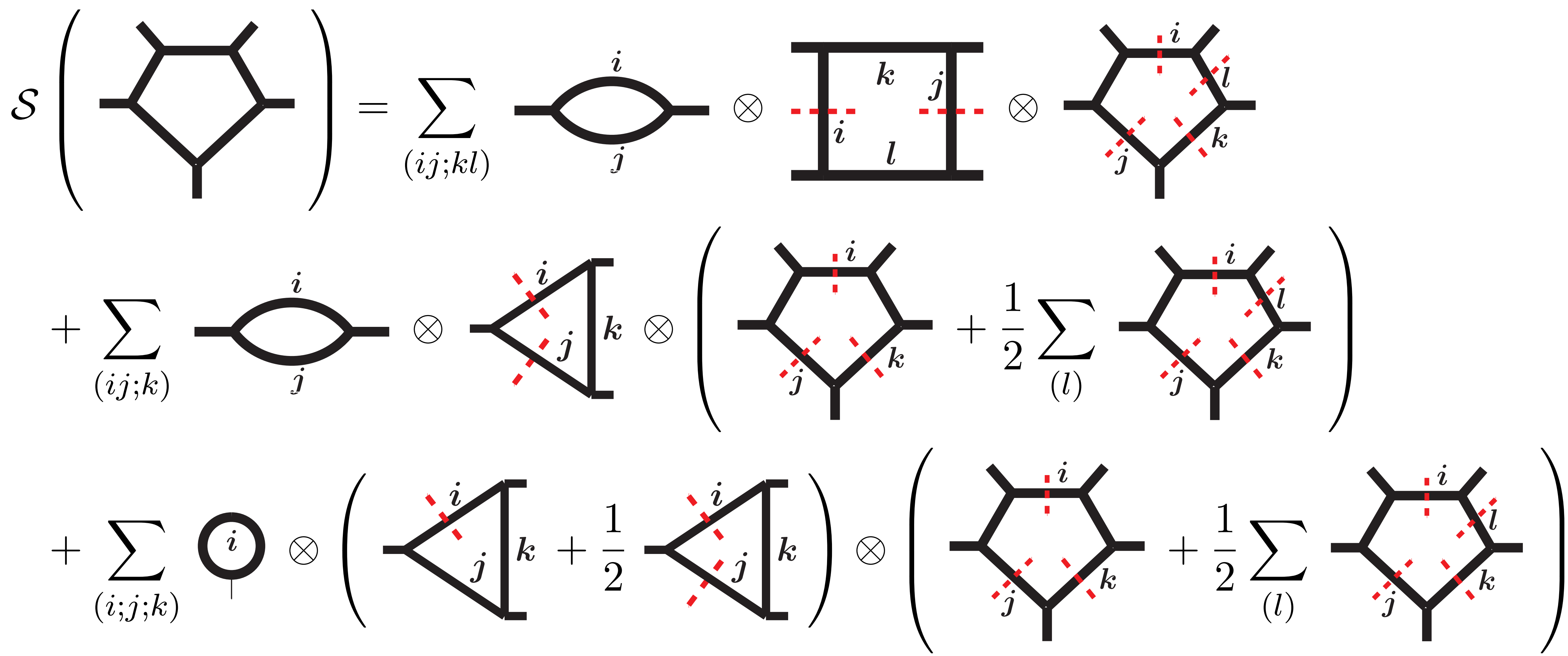


Dual conformally invariant

- Symbols only involve:
  - ➡ Bubble and tadpole integrals.
  - ➡ Max-, NMax-, NNMax-cuts from differential equation.

(See also [Spradlin, Volovich; Arkani-Hamed, Yuan; Herrmann, Parra-Martinez])

One-loop symbols



Conclusion

- Strong evidence that there is a coaction which:
  - ➡ acts on integrals associated to positive geometries,
  - ➡ is consistent with expansion in DimReg.
  - ➡ In some cases this was proven rigorously! [Brown, Dupont]
- When applied to one-loop, it results in a very compact representation of the coaction in terms of one-loop integrals and their cuts.
  - ➡ Non-trivial role played by relations between cuts/contours.
  - ➡ Proven for finite box and triangle integrals. [Tapuskovic]
- Ruth's talk: extension to other classes of positive geometry and two-loop integrals.