Power series and computation of multi-loop/multi-leg integrals

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Based on FM [arXiv:1907.13234]

and

Abreu, Ita, FM, Page, Tschernow, Zeng [arXiv: today],
Frellesvig, Hidding, Maestri, FM, Salvatori [arXiv:1911.06308],
Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, FM, Salvatori, Smirnov [arXiv:1907.13156]

Zoomplitudes 2020

Feynman integrals, a necessary evil

- Computing cross sections/scattering amplitudes in perturbation theory requires the computation of multi-loop/multi-leg Feynman integrals
- This is one of the main bottlenecks in making precise predictions for collider experiments (e.g. LHC)
- Much progress made in recent decades, but still an open problem

State-of-the-art technology

- Analytic methods (direct integration, differential equations, …)

  Identify a space of known (special) functions, and fit Feynman integrals in it:

  - Pro: Efficient numerical routines
  - Cons: Not algorithmic

We need an algorithm for efficient numerical evaluations (this talk)

- Numerical methods (Sector decomposition, Loop-Tree duality, …)
  - Pro: Algorithmic
  - Cons: Low precision, long evaluation time, numerical instabilities

Differential equations (DE)

- By using integration-by-parts identities (IBPs) we can reduce (scalar) Feynman integrals to a basis of "Master Integrals" (MIs)

  [Tkachov 1981][Chetyrkin, Tkachov 1981][Laporta 2000]

- This implies that we can define a system of first order linear differential equations for the MIs, of the form

vector: each component is a MI

$$\frac{\partial \mathbf{F}(\epsilon, x)}{\partial x} = \mathbf{A}(\epsilon, x)\mathbf{F}(\epsilon, x)$$

rational matrix

- Conjecture: all Feynman integrals admit a canonical basis such that

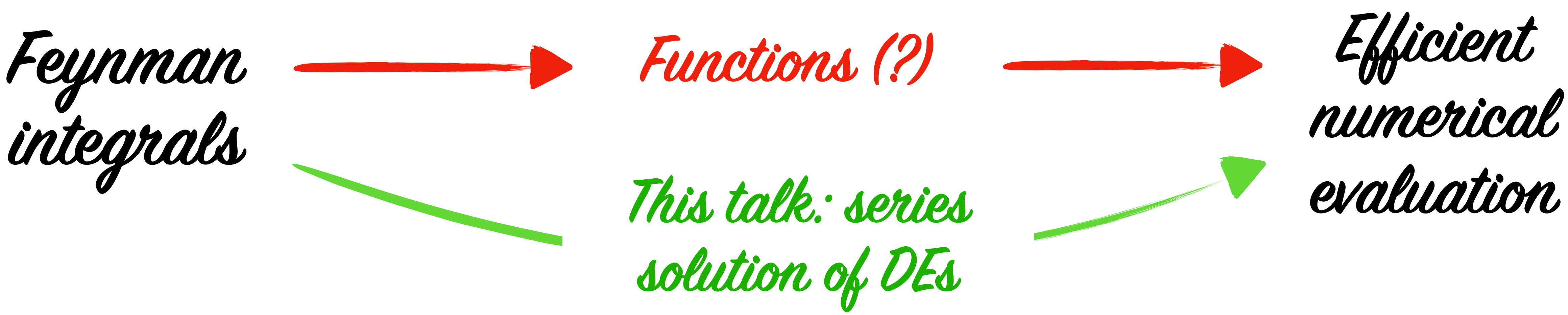
  [Henn 2013]

$$\frac{\partial \mathbf{I}(\epsilon, x)}{\partial x} = \epsilon\, \mathbf{M}(x)\mathbf{I}(\epsilon, x)$$

Solution in terms of known functions is difficult in general

This talk

- By-pass the "functions" step, and land directly to a representation suited for (efficient) numerical evaluations



Series solutions in the literature

- Single scale: series expansions have been mostly applied to single scale problems N3LO: [Anastasiou, Duhr, Dulat, Herzog, Mistlberger 2015][Mistlberger 2018]

- Several scales: series expansions have been performed with respect to one variable (parametrising special kinematic configurations)
  small Higgs p_T: [Bonciani, Degrassi, Giardino, Groeber 2018]
  small bottom mass: [Kudashkin, Melnikov, Tancredi, Wever 2017, 2018]

  - The coefficients of the series are special functions (still hard)

special functions (!)

$$\sum_{j_1 \in S} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{N_{i,k}} \mathbf{c}_{i,k}^{(j_1,j_2,j_3)} (t-t_k)^{j_1+j_2} \log(t-t_k)^{j_3}$$

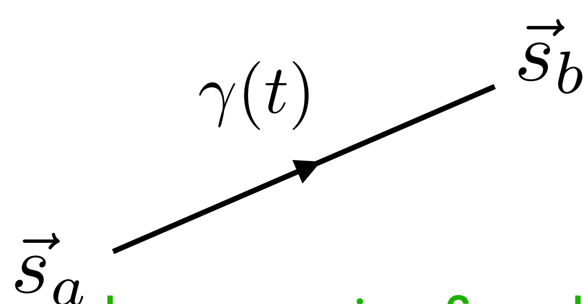
small perturbation around a kinematic limit

- In the multivariate case, it is desirable to have a systematic approach to obtain results in all points of the kinematic regions

key idea: reduce to one scale and expand

Set of DEs (one for each Mandelstam): $$\frac{\partial \mathbf{F}(\epsilon, \vec{s})}{\partial s_n} = \mathbf{A}(\epsilon, \vec{s})\mathbf{F}(\epsilon, \vec{s}), \quad n \in \{1, \dots, N\}$$

- Fix two points and define a contour connecting them



new point, to be computed

$$\vec{s}(t) = \vec{s}_a + (\vec{s}_b - \vec{s}_a)t, \quad t \in [0, 1]$$

known point, fixes boundary constants

- Differential equations along the contour given by:

$$\frac{d}{dt}\mathbf{F}(\epsilon, t) = \mathbf{A}(\epsilon, t)\mathbf{F}(\epsilon, t) \qquad \text{with} \qquad \mathbf{A}(\epsilon, t) = \sum_{n=1}^{N} \mathbf{A}_n(\epsilon, \vec{s}(t))\frac{ds_n(t)}{dt}$$

- One-dimensional problem: transport the solution by series expanding along the contour (done on-the-fly for every new point)

[FM 2019]

(local) series solution

- Canonical polylogarithmic case:

DEs: $\frac{d}{dt}\mathbf{I}(t,\epsilon) = \epsilon\mathbf{A}(t)\mathbf{I}(t,\epsilon)$ solution: $\mathbf{I}^{(i)}(t) = \int \mathbf{A}(t)\mathbf{I}^{(i-1)}(t)dt + \mathbf{c}^{(i)}$

- series expand the DEs matrix around a point

$$\mathbf{A}(t) = \sum_{i=-2}^{\infty} \mathbf{A}_{i,k}(t-t_k)^{\frac{i}{2}},$$

constant matrices

$$\mathbf{I}_k^{(i)}(t) = \sum_{j=-2}^{\infty} \mathbf{A}_{j,k} \int (t-t_k)^{\frac{j}{2}} \mathbf{I}_k^{(i-1)}(t)dt + \mathbf{c}_k^{(i)}$$

- solution is trivial: integrals always of the form $\int (t-t_k)^{\frac{i}{2}} \log(t-t_k)^j dt,$

$$\mathbf{I}_k^{(i)}(t) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{N_{i,k}} \mathbf{c}_k^{(i,j_1,j_2)}(t-t_k)^{\frac{j_1}{2}} \log(t-t_k)^{j_2},$$

constants

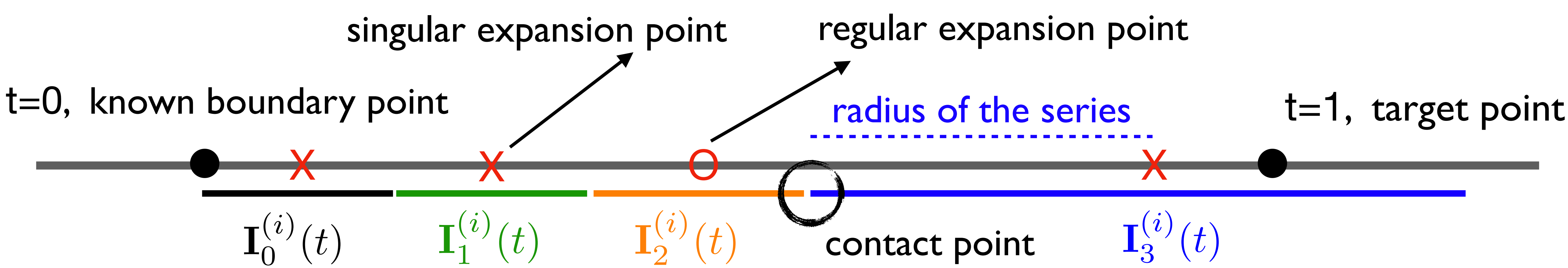
- General case: series solution found by using the Frobenius method (note: Frobenius method only applies to single-scale DEs)

For a review see [Coddington 1955]

Patching the contour

- Local series solution converges up to the closest singular point: need multiple series to patch the contour

- Truncated series: to ensure fast convergence, radius set to half distance between the expansion point and closest singularity



- The series depend on boundary constants fixed by using boundary point and continuity at the contact points.

  By setting each series to zero outside its radius:

$$\mathbf{I}^{(i)}(t) = \mathbf{I}_0^{(i)}(t) + \mathbf{I}_1^{(i)}(t) + \mathbf{I}_2^{(i)}(t) + \mathbf{I}_3^{(i)}(t), \quad t \in [0,1]$$

Analytic continuation

- Solution (around a physical threshold) always of the form

half-integer

$$\mathbf{I}_k^{(i)}(t) = \sum_{j_1 \in S} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{N_{i,k}} \mathbf{c}_k^{(i,j_1,j_2,j_3)} (t-t_k)^{j_1+j_2} \log(t-t_k)^{j_3}$$

finite set of non-integer numbers

need analytic continuation

- The analytic continuation of the series is straightforward:

Assume expansion point corresponds to a physical threshold in the s_n channel

Feynman prescription:

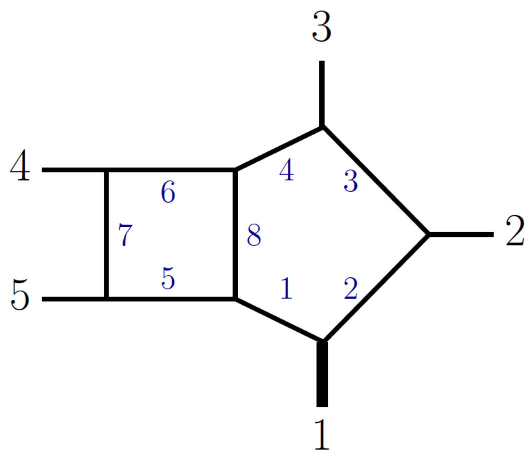
$$s_n(t) = \alpha_n + \beta_n t \quad \to \quad \alpha_n + \beta_n t + i\delta$$

equivalent to:

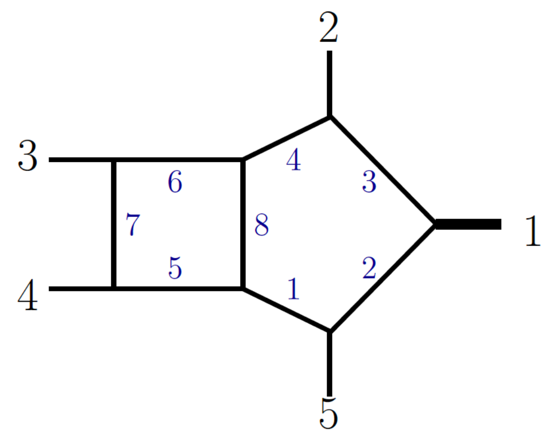
$$t \to t + i\,\mathrm{sign}(\beta_n)\delta, \quad \delta > 0$$

All planar integrals for 5 points 1 mass

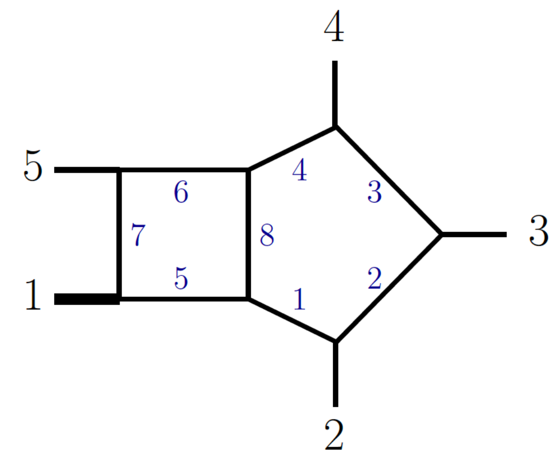
[Abreu, Ita, FM, Page, Tschernow, Zeng, today]



74 MI's
38 letters



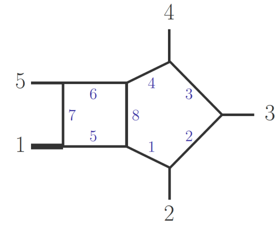
75 MI's
48 letters

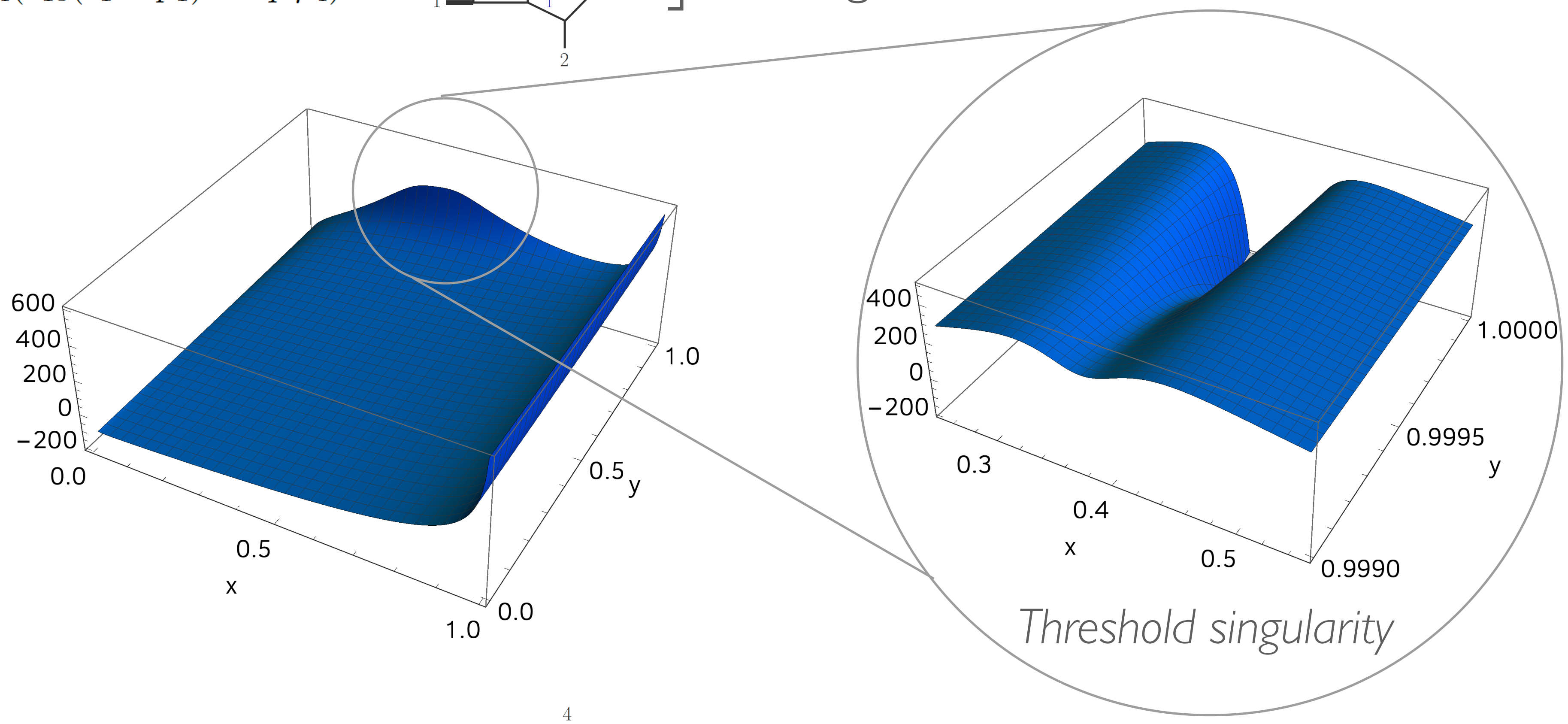


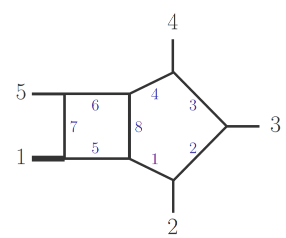
86 MI's
49 letters

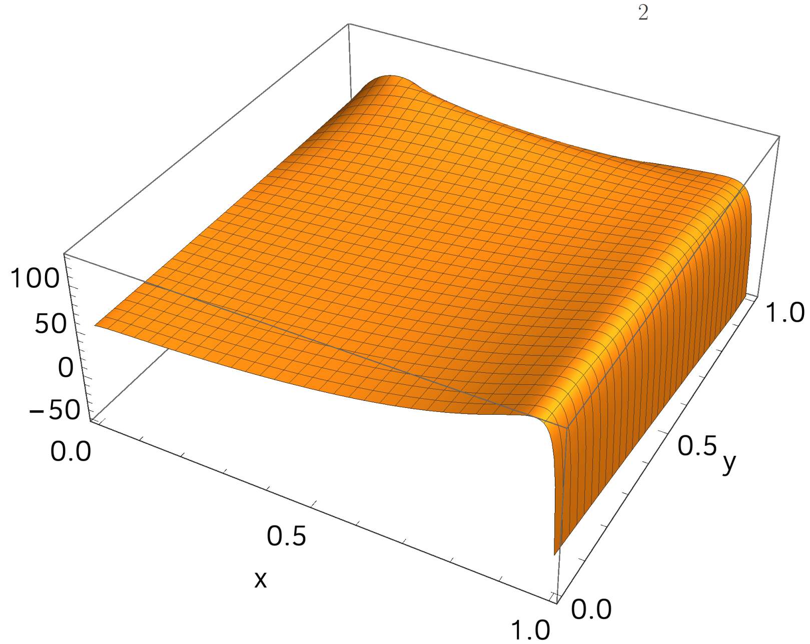
- Vector-boson + 2 jets production, very challenging, 6 physical scales !
- Differential equations in d-log canonical form
- Algebraic alphabet, analytic solution out of reach (for the moment?)
- Series solution is fully algorithmic. Results in all kinematic regions. Boundary conditions computed by requiring absence of spurious logarithmic singularities.

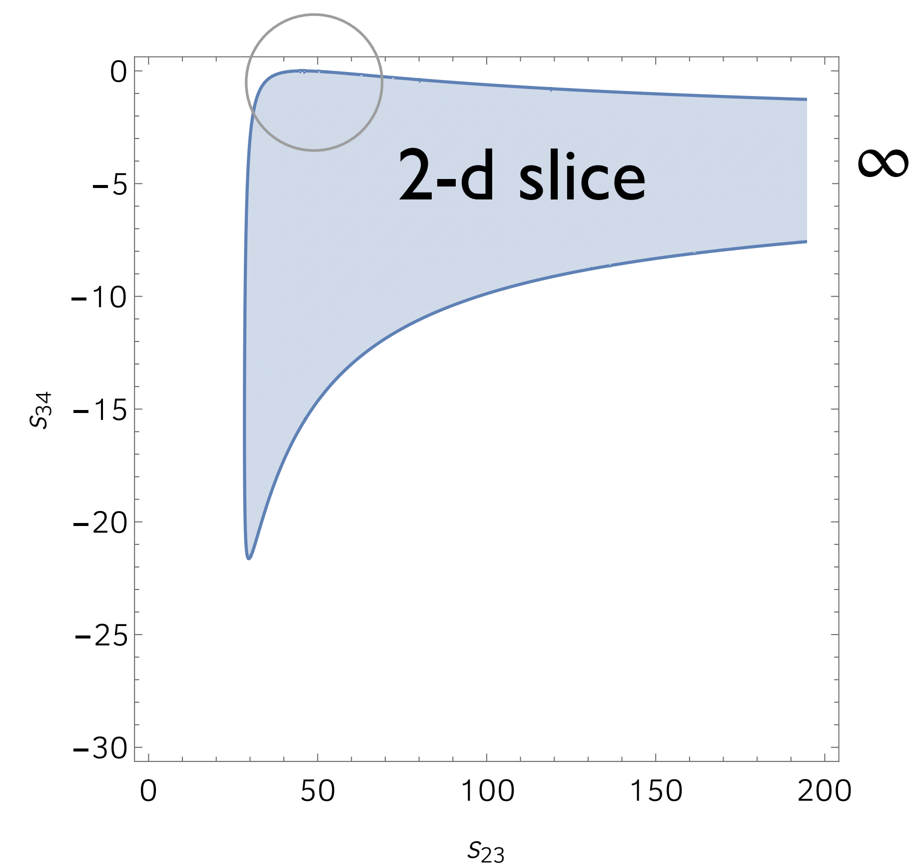
Plots over 2-d slice of the physical region (sampled 400k points)

Re[$\epsilon^4 s_{23} s_{34}(s_{15}(\ell_1 - p_1)^2 - q^2\rho_4)$ X ] @ weight 4



Im[$\epsilon^4 s_{23} s_{34}(s_{15}(\ell_1 - p_1)^2 - q^2\rho_4)$ X ] @ weight 4

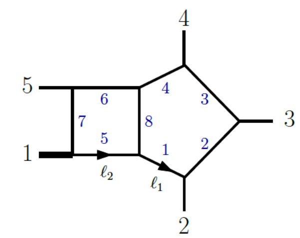




weight 4 functions at the physical point $(p_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}) = \left(\frac{137}{50}, -\frac{22}{5}, \frac{241}{25}, -\frac{377}{100}, \frac{13}{50}, \frac{249}{50}\right)$

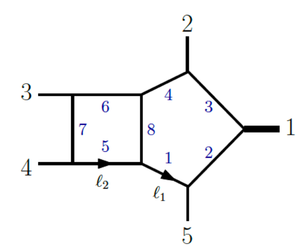
128 digits displayed

$$\epsilon^4 s_{23} s_{34}(s_{15}(\ell_1 - p_1)^2 - q^2 \rho_4) \quad \mathrm{X}$$



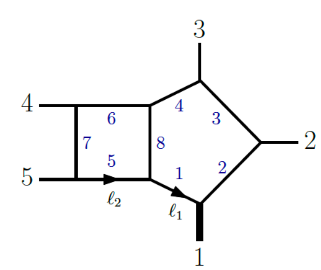
+ 11.90852968084159332956737844434123149462154481781376319628152875108573403240172119021456072482970507632118904927772802888920384959
- 143.83838235097336513553728991658286648264414416047763179372404771611422704901446783351109298575962776931534808819900977030065443464

$$\epsilon^4 s_{34}(s_{15} s_{12} - q^2 s_{34})(\ell_1 - p_4)^2 \quad \mathrm{X}$$



+ 44.16216574473530086723311855418285332220947385104364702071229029261021307762640909660570697034340991726669529989754679457820400754
- 46.21874613385033996994440307755667843436468684075080312553966214577041860195711346093023106313832993798068588174797894436835940420

$$\epsilon^4 s_{45} s_{12} s_{23}(\ell_1 - p_5)^2 \quad \mathrm{X}$$

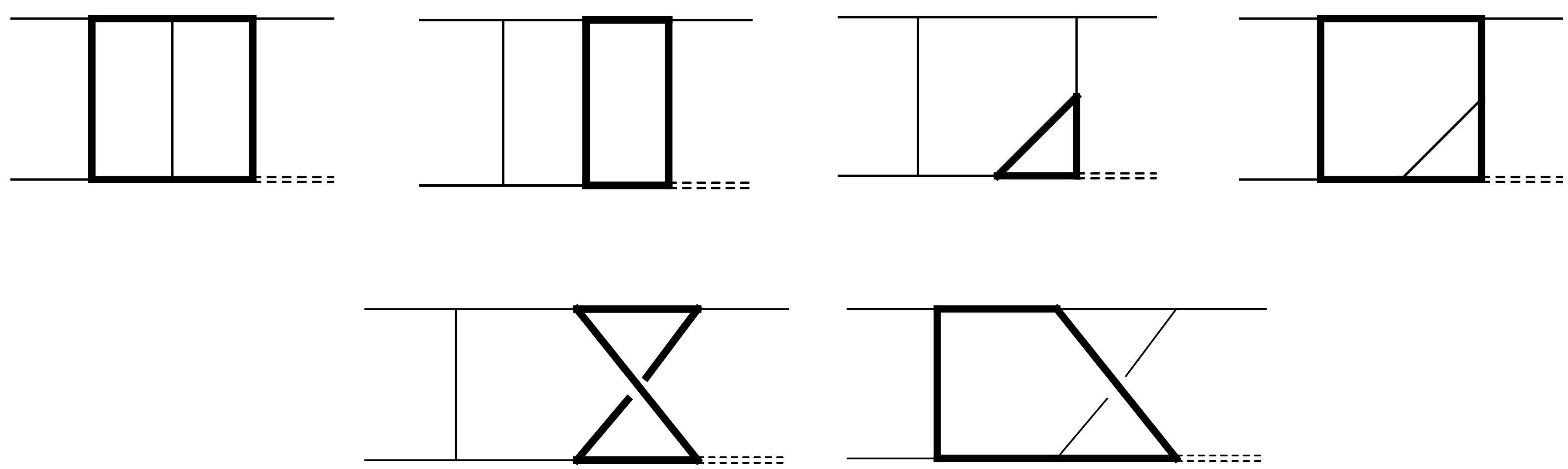


+ 29.80276365179310881202389321759335130735012172284500683744950383361580854859249253152495792891906443533746205253030322662311313030
+ 273.86627846266515113913295225572416419016316389639992895351266869752653041314721274860621652883763233020089289165122764272839009922

H+3 partons amplitude

- 6 integral (elliptic) families (with exact mass dependence)

[Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, FM, Salvatori, Smirnov 2016-2019]

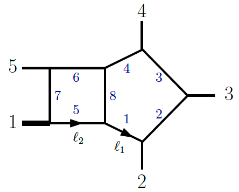


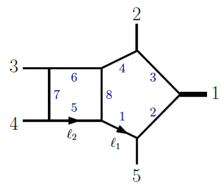
- QCD NNLO Higgs inclusive cross section and the NLO corrections H+Jet production (including bottom and top mass effects)

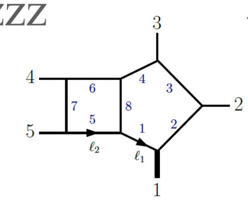
All the required master integrals are now known !

See [Czakon, Niggetiedt, 2020] for the 3-loop form factor

Timing and precision


zzz


zmz

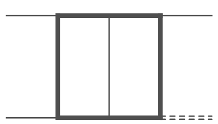

mzz

Time per point per MI (up and including weight 4, one CPU thread)

	Family	MI's	time per MI (s)	total time (s)	truncation order
32 digits	zzz	86	2.08	179	$70 < n_k < 140$
	zmz	75	2.24	168	
	mzz	74	1.69	125	

time per point: averaged over ~10,000 SHERPA Monte Carlo phase-space points for LO cross section

Series

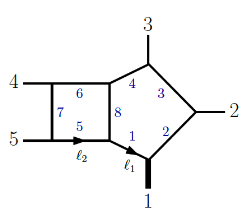

A (elliptic)

Series

	Family	MI's	time per MI (s)	total time (s)	truncation order
32 digits	A (Series)	73	2.21	162	$70 < n_k < 140$
2 digits	A(FIESTA 4.1)	73	821	60000	
2 digits	A(SecDec 1.4.3 QMC)	73	342	25000	

SecDec

FIESTA


mzz

Series

	Family	MI's	time per MI (s)	total time (s)	truncation order
128 digits	mzz (Series)	74	16	1072	$140 < n_k < 240$
128 digits	mzz (MPLs, PTW)	74	82	6090	

MPLs [Papadopoulos, Tommasini, Wever, 2015]

- timing is order 1 second per integral with quadruple precision, on 1 cpu thread
- timing and precision comparable to (generic) polylogarithmic results

Conclusions and Outlook

- We discussed a systematic method to compute multi-loop / multi-leg integrals based on series solutions of differential equations

- The method is efficient, allows for high numerical precision in all kinematic regions, suitable for Monte Carlo integrations

- We presented new results for several state-of-the-art integrals out of reach with analytic methods (for the moment ?)

- We believe that this method will have a central role in the computation of several LHC observables