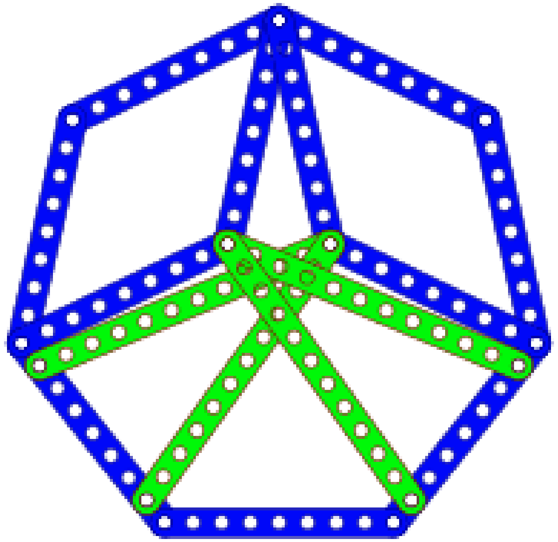
Lifting Heptagon Symbols to Functions



Lance Dixon (SLAC)

LD, Y.-T. (Andy) Liu, 20mm.nnnnn

Amplitudes 2020

image:wikiwand/Meccano

Scattering Amplitudes in Planar N=4 SYM

- Maximally supersymmetric N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in the large N_c (planar) limit
- Structure of n-point amplitudes is very rigid:

$$A_n = \sum_i rational_i \times transcendental_i$$

- For planar N=4 SYM, understand rational structure quite well, focus on the transcendental functions.
- Space of functions is so restrictive, and physical constraints are so powerful, can write L loop answer as linear combination of known weight $2L$ polylogarithms.
- Unknown coefficients found simply by solving linear constraints (sometimes a large number)

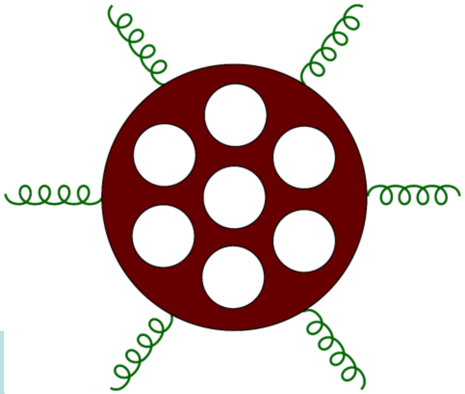
Hexagon function bootstrap

3 LD, Drummond, Henn, 1108.4461, 1111.1704;

4,5 Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;

6,7 Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890 and 1906.07116; LD, Dulat, 20mm.nnnnn (NMHV 7 loops)

Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops



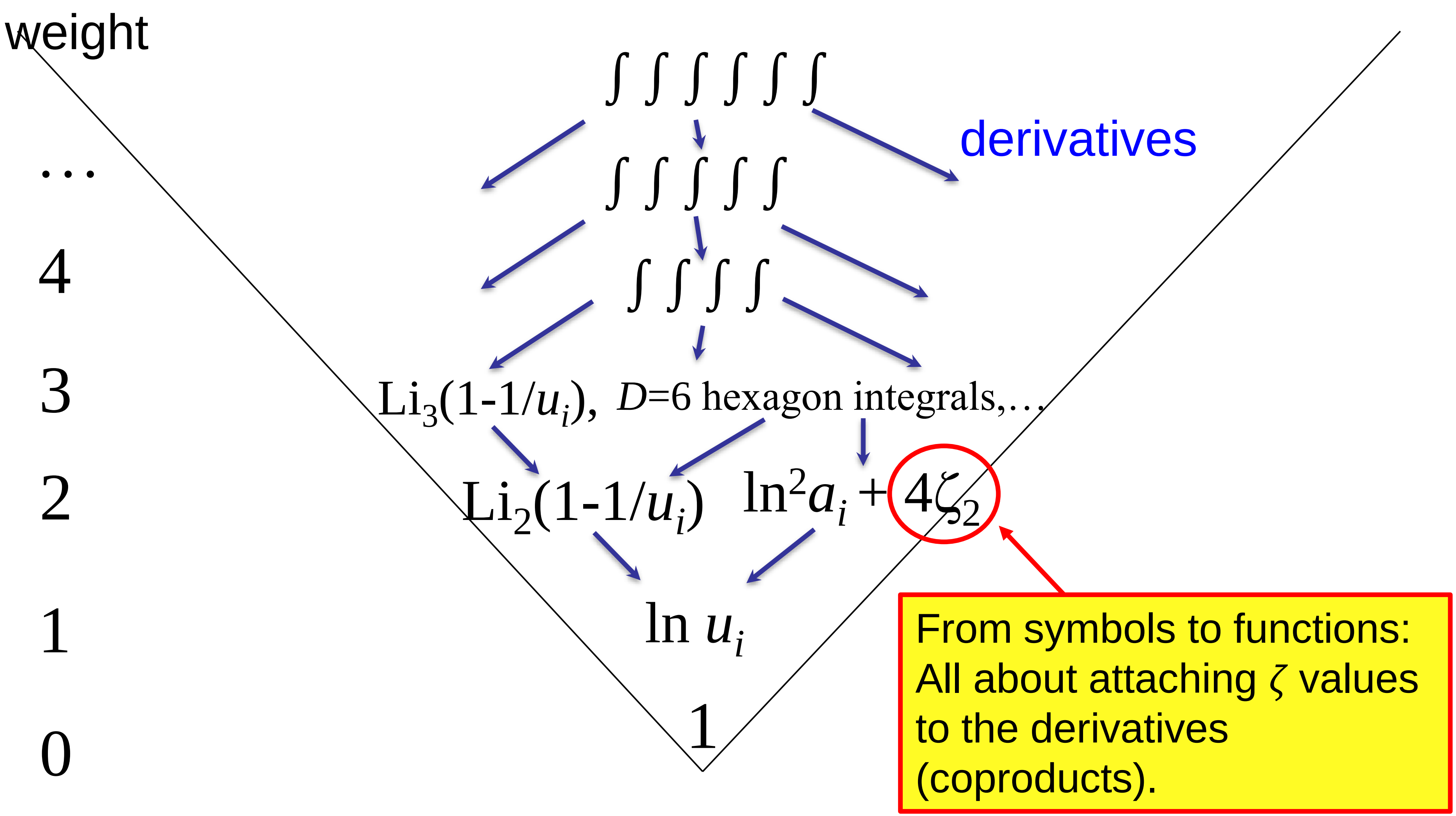
First step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

But at function level it was mostly limited to 6 point amplitudes!

High loop orders beyond 6 points

- Cluster algebras provide strong clues to right polylogarithmic function space
  Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Foster, Gurdogan, 1710.10953
- Symbol of 3-loop MHV 7-point amplitude bootstrapped first.
  More rigid than 6 points; fewer constraints needed!
  Drummond, Papathanasiou, Spradlin 1412.3763
- With Steinmann relations, could go to 4-loop MHV and 3-loop non-MHV symbol LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976
- Extended Steinmann → 4-loop NMHV symbol
  Drummond, Gurdogan, Papathanasiou, 1812.04640
- **Still need to go from symbols → actual functions!**

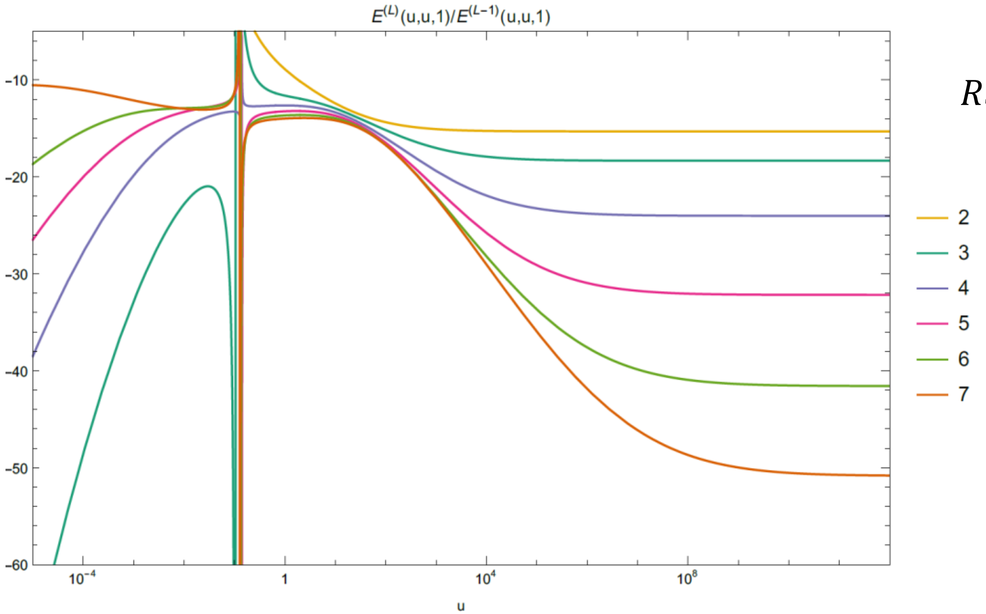
Heuristic view of hexagon or heptagon function space



Why functions?

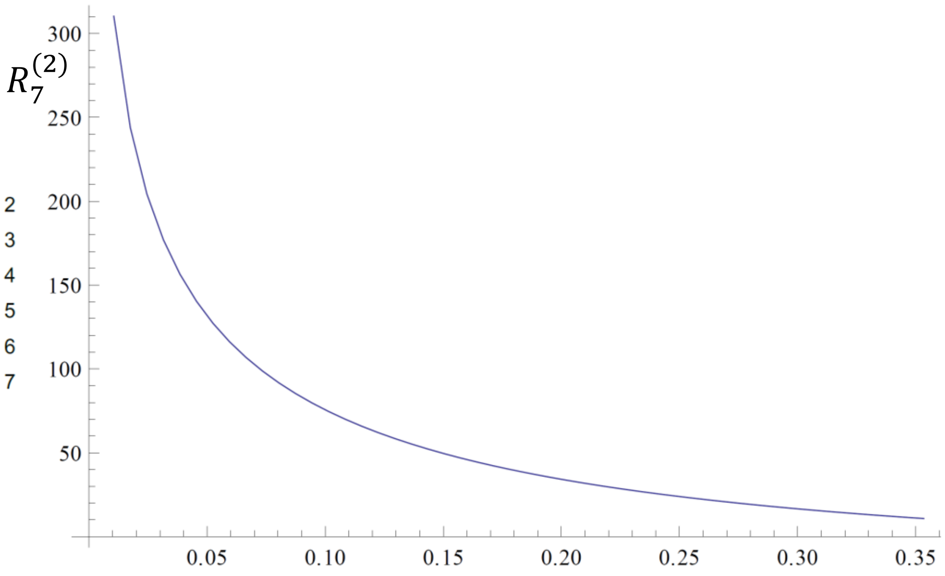
Motivation 1: "LHC doesn't measure symbols"

Lots of plots available for $n = 6$, almost nothing for $n = 7$



$n = 6,\ L = 1,2,3,4,5,6,7$

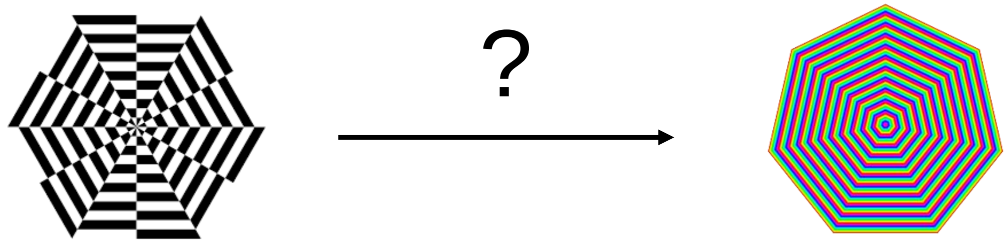
LD, Dulat, 20mm.nnnnn; see also Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890



$n = 7,\ L = 2$

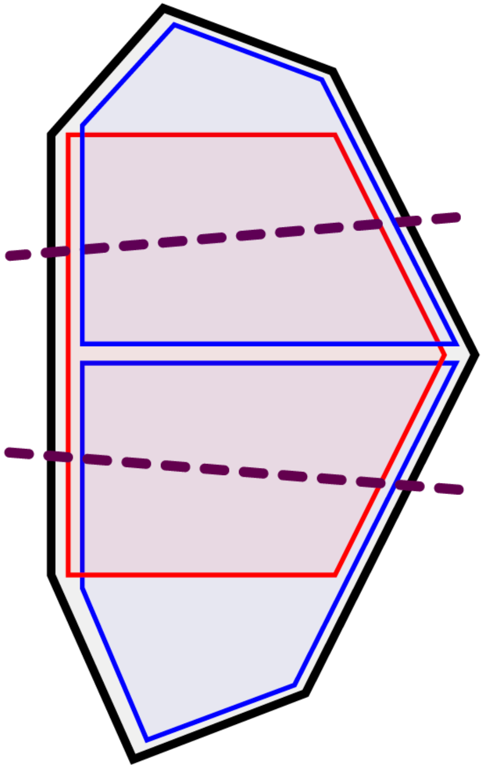
Golden, Spradlin, 1406.2055;
see also Golden, McLeod, 1810.12181;
Bourjaily, Volk, von Hippel, 1912.05690

Motivation 2: Every Superhero/amplitude needs an Origin Story



- For $n = 6$, when all 3 cross ratios $u_i \to 0$, logarithm of MHV amplitude is remarkably simple, $\propto c(\zeta)\ln^2 u_i$ through $L = 7$
  Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1960.07116
- Zeta-valued coefficients now understood to all orders
  Basso, LD, Papathanasiou, 2001.05460; hexagon origin story next by B. Basso
- **How about $n = 7$?**
- Analogous origin (due to Gram det): first 6 $u_i \to 0$, $u_7 \to 1$
- MHV symbol, known to 4 loops, is consistent with this, but provides no ζ-valued information
  LD, Drummond, Harrington, McLeod, Papathanasiou, Spradlin, 1612.08976

Motivation 3: Near collinear (OPE) limit



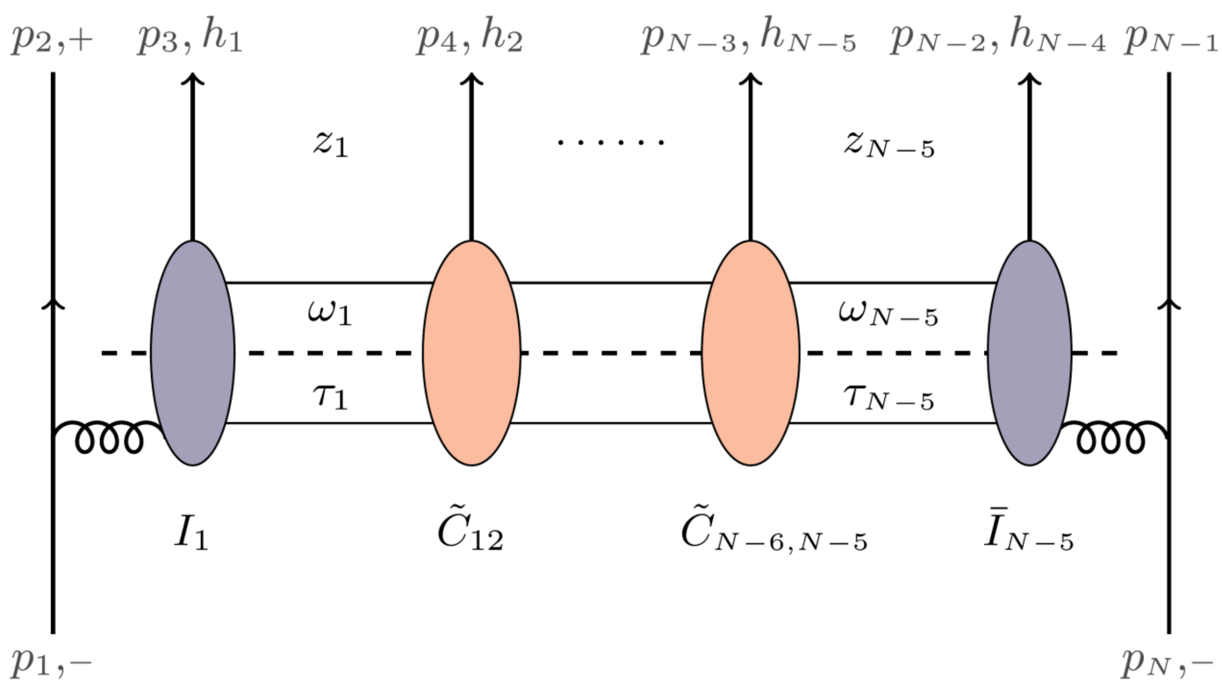
Test full Pentagon OPE framework
Basso, Sever, Vieira, 1303.1396,…
to higher loop orders at function level

$$\mathcal{W}_{\rm hep} = \sum_{\mathbf{a},\mathbf{b}} \int d\mathbf{u}\, d\mathbf{v}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v}) P_{\mathbf{b}}(\bar{\mathbf{v}}|0)$$
$$e^{-E(\mathbf{u})\tau_1 + ip(\mathbf{u})\sigma_1 + im_1\phi_1 - E(\mathbf{v})\tau_2 + ip(\mathbf{v})\sigma_2 + im_2\phi_2}$$

Motivation 4: Multi-Regge limit

- Test recent all orders conjecture for central emission block, which first appears at $n = 7$

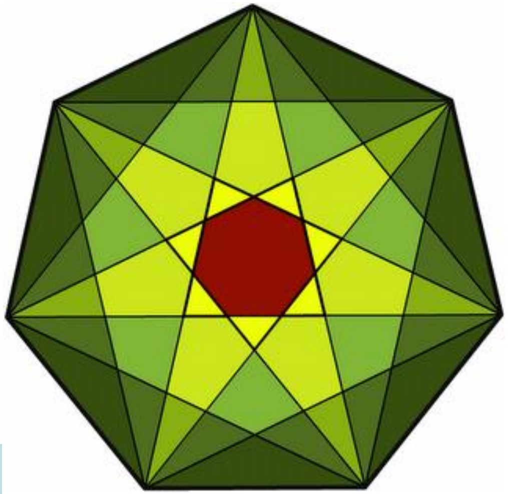
Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek, 1912.00188



Motivation 5: Coaction Principle

- Restricted set of zeta values & functions appear for $n = 6$
- For example, no independent ζ_2 or ζ_3
  Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1906.07116
- In line with previous observations of a **coaction principle** for ϕ^4 theory, electron g-2
  Schnetz, 1302.6445; Brown, 1512.06409; Panzer, Schnetz, 1603.04289; Schnetz, 1711.05118
- **Lower loop/weight "missing" values constrain/predict missing values at higher loops.**
- **What happens for $n = 7$?**

Outline of rest of talk



1. Iterated integral/symbol refresher
2. “co” surface interpolating between origin and soft/collinear limits
3. Fixing ζ values using branch cut conditions
4. Fixing amplitudes
5. Plots & comments
6. Conclusions

Iterated integrals

Chen; Goncharov; Brown

- Generalized polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives: $$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d\ln s_k$$

$\mathcal{S}$ = finite set of rational expressions, "symbol letters", and

$f^{s_k} \equiv \{n-1,1\}$ component of a "coproduct" Δ

f^{s_k} are also pure functions, weight n-1. [still contain ζ values]

- Iterate: $d\,f^{s_k} \Rightarrow f^{s_j, s_k} \equiv \{n-2,1,1\}$ component
- Symbol = {1,1,…,1} component (maximally iterated) [no ζ's]

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Symbol letters = polylog arguments

Projectively invariant ratios of 4-brackets of momentum twistors Z_i
Hodges, 0905.1473

$$< ijkl > \equiv \det(Z_i Z_j Z_k Z_l)$$

- **Hexagon, 3 variables (u, v, w), 9 letters:**

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$u = \frac{<1236><3456>}{<1346><2356>} \qquad 1-u = \frac{<1356><2346>}{<1346><2356>} \qquad y_u = \frac{<1236><2456><1345>}{<1235><1246><3456>}$$

+ 6 more from cyclic permutations

Goncharov, Spradlin, Vergu, Volovich, 1006.5703;
LD, Drummond, Henn, 1108.4461

Heptagon: 6 variables, u_i , 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle},$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

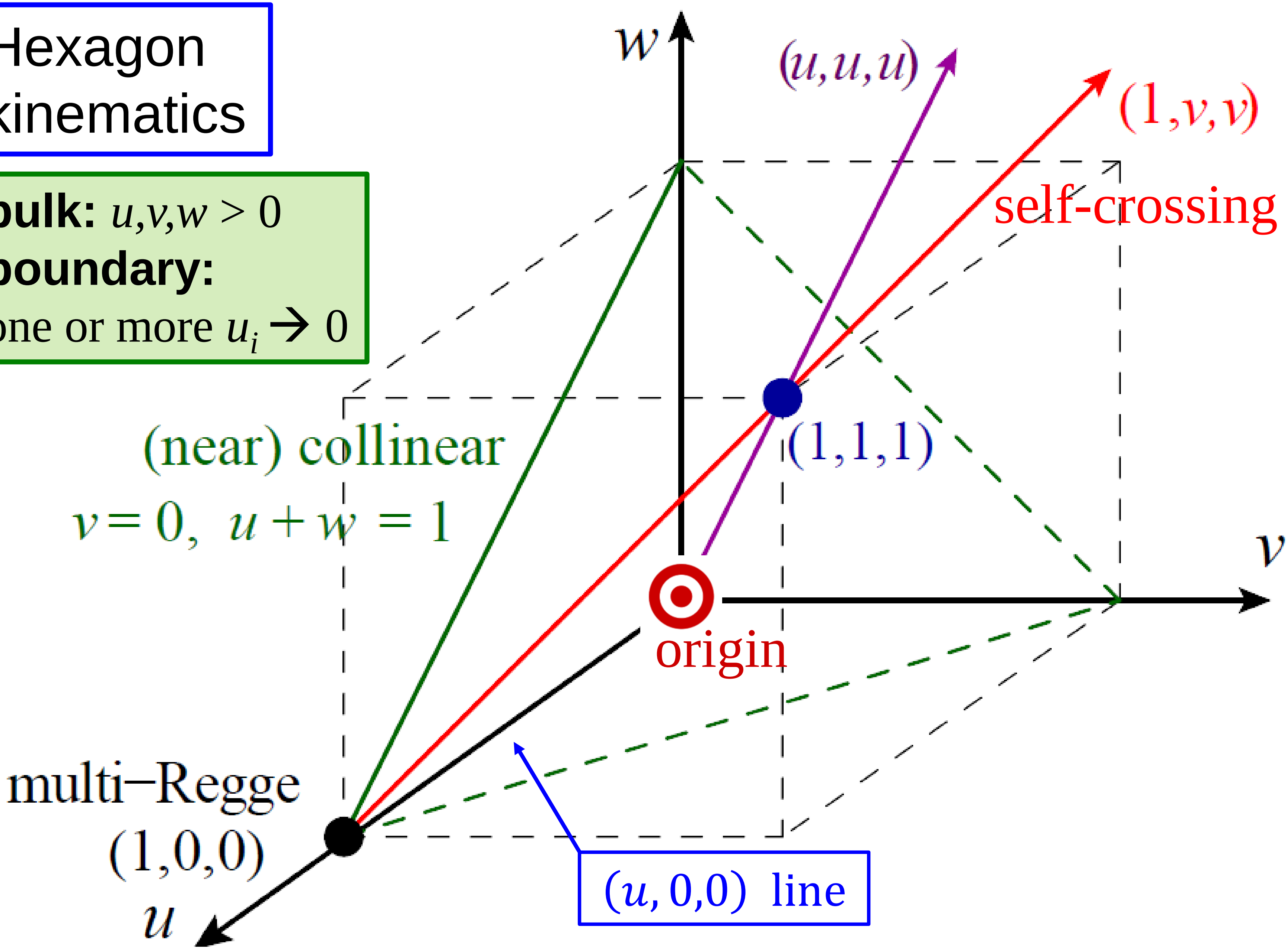
$$u_1 = \frac{a_{17}}{a_{13}a_{14}}$$

plus cyclic, $i \rightarrow i+1 \pmod 7$, $a_{ji} \rightarrow a_{j,i+1}$ **(6 x 7 = 42 letters)**

Gram determinant 4th order constraint $G(u_i) = 0$

Hexagon kinematics

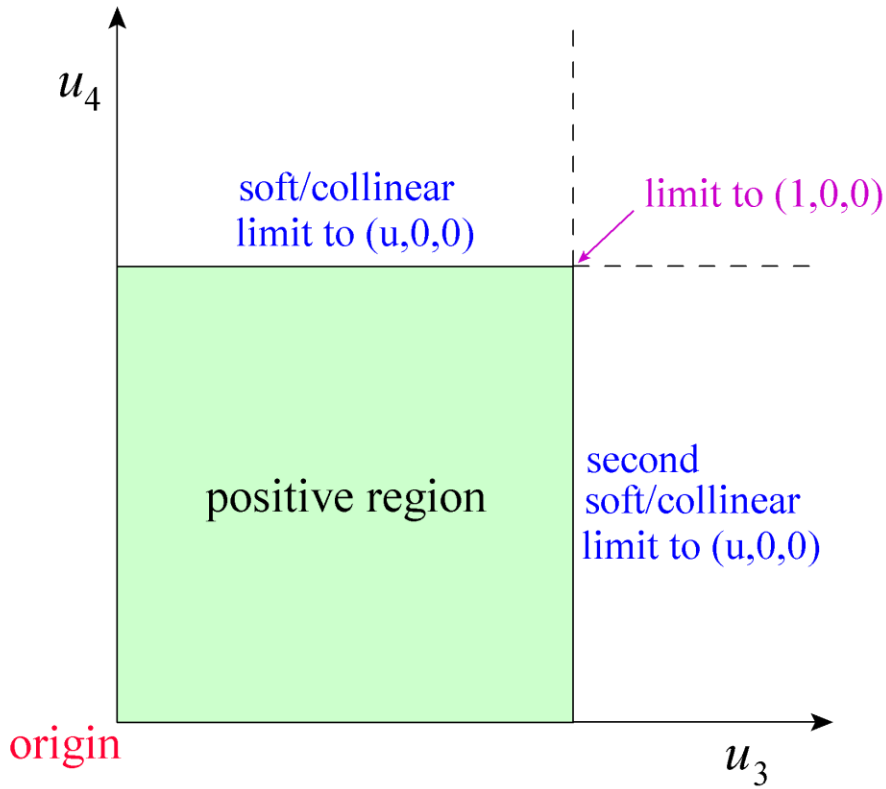
bulk: $u,v,w > 0$
boundary:
one or more $u_i \to 0$



Very useful 2d slice of heptagon kinematics:

"co" surface on boundary

interpolates between soft/collinear limits and origin



$u_{1,2,5,6} << 1$ $\qquad u_7 = 1$

Symbol letters on co surface

$$\{u_1, u_2, u_5, u_6, u_3, 1-u_3, u_4, 1-u_4, 1-u_7\}$$

- Remarkably simple! In fact, coefficients of $1-u_7$ coproduct always vanish, as a branch-cut constraint.
- Function space is just single-argument harmonic polylogs:

$$\{\ln(u_i), i = 1,2,5,6\} \otimes \{H_{\vec{w}}(u_3)\} \otimes \{H_{\vec{w}}(u_4)\}, \quad w_k \in \{0,1\}$$

- **“co” surface is two copies of 6-point $(u, 0,0)$ line**

co surface as a triple scaling limit

- BSV parametrization: $T_i, S_i, F_i, i = 1{,}2$

Basso, Sever, Vieira
1306.2058

$$
(Z_1 \ \ldots \ Z_7) = \begin{pmatrix} \frac{S_1}{\sqrt{F_1}} & 1 & -1 & -S_2\sqrt{F_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{T_2\sqrt{F_2}} & \frac{S_2+T_2F_2}{T_2S_2\sqrt{F_2}} & 1 & \frac{1}{S_1\sqrt{F_1}} \\ \frac{\sqrt{F_1}}{T_1} & 0 & 0 & -\frac{1}{T_2\sqrt{F_2}} & -\frac{1}{T_2\sqrt{F_2}} & 0 & \frac{\sqrt{F_1}}{T_1} \\ T_1\sqrt{F_1} & 0 & 1 & \frac{1+T_2S_2F_2+T_2^2}{T_2\sqrt{F_2}} & \frac{1}{T_2\sqrt{F_2}} & 0 & 0 \end{pmatrix}
$$

- OPE limit: $T_i \to \epsilon\, T_i, \quad \epsilon \to 0$

BSV, 1407.1736

- Double scaling limit: $T_i \to \epsilon\, T_i, \quad F_i \to \frac{F_i}{\epsilon}, \quad \epsilon \to 0$
- co limit: $T_i \to \epsilon\, T_i, \quad S_i \to \frac{S_i}{\epsilon}, \quad F_i \to \frac{F_i}{\epsilon^2}, \quad \epsilon \to 0$
- Suggests studying similar limits for $n > 7$

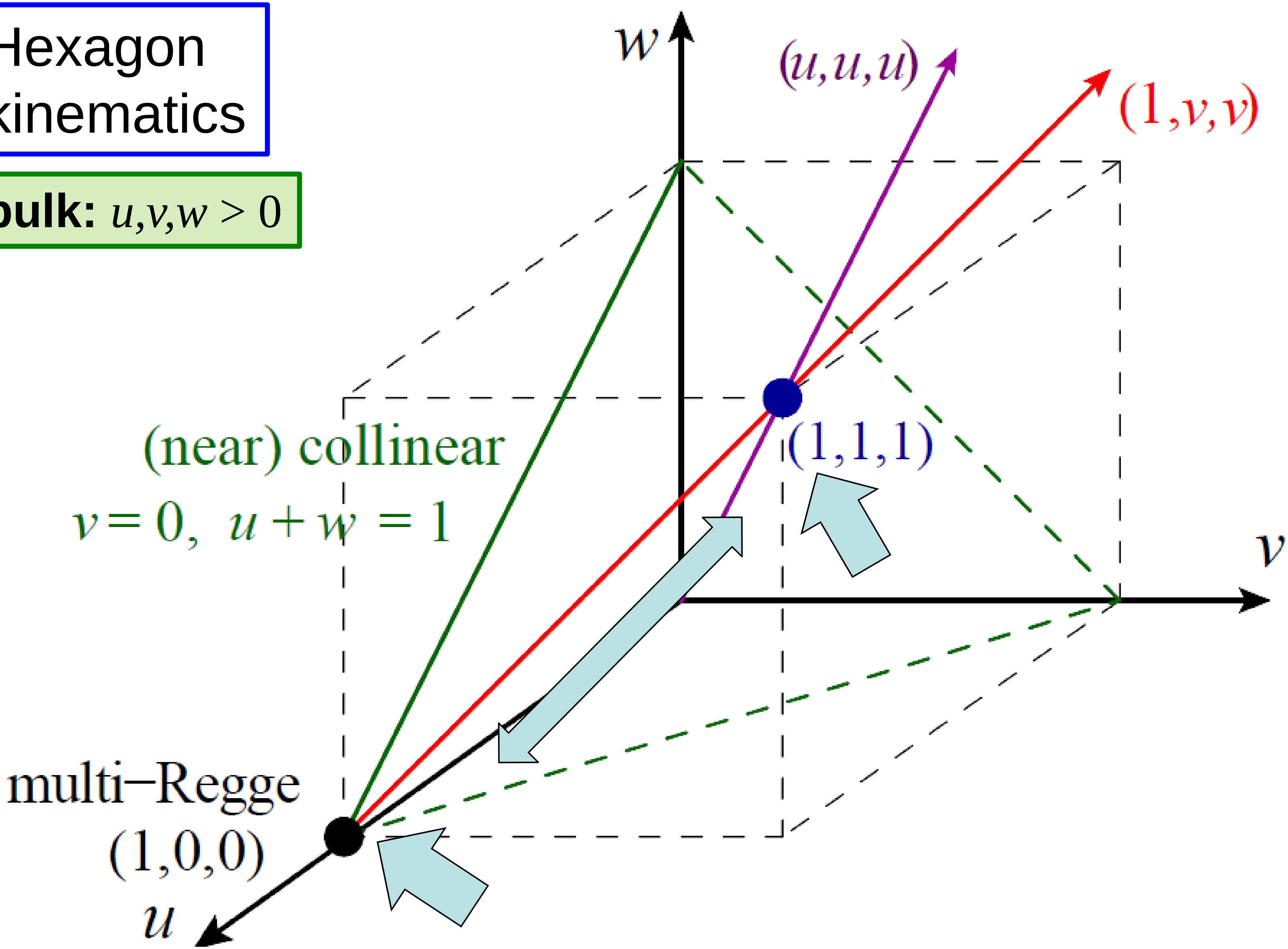
Branch cut conditions

- **The way to fix all ζ values, build consistent functions**
  LD, Drummond, von Hippel, Pennington, 1308.2276
- Functions $\boldsymbol{\zeta_m \ln s_k}$ for $s_k \neq u_k$ have unphysical branch cuts. Not independent functions, but get "attached" to other functions to make their branch cuts all physical.
- **Where to fix?** Near $\boldsymbol{s_k} = 0$, initially.
- **Hexagon natural base point:** $(u, v, w) = (1,1,1)$
- Finite, dihedrally invariant, all hexagon functions evaluate to multiple zeta values (MZV's) there.
- **"Bulky" choice of branch cut conditions**: $\text{no} \;\; \ln(1-u_i) \;\; \text{at} \;\; (1,1,1)$
  $\text{no} \;\; \ln(y_u) \text{ or } \ln(y_v) \;\; \text{at} \;\; (u,u,1)|_{u\to 0}$
- Transport results back up $(u,u,1)$ line to $(1,1,1)$; remarkably only restricted set of MZV's appear there: $1,\ \zeta_2,\ [no\ \ \zeta_3],\ \zeta_4,\ 5\zeta_5 - 2\zeta_2\zeta_3, \ldots$

Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1906.07116

Hexagon kinematics

bulk: $u,v,w > 0$



7-point branch cut conditions

- **No simple bulk analog of (1,1,1) for heptagons, so work on boundary (co surface)**
- Vanishing of $\ln(1-u_7)$ on entire "co" surface
- No $\ln(1-u_3)$ at $u_3 = 1$
- No $\ln(1-u_4)$ at $u_4 = 1$
- Also one branch cut condition on nearby "soft" surface (or can apply additional "co" conditions)
- Parity-odd functions vanish at $u_3 = u_4 = 1$

  because this point touches 6-point $\Delta = 0$ surface
- **These conditions, along with integrability and extended Steinmann relations, suffice to fix all ζ values and fully define heptagon functions**
- Big computational assist from previous symbol level information, complete through weight 6

  LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976

Many interesting surfaces intersect "co"

- For example, cancellation of spurious poles in the NMHV amplitude implies
  LD, Drummond, Harrington, McLeod, Papathanasiou, Spradlin, 1612.08976

$$\text{Spurious I:}\quad E_{47}\big|_{<1356>\,=\,0} = 0$$

$$\text{Spurious II:}\quad (E_{23} - E_{25})\big|_{<1467>\,=\,0} = 0$$

- $< 1356 > = 0$ intersects "co" at $u_4 = 1$
- $< 1467 > = 0$ intersects "co" at $u_3 = 1$
- Easy to impose these constraints at function level, and uniquely determine MHV and NMHV amplitudes, using also their soft/collinear limits

$R_7^{(2)}$

Golden, Spradlin, 1406.2055

- For generic kinematics, Golden-Spradlin formula has 49,677 terms, 112 featuring the non-classical polylog

$$L_{2,2}(x,y) \equiv \frac{1}{2}\int_0^1 \frac{dt}{t}\Big[\mathrm{Li}_2(-tx)\,\mathrm{Li}_1(-ty) - \mathrm{Li}_2(-ty)\,\mathrm{Li}_1(-tx)\Big]$$

- When we drop onto the co surface, the 112 $L_{2,2}$ functions collapse to 3 pairs, which can all be removed using the identity

$$\begin{aligned} L_{2,2}(x,y) + L_{2,2}(1/y,1/x) = {} & 3\,\mathrm{Li}_4(x/y) + \mathrm{Li}_4(-x) - \mathrm{Li}_4(-y) \\ & - \ln(x/y)\,\mathrm{Li}_3(x/y) - \ln y\,\mathrm{Li}_3(-x) + \ln x\,\mathrm{Li}_3(-y) \\ & + \frac{1}{4}(\ln^2 y + 2\zeta_2)\,\mathrm{Li}_2(-x) - \frac{1}{4}(\ln^2 x + 2\zeta_2)\,\mathrm{Li}_2(-y) \\ & + \frac{1}{24}\ln^2 y\,\ln(x/y)\,(\ln y - 3\,\ln x) - \frac{\zeta_2}{4}\ln(x/y)\,(\ln x - 3\,\ln y) - 3\,\zeta_4 \end{aligned}$$

$R_7^{(2)}$ on the co surface

- Using also standard classical polylog identities, Golden-Spradlin expression collapses to:

$$
\begin{aligned}
R_7^{(2)}(u_{1,2,5,6} \ll 1,\ u_7 = 1) \ = \ & 6\,\mathrm{Li}_4\left(\frac{-u_3}{1-u_3}\right) - \frac{1}{2}\left[\mathrm{Li}_2\left(\frac{-u_3}{1-u_3}\right)\right]^2 - 2\,\ln\left(\frac{u_3}{u_1u_2u_4}\right)\ \mathrm{Li}_3\left(\frac{-u_3}{1-u_3}\right) \\
& + \left[\ln\left(\frac{1-u_3}{u_1u_2u_4}\right)\ln u_3 + \ln(u_1u_4)\ln u_2 - \frac{1}{2}\ \ln^2(1-u_3) - 3\,\zeta_2\right]\mathrm{Li}_2\left(\frac{-u_3}{1-u_3}\right) \\
& + \frac{1}{8}\ln^4(1-u_3) - \left[\frac{1}{6}\ \ln^2(1-u_3) + \zeta_2\right]\ln(1-u_3)\ \left[\ln u_3 + 2\ \ln(u_1u_2u_4)\right] \\
& + \frac{1}{2}\ \ln^2(1-u_3)\ \left[\ln u_1\ \ln u_2 + \ln(u_1u_2)\ \ln u_3 + \ln(u_2u_3)\ \ln u_4 + 3\,\zeta_2\right] \\
& - \ln(1-u_3)\ \ln u_3\ \ln u_2\ \ln(u_1u_4) \\
& + \frac{\zeta_2}{2}\left[\left(\ln u_4 + 2\ \ln(u_1u_2u_5)\right)\ln u_3 + 2\ \ln u_1\ \ln u_2\right] + \frac{17}{4}\zeta_4 \\
& + \{\ u_1 \leftrightarrow u_6,\ \ u_2 \leftrightarrow u_5,\ \ u_3 \leftrightarrow u_4\ \}
\end{aligned}
$$

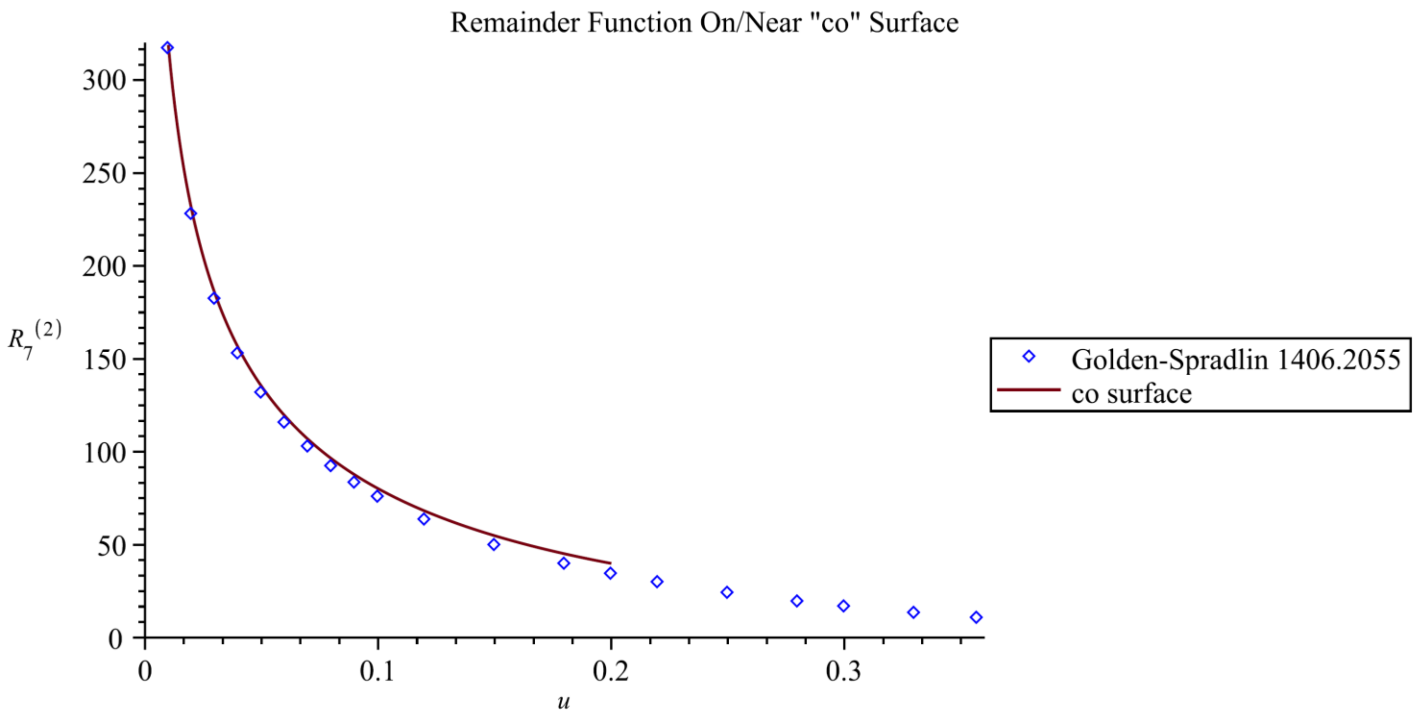
only terms surviving at origin, $u_3, u_4 \rightarrow 0$

“Diagonal line” in co surface

$$R_7^{(2)}(u,u,u,u,u,u,1) = 12\,\mathrm{Li}_4\left(\frac{-u}{1-u}\right) - \left[\mathrm{Li}_2\left(\frac{-u}{1-u}\right)\right]^2 + 8\,\ln u\,\mathrm{Li}_3\left(\frac{-u}{1-u}\right)$$
$$- \Big(2\,\ln^2 u - 2\,\ln u\,\ln(1-u) + \ln^2(1-u) + 6\zeta_2\Big)\mathrm{Li}_2\left(\frac{-u}{1-u}\right)$$
$$- \frac{1}{12}\,\ln(1-u)\Big(48\,\ln^3 u - 60\,\ln^2 u\,\ln(1-u) + 28\,\ln u\,\ln^2(1-u) - 3\,\ln^3(1-u)\Big)$$
$$- \zeta_2\,\ln(1-u)\Big(14\,\ln u - 3\,\ln(1-u)\Big) + 9\,\zeta_2\,\ln^2 u + \frac{17}{2}\zeta_4$$

- Doesn’t really make sense unless $u \ll 1$, since Gram determinant $G(u) \propto u$ doesn’t vanish. But we can still plot it for smallish u

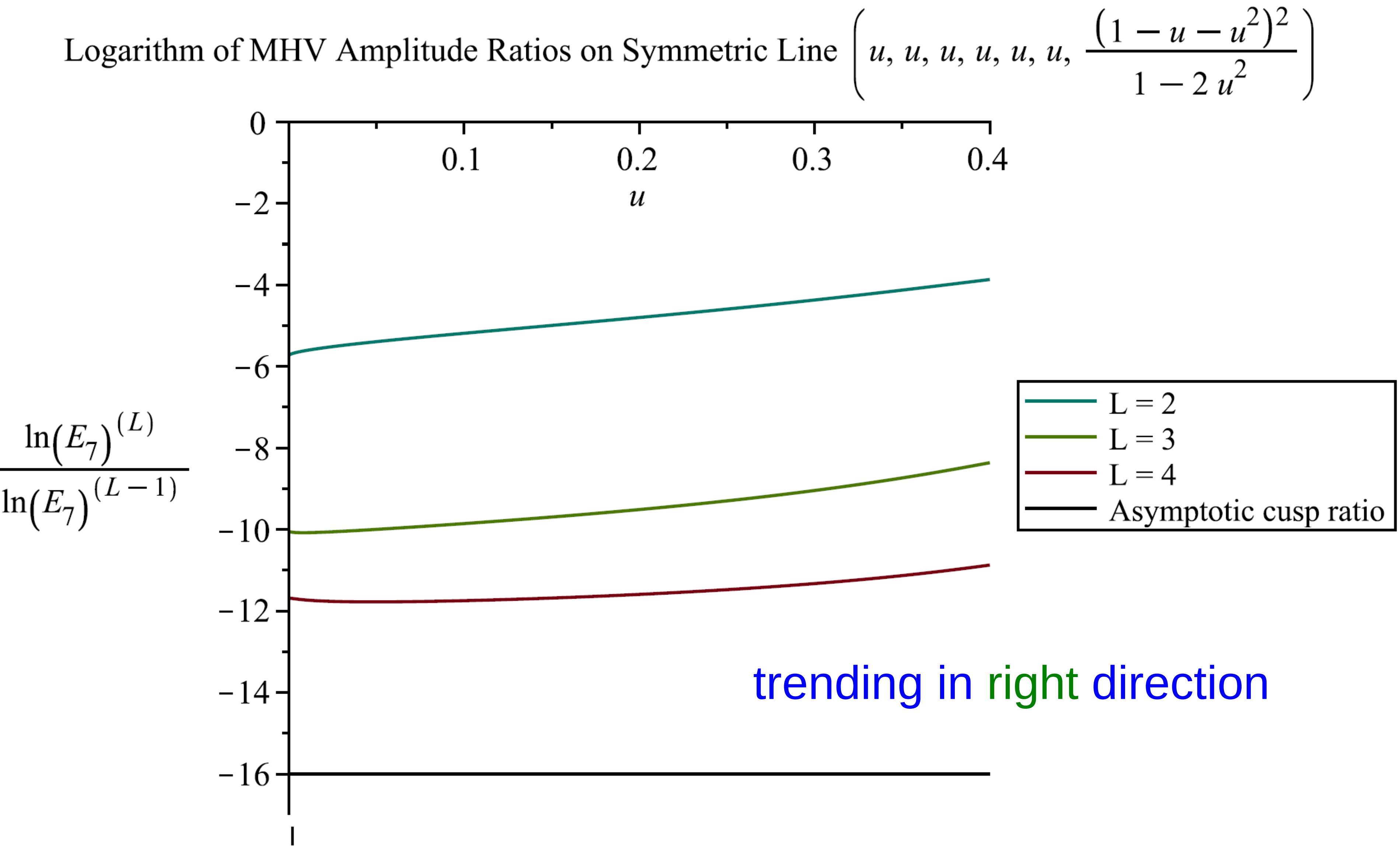
$R_7^{(2)}$ on $\left(u,u,u,u,u,u,\frac{(1-u-u^2)^2}{1-2u^2}\right)$ *vs.* on co surface $(u,u,u,u,u,u,1)$



- In fact, we can compute any amplitude on $\left(u,u,u,u,u,u,\frac{(1-u-u^2)^2}{1-2u^2}\right)$, as a power series expansion around $u = 0$.
- Integration constants fixed at origin, $(0,0,0,0,0,0,1)$.
- Forty terms suffice to converge past end of positive region at $u = 0.35689\ldots$

“High” Order Perturbative Behavior

- Planar N=4 SYM has finite radius of convergence of perturbative expansion (unlike QCD, QED, whose perturbative series are asymptotic)
- For BES solution to cusp anomalous dimension, using coupling $g^2 = \frac{\lambda}{16\,\pi^2}$, this radius is $\frac{1}{16}$
- Ratio of successive terms $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}} \to -16$
- Based on 6-point experience, expect ratios of successive loop orders for amplitudes to also approach -16, at least in the bulk

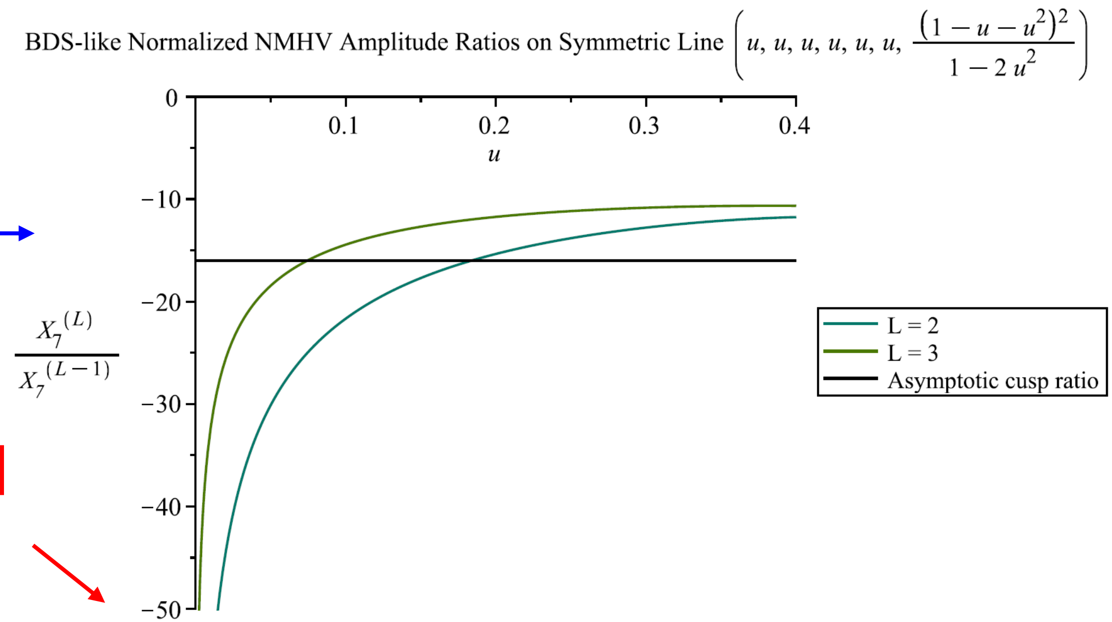


trending in right direction

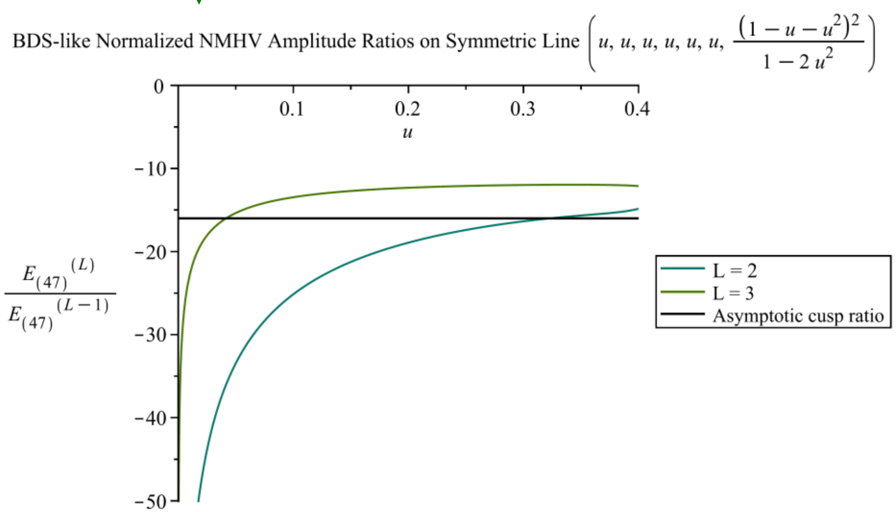
NMHV successive loop ratios, through 3 loops

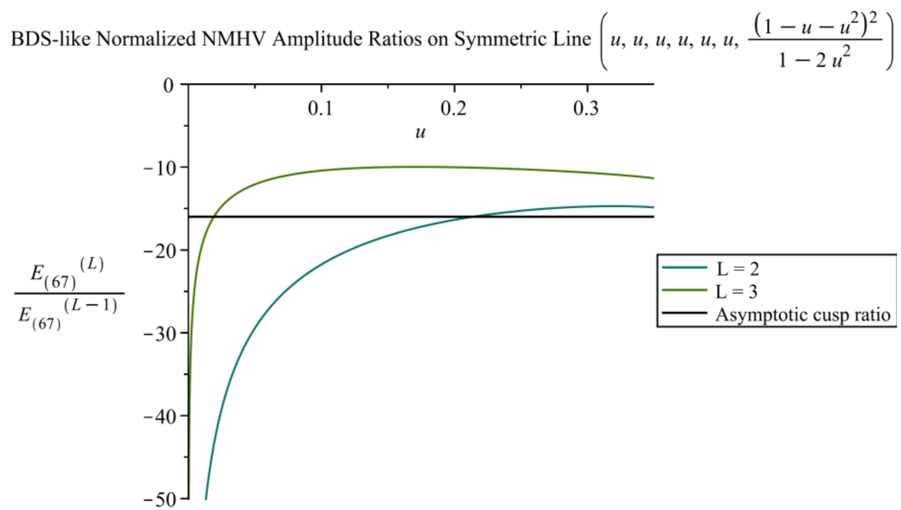
3 of 15 components shown

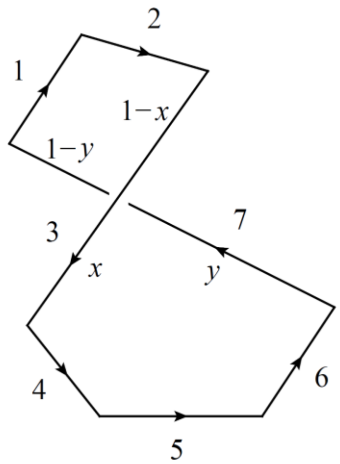
tree



(47) = [12356]   (67) = [12345]







"Self-crossing" surface

→ 4 parameter surface

LD, Esterlis, 1602.02107

$$u_7 = 1\,, \qquad u_3 = 1 - \frac{u_1 u_5}{u_6}\,, \qquad u_4 = 1 - \frac{u_2 u_6}{u_1}$$

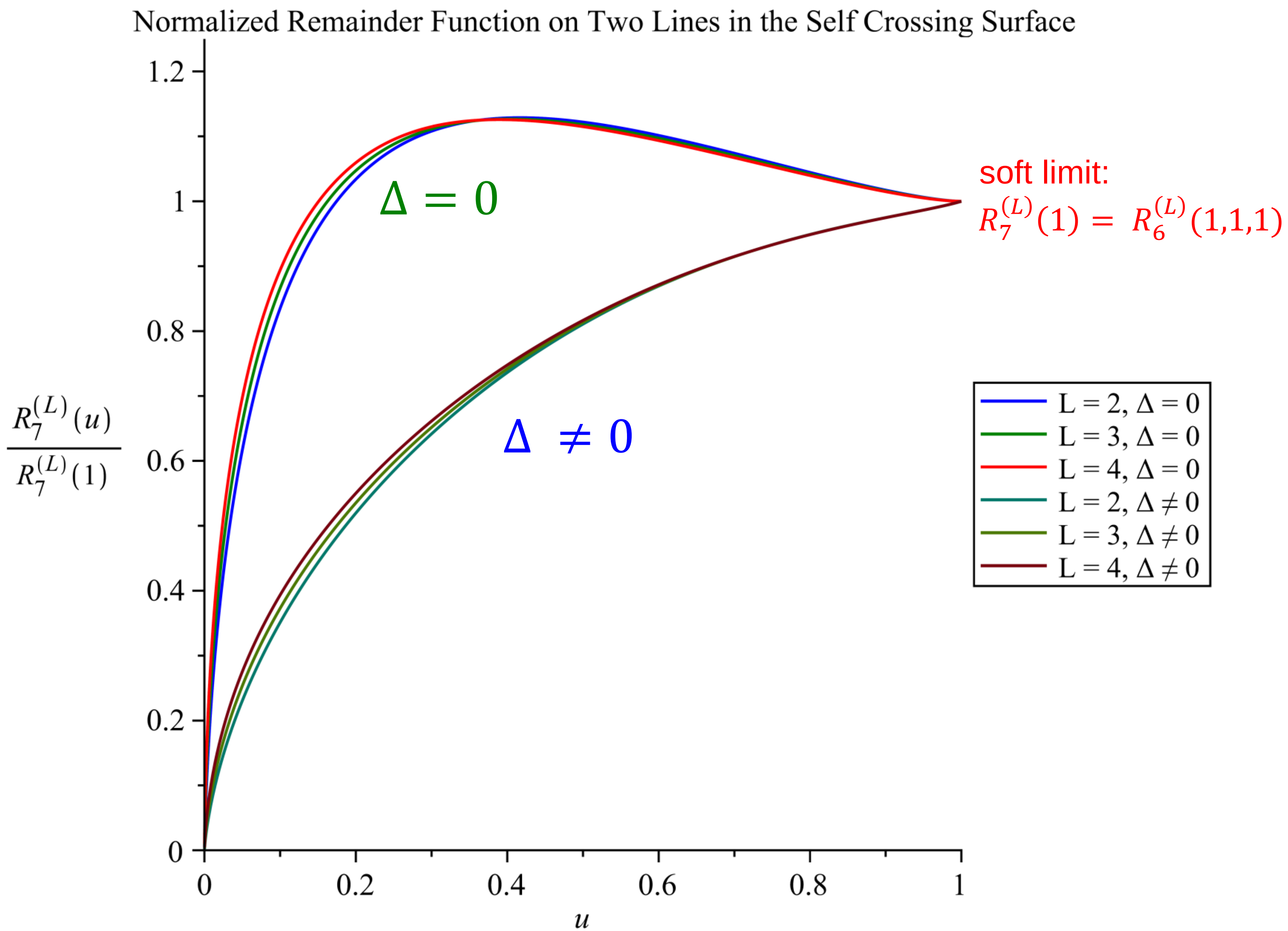
- Work on Euclidean sheet, as another example of bulk kinematics
- Intersects co surface at $u_3 = u_4 = 1$

- For simplicity, consider two different 1D "lines" in this surface:

$(u, u, 1-u, 1-u, u, u, 1)$ and $(\frac{4u}{(1+u)^2}, u, 1-u, 1-u, u, \frac{4u}{(1+u)^2}, 1)$

$\Delta \neq 0$ $\qquad\qquad$ $\Delta = 0$

- Parity-odd functions nonvanishing vs. vanishing
- Both lines intersect co surface at $u = 0$, and leave Euclidean region at same soft limit point, $u = 1 \implies (1,1,0,0,1,1,1)$.
- Symbol alphabet in $\Delta \neq 0$ case still has square roots, but can integrate up as power series around both $u = 0$ and $u = 1$.

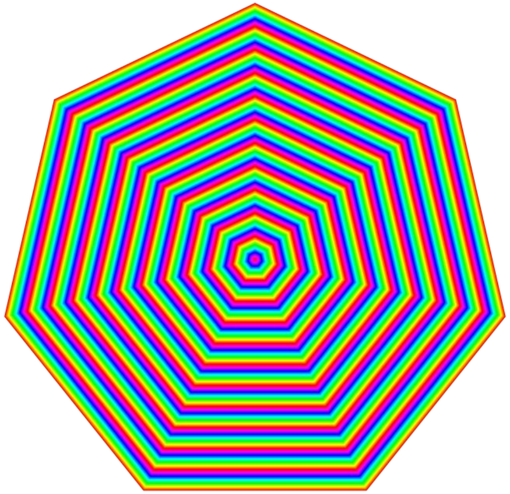

Normalized Remainder Function on Two Lines in the Self Crossing Surface
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
u
$\frac{R_7^{(L)}(u)}{R_7^{(L)}(1)}$
$\Delta = 0$
$\Delta \neq 0$
soft limit:
$R_7^{(L)}(1) = R_6^{(L)}(1,1,1)$
L = 2, Δ = 0
L = 3, Δ = 0
L = 4, Δ = 0
L = 2, Δ ≠ 0
L = 3, Δ ≠ 0
L = 4, Δ ≠ 0

Coaction Principle?

- By taking coproducts of 4 loop MHV and 3 loop NMHV amplitudes, we find that ζ_3 is an independent constant function in the heptagon function space
- Big surprise, because ζ_3 is not independent at 6 points; it does not even appear as a value of any hexagon function at $(u, v, w) = (1,1,1)$! CDDvHMP, 1906.07116
- ζ_2 is not independent, just as at 6 points
  → coaction principle can be nontrivial (at function level)
- ζ_4 is independent, just as at 6 points
- Can't yet determine independence of ζ_5 or $\zeta_2\zeta_3$; need at least 4 loop NMHV amplitude
- But so far it seems like 7 points and 6 points will behave quite differently

Lightning dive to the origin

in progress also with B. Basso and G. Papathanasiou



6-points: $\ln \mathcal{E}^{MHV}$ is quadratic in logarithms through at least 9 loops, and almost certainly to all orders

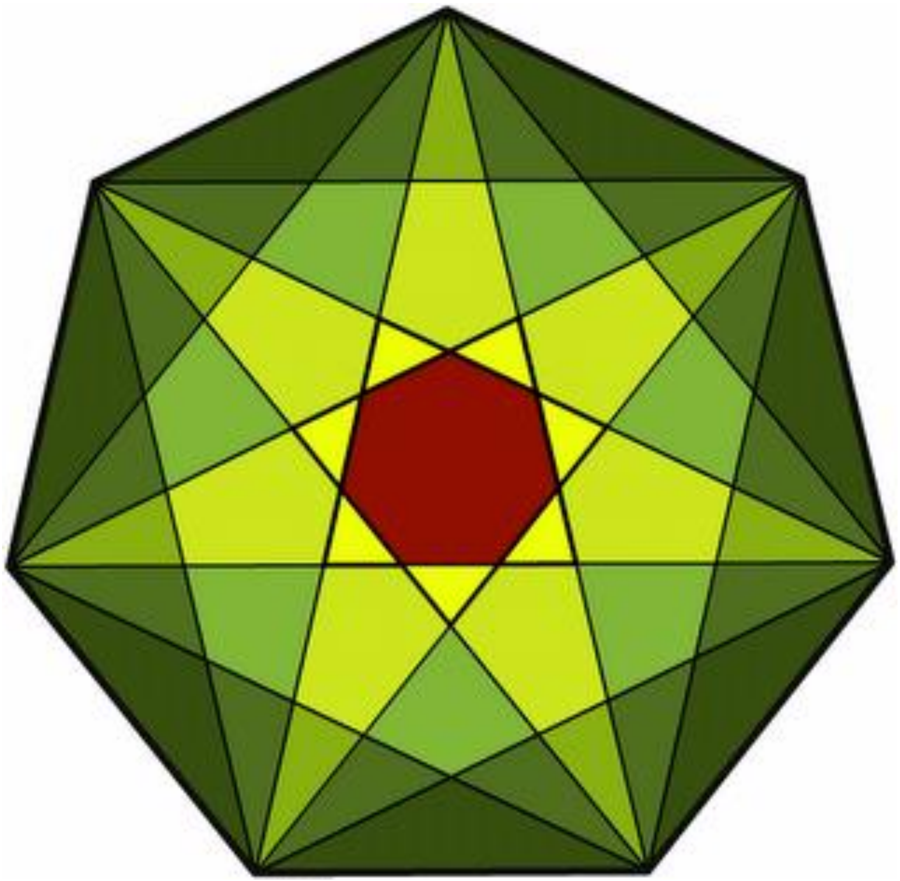
CDDvHMP, 1903.10890, next talk by Benjamin Basso

7-points: $\ln \mathcal{E}^{MHV}$ is quadratic in logarithms through at least 4 loops

Conclusions

- Heptagon function bootstrap now well underway
- Helped greatly to find a simple “co” surface on the boundary where enough branch cut conditions can be imposed to fix all zeta values
- Can integrate up along different lines into the bulk
- co surface touches many interesting regions (origin, MRK, OPE, …) and will allow zeta values to be fixed there too
- Similar multi-scaling surfaces may be very useful for $n > 7$ too!

Extra Slides



Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

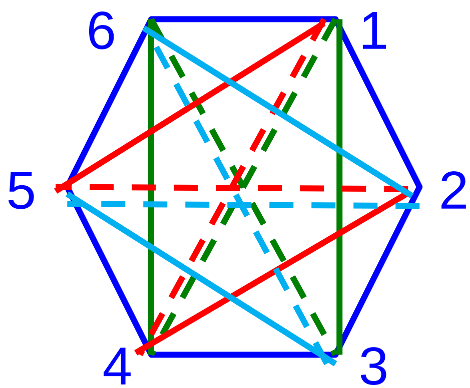
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for n = 4,5

$n = 6$ → precisely 3 ratios:

$$u_1 = u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$u_2 = v = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

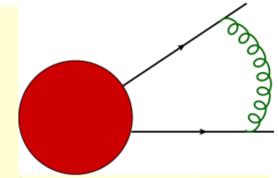
$$u_3 = w = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



$n = 7$ → 6 independent ratios

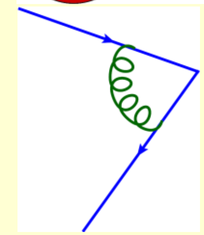
Removing Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$

  – known to all orders in planar N=4 SYM:
  Beisert, Eden, Staudacher, hep-th/0610251



- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708

- Normalized [MHV] amplitude is finite, dual conformal invariant.
- BDS-like also maintains important relation due to causality (Steinmann).

$$\mathcal{E}_n(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_n(s_{i,i+1},\epsilon)}{\mathcal{A}_n^{\rm BDS-like}(s_{i,i+1},\epsilon)} = \exp[\mathcal{R}_n + \frac{\Gamma_{\rm cusp}}{4}\mathcal{E}_n^{(1)}]$$

BDS-like ansatz ($n = 6$)

$$\frac{\mathcal{A}_6^{\text{BDS-like}}}{\mathcal{A}_6^{\text{MHV}(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon) \frac{1}{2} \hat{M}_6(L\epsilon) + C^{(L)} \right)\right]$$

where $f^{(L)}(\epsilon) = \frac{1}{4}\gamma_K^{(L)} + \epsilon \frac{L}{2}\mathcal{G}_0^{(L)} + \epsilon^2 f_2^{(L)}$ are constants, and

$$\begin{aligned} \hat{M}_6(\epsilon) &= M_6^{1-\text{loop}}(\epsilon) + Y(u,v,w) \\ &= \sum_{i=1}^{6} \left[-\frac{1}{\epsilon^2}(1 - \epsilon \ln s_{i,i+1}) - \ln s_{i,i+1} \ln s_{i+1,i+2} + \frac{1}{2} \ln s_{i,i+1} \ln s_{i+3,i+4} \right] + 6\zeta_2 \end{aligned}$$

- Y is dual conformally invariant part of one-loop amplitude $M_6^{1-\text{loop}}$ containing all 3-particle invariants:

$$Y(u,v,w) = -\mathcal{E}^{(1)} = -\text{Li}_2\left(1 - \frac{1}{u}\right) - \text{Li}_2\left(1 - \frac{1}{v}\right) - \text{Li}_2\left(1 - \frac{1}{w}\right)$$

- More minimal BDS-like ansatz contains all IR poles, but **no 3-particle invariants**.

Branch cut condition

- All massless particles → all branch cuts start at origin in

$$s_{i,i+1}, \quad s_{i,i+1,i+2}$$

- 6-point: Branch cuts all start from 0 or ∞ in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad \text{or } v \text{ or } w$$

→ Only 3 weight 1 functions, not 9: $\{ \ln u, \ln v, \ln w \}$

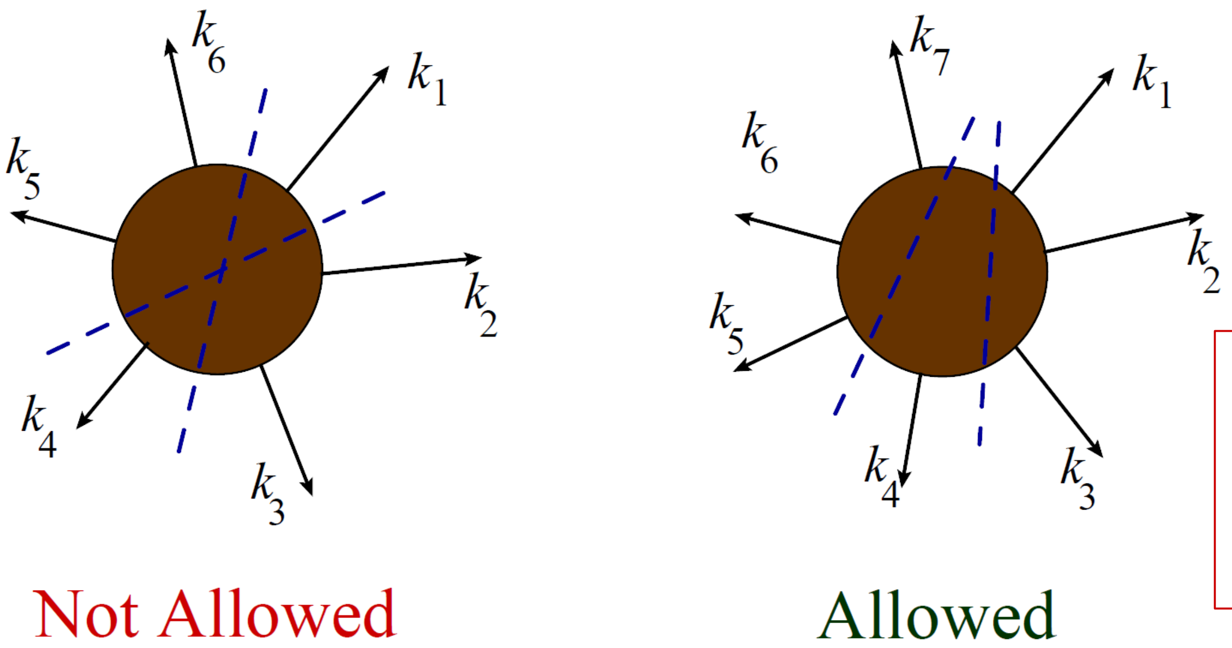
- 7-point: Only 7 weight 1 functions, not 42 $\{\ln u_i\}$

- Discontinuities commute with branch cuts
- Require derivatives of higher weight functions to obey branch-cut condition too → very powerful constraint.

Steinmann relations

Steinmann, Helv. Phys. Acta (1960)   Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts:



can't apply to 2 particle cuts in massless case because they are not independent

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

Violated by BDS normalized amplitudes!

Steinmann constraints ($n = 6$)

S. Caron-Huot, LD, M. von Hippel, A. McLeod, 1609.00669

$$\text{Disc}_{s_{234}}\left[\text{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

+ cyclic conditions

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}}$$

$\ln^2 u$ **NO**

$\ln^2 \frac{uv}{w}$ **OK**

$$\frac{uv}{w} = \frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}$$

Weight 2 functions restricted to 6 out of 9:

$$\text{Li}_2(1-1/u) \quad \text{Li}_2(1-1/v) \quad \text{Li}_2(1-1/w)$$

$$\ln^2\frac{uv}{w} \quad \ln^2\frac{vw}{u} \quad \ln^2\frac{wu}{v}$$

Analogous constraints for $n=7$

LD, J. Drummond, T. Harrington, A. McLeod, G. Papathanasiou, M. Spradlin, 1612.08976