The Origin of the Six-Gluon Amplitude in Planar N=4 SYM

Benjamin Basso
LPENS

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Based on work with
Lance Dixon and Georgios Papathanasiou

Gluon scattering amplitudes in planar N=4 SYM

Powerful techniques to compute amplitudes at various values of coupling constant

$$g^2 = \lambda/(4\pi)^2$$

Perturbative amplitude bootstrap

Most efficient way of building amplitudes at weak coupling
(rely on knowledge of relevant space of functions) *(Lance's talk)*

Pentagon flux-tube OPE

Work at finite coupling around the collinear limit
(rely on integrability)

String theory

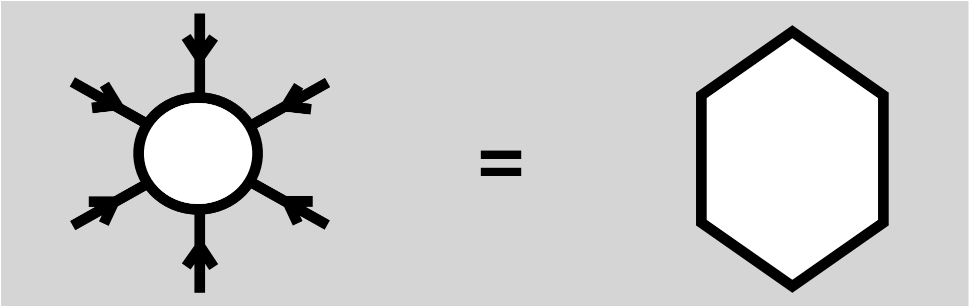
Strong coupling description in terms of minimal surfaces
(rely on gauge-string duality)

But the dream of re-summing / plotting an amplitude at finite coupling is not fulfilled yet

Closest target: 6-gluon MHV amplitude

Same as **Hexagon Wilson Loop**

[Alday,Maldacena'07]
[Drummond,Korchemsky,Sokatchev'07]
[Brandhuber,Heslop,Travaglini'07]
[Drummond,Henn,Korchemsky,Sokatchev'07]



Smaller polygons are "trivial" owing to dual conformal symmetry and controlled by BDS ansatz / cusp anomalous dimension $\Gamma_{\rm cusp}$

Subtracting BDS / divergences in a friendly way - with $\mathcal{E}^{(1)} = \sum_{i=1,2,3} \mathrm{Li}_2(1-1/u_i)$

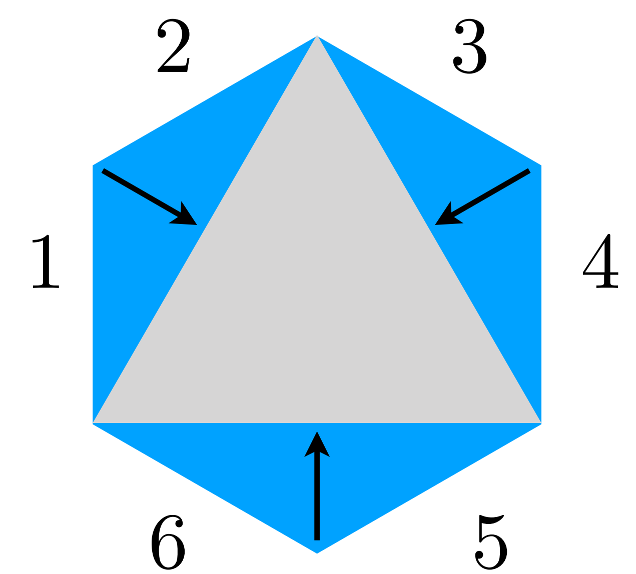
$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{ij},\epsilon)}{\mathcal{A}_{6,\text{BDS-like}}(s_{ij},\epsilon)} = \exp\left[\mathcal{R}_6 + \frac{1}{4}\Gamma_{\rm cusp}\,\mathcal{E}^{(1)}\right]$$

It is IR finite and function of 3 Hexagon cross ratios

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}\,,\quad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}\,,\quad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

The Origin

Triple collinear limit aka “Origin”



$$u_1, u_2, u_3 \to 0$$

$$s_{12}, s_{34}, s_{56} \to 0$$

Observation: 6-gluon amplitude shows Sudakov-like suppression

[Alday, Gaiotto,Maldacena'09]
[Caron-Huot,Dixon,Dulat,McLeod,vonHippel,Papathanasiou'19]

$$\log \mathcal{E} = -\frac{\Gamma_{\text{oct}}}{24} \log^2(u_1 u_2 u_3) - \frac{\Gamma_{\text{hex}}}{24} \sum_i \log^2 \left(u_i/u_{i+1}\right) + C_0$$

Quadratic in logs and controlled by a few functions of the coupling alone

$$\Gamma_{\text{oct}} = \Gamma_{\text{oct}}(g^2)\,,\ \Gamma_{\text{hex}} = \Gamma_{\text{hex}}(g^2)\,,\ C_0 = C_0(g^2)$$

Opportunity to study an amplitude at finite coupling and to practice re-summation

Perturbation theory

Weak coupling evidence and data

Hexagon function bootstrap enabled calculation of six-gluon amplitude through 7 loops throughout the entire kinematical space

[Dixon,Drummond,Henn'11]
[Dixon,Drummond,vonHippel,Pennington'13]
[Caron-Huot,Dixon,McLeod,vonHippel'16]
[Caron-Huot,Dixon,Dulat,McLeod,vonHippel,Papathanasiou'19]

First 5 loops:

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
Γ_{oct}	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$	$\frac{126976}{3}\zeta_8$
Γ_{cusp}	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6-32\zeta_3^2$	$\frac{28384}{3}\zeta_8+128\zeta_2\zeta_3^2+640\zeta_3\zeta_5$
Γ_{hex}	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6-24\zeta_3^2$	$\frac{18287}{6}\zeta_8+48\zeta_2\zeta_3^2+480\zeta_3\zeta_5$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6+2\zeta_3^2$	$\frac{67645}{32}\zeta_8+6\zeta_2\zeta_3^2-40\zeta_3\zeta_5$	$-\frac{4184281}{160}\zeta_{10}-65\zeta_4\zeta_3^2-120\zeta_2\zeta_3\zeta_5+228\zeta_5^2+420\zeta_3\zeta_7$

All look very similar and written in terms of Riemann zeta values (no complicated MZVs)

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

Emerging pattern: the anomalous dimension Γ_{oct} is simpler

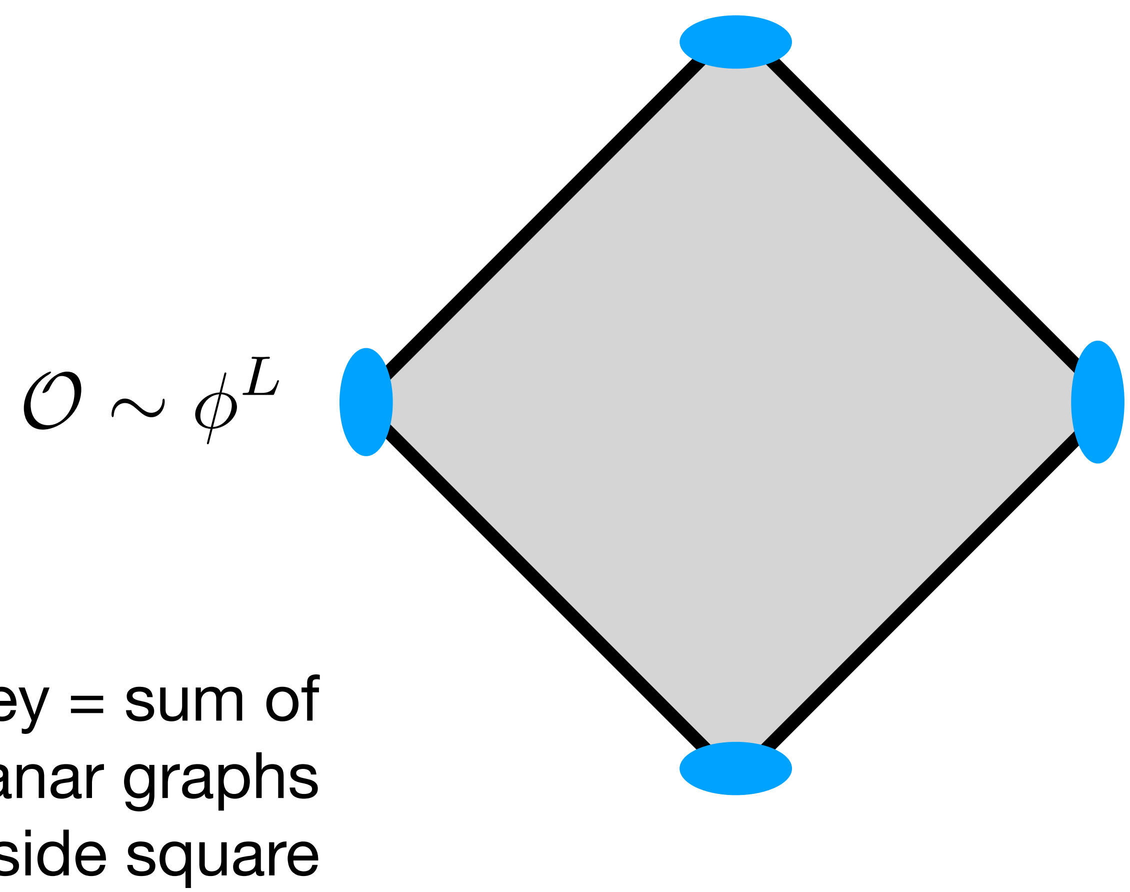
Easy coefficient

Immediate observation can be made (through at least 7 loops)

$$\Gamma_{\rm oct} = \frac{2}{\pi^2}\log\cosh(2\pi g) = 4g^2 - \frac{8\pi^2 g^4}{3} + \frac{128\pi^4 g^6}{45} - O(g^8)$$

Moreover: same anomalous dimension controls the light-like limit of large-charge 4pt function - dubbed "the Octagon"

[Coronado'18]
[Kostov,Petkova,Serban'19]
[Belitsky,Korchemsky'19]



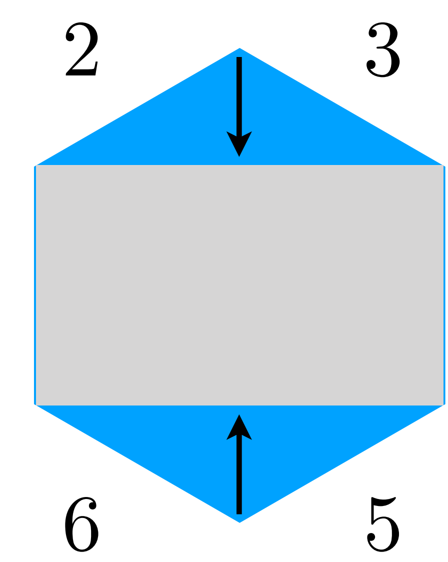
Large-charge BPS operators at every corner

Double logarithmic scaling for light-likeframe controlled by $\Gamma_{\rm oct}$

Determined recently at finite coupling

Need hammer for other quantities…

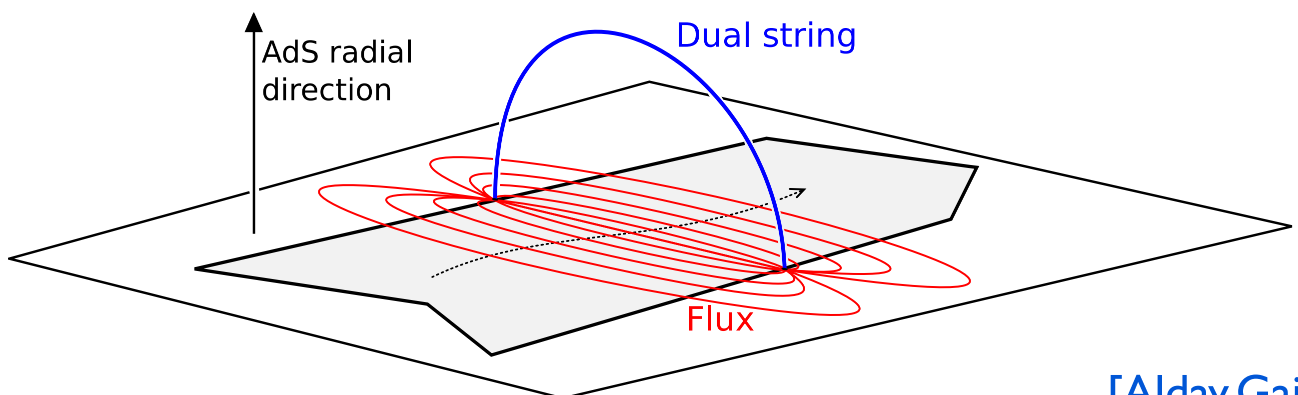
Collinear limit and flux tube



$$s_{56} \to 0$$

$$\tau \,, \sigma \,, \varphi \qquad u_2 \to 0$$

Philosophy: Cusps on opposite sides of square can be thought of as producing excitations propagating from one side to the other



[Alday,Gaiotto,Maldacena,Sever,Vieira'10]

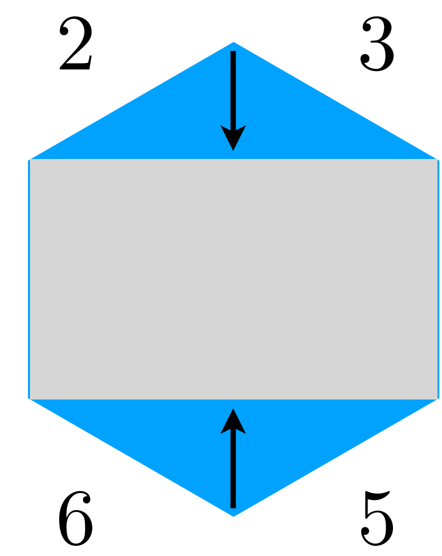
WL = sum over complete basis of intermediate states

$$\mathcal{W}_6 = \sum_{\psi} P(0|\psi)P(\psi|0)e^{-E\tau+iP\sigma+a\varphi}$$

$\tau \,, \sigma \,, \varphi$ parameterize the cross ratios, e.g. $u_2 \sim e^{-2\tau} \to 0$

Need hammer for
other quantities…

Collinear limit and flux tube



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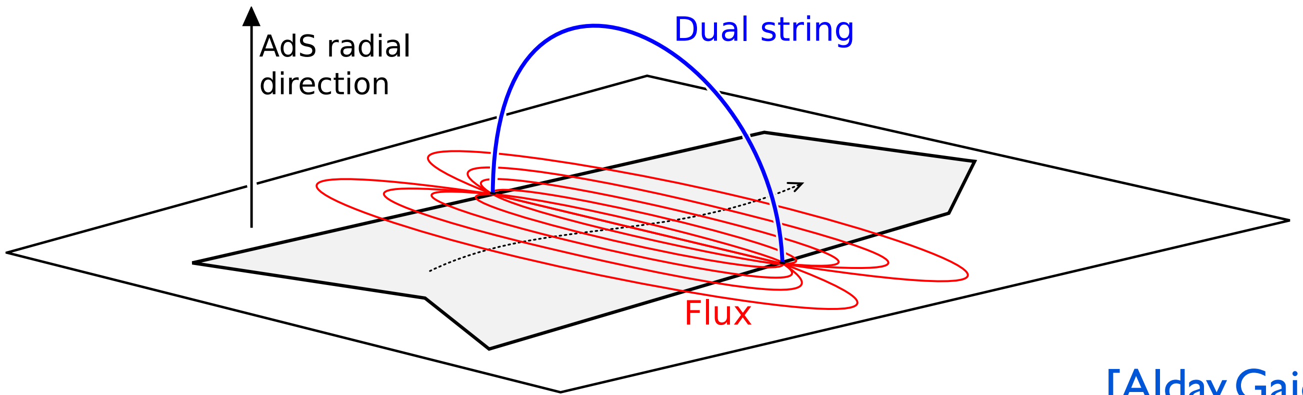


[Alday,Gaiotto,Maldacena,Sever,Vieira'10]

Flux tube states?

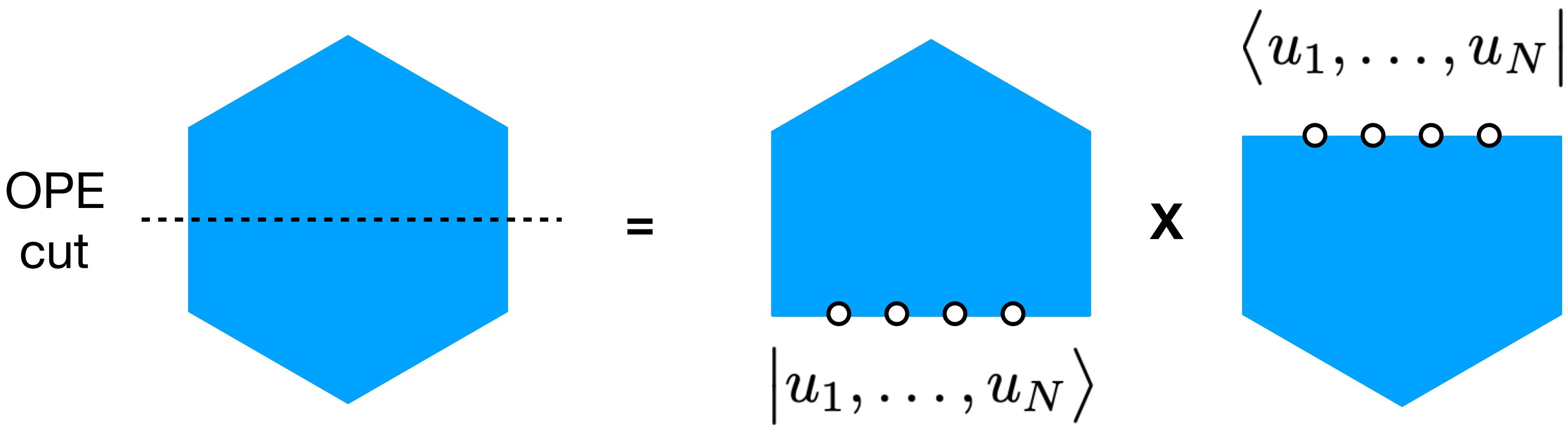
Ground state: GKP string with energy density (tension) = $\Gamma_{\rm cusp}$

Excitations: fluctuations along the long string carrying **energy = twist**

Pentagon OPE

[BB,Sever,Vieira'13]

Fundamental OPE building blocks are pentagon WLs with insertions



State with N excitations carrying rapidities (momenta) u1, ..., uN

Pentagon transitions give us control on the OPE series for WLs

They can be bootstrapped thanks to integrability

Distance OPE - Origin?

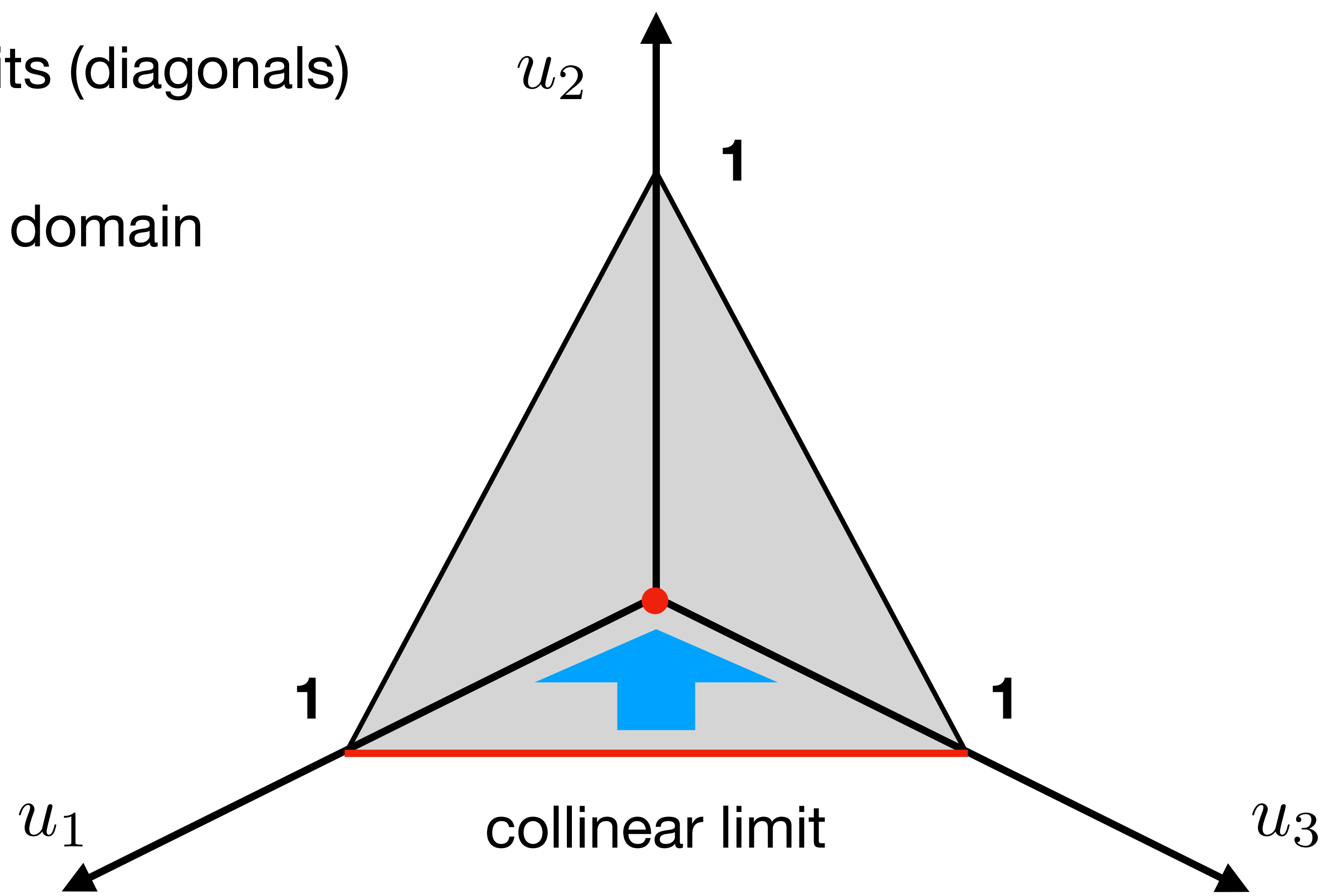
Far from collinear limits (diagonals)

Edge of convergence domain
of OPE series

Must re-sum

But what exactly?

Fortunately not
everything



$\varphi \to \infty$

Relevant states here are **lightest charged** states (with charge = helicity)

Tower of gluonic excitations $\mathcal{D}_z^{a-1} F_{+z}$ with helicity $a = 1, 2, 3, \ldots$

Gluons dominance

[BB,Sever,Vieira'14]
[Drummond,Papathanasiou'15]

Gluonic contributions to Hexagon WL

$$\mathcal{W}_6 = \sum_{N\geqslant 0} \frac{1}{N!} \sum_{a_1,\ldots,a_N\geqslant 1} e^{\varphi\sum_i a_i} \int \frac{du_1\ldots du_N}{(2\pi)^N} \frac{\prod_i \mu_i}{\prod_{i<j} P_{ij}P_{ji}} e^{-\tau\sum_i E_i + i\sigma\sum_i p_i}$$

Simply related to subtracted amplitude $\mathcal{W}_6 = \mathcal{E}\, e^{\frac{1}{2}\Gamma_{\text{cusp}}(\sigma^2+\tau^2+\zeta_2)}$

All ingredients can be determined at any coupling using integrability

At weak coupling the series truncates quickly as the N-gluon contributions $\sim g^{2N^2}$

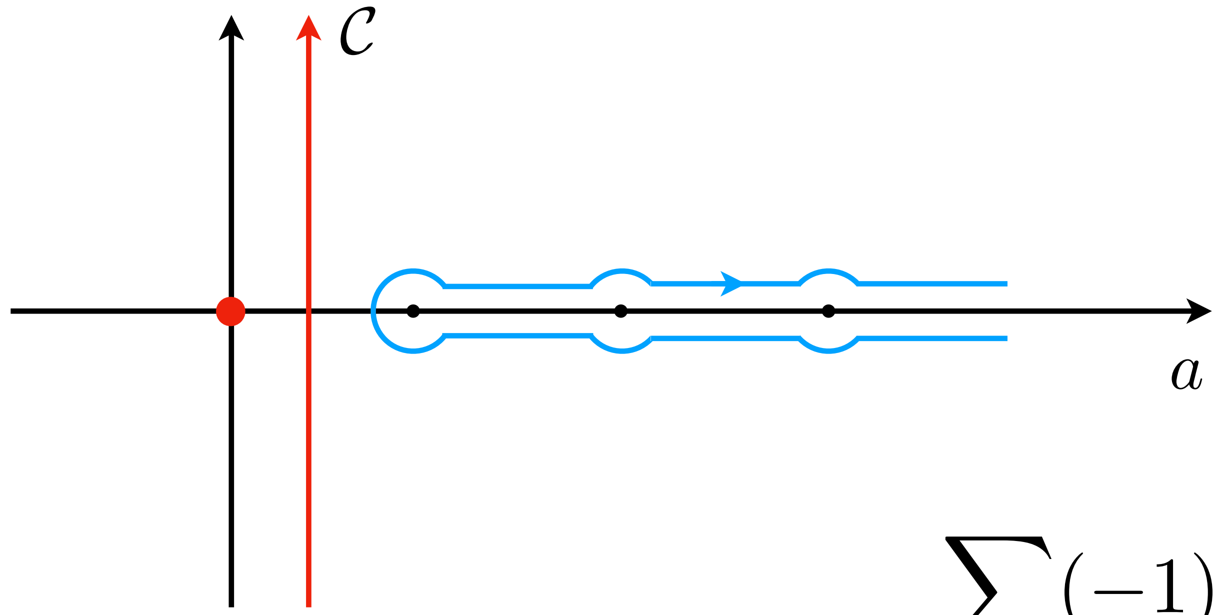
1-particle exchange is enough through 3 loops,
2-particle exchange is enough through 8 loops,
etc.

Integrals / sums can be expressed in terms of HPLs [Papathanasiou'13]

Sommeferld-Watson transform

Limit $\varphi \to \infty$ entails summation over helicity $a = 1, 2, 3, \ldots$

Method: convert sum into contour integral (like in Regge theory)



$$\sum_{a \geqslant 1} (-1)^a e^{a\varphi} f(a) = \int_{\mathcal{C}} da\, e^{a\varphi} \frac{i f(a)}{2 \sin(\pi a)}$$

$\varphi \to \infty$ is dominated by nearest singularity ("Regge pole") with $\mathrm{Re}\, a < 1$

In case at hand, the logarithms come from singularity at $a = 0$

Example

Single-particle exchange at one loop

$$\mathcal{W}_6 = 1 + g^2 \sum_{a=1}^{\infty} (-1)^a e^{a(\varphi-\tau)} \int \frac{du}{2\pi} \frac{\Gamma(\frac{a}{2}+iu)\Gamma(\frac{a}{2}-iu)}{(u^2+\frac{a^2}{4})\Gamma(a)} e^{2iu\sigma} + O(g^4)$$

Integrand is smooth and small at $a = 0$

Regge pole arises because of singularities pinching contour $u = \pm ia/2$

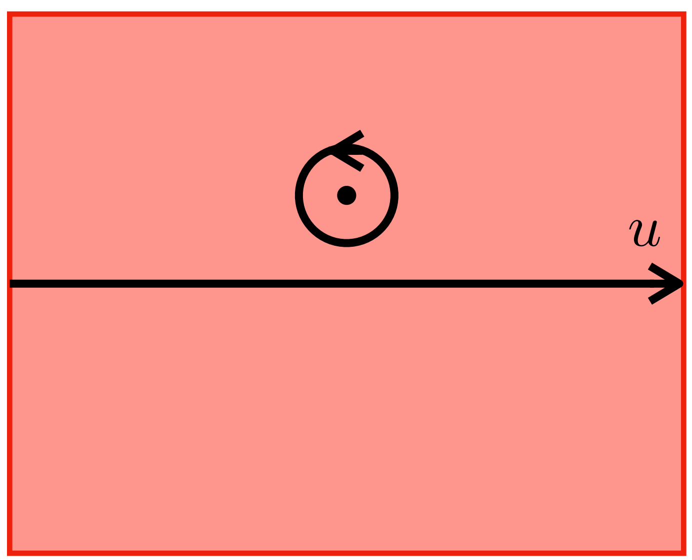
Recipe: pick up the residue at $u = ia/2$ and then at $a = 0$

Same OPE integrand as for expansion around collinear limit
but integrated along different contour of integration

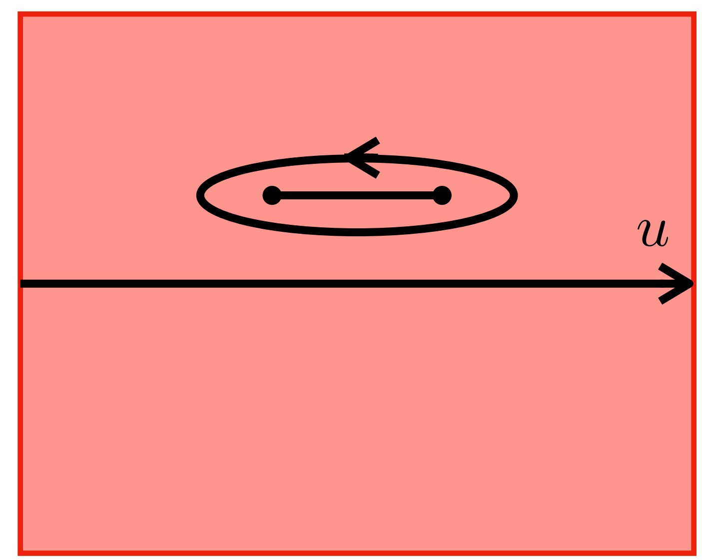
Lift to finite coupling

The pole in rapidity is replaced by a Zhukowski cut

$$u \quad \longrightarrow \quad x(u) = \frac{u + \sqrt{u^2 - (2g)^2}}{2}$$



Weak coupling



Finite coupling

The song remains the same: 1) take contour integral in u space 2) read off residue at a = 0

Bonus: new representation shows up by passing to the second sheet

Quasi Gaussian integral

Get (infinite-dimensional) integral representation

[BB,Dixon,Papathanasiou'20]

$$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i \, F(\xi) \, e^{-\xi M \xi + 2ig\tau\xi_1}$$

Gaussian weight is controlled by celebrated BES kernel

[Beisert,Eden,Staudacher'06]

$$M = 1 + \mathbb{K} \quad \longrightarrow \quad \Gamma_{\text{cusp}} = 4g^2 \left[\frac{1}{1+\mathbb{K}}\right]_{11}$$

with matrix element given by integral of product of Bessel functions

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j} \int_0^{\infty} \frac{dt}{t} \frac{J_i(2gt) J_j(2gt)}{e^t - 1}$$

Strictly Gaussian integral

The integrand F is a (complicated) Fredholm determinant $F(\xi) = F(\xi, \varphi, \tau, \sigma)$

Nicely we observed (in PT) that it becomes Gaussian at the origin $\varphi \to \infty$

$$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i \; e^{-\xi(M+\delta M)\xi + 2ig\tau\xi_1}$$

Entirely Gaussian but with a **modified** kernel $M + \delta M$

Integrals can then be done explicitly

* Log of amplitude is quadratic in logs of cross ratios

** Anomalous dimensions expressed as inverse element of shifted kernel

One kernel to rule them all

Partition BES kernel into 4 blocks and introduce angle

$$(\mathbb{K}_{\circ\circ})_{ij} = \mathbb{K}_{2i-1,2j-1}$$

$$(\mathbb{K}_{\circ\bullet})_{ij} = \mathbb{K}_{2i-1,2j}$$

etc

$$\mathbb{K}(\alpha) = 2\cos\alpha \begin{bmatrix} \cos\alpha\,\mathbb{K}_{\circ\circ} & \sin\alpha\,\mathbb{K}_{\circ\bullet} \\ \sin\alpha\,\mathbb{K}_{\bullet\circ} & \cos\alpha\,\mathbb{K}_{\bullet\bullet} \end{bmatrix}$$

with original undeformed case corresponding to $\alpha = \pi/4$

All anomalous dimensions encoded in

$$\Gamma_{\alpha} = 4g^2 \left[\frac{1}{1+\mathbb{K}(\alpha)}\right]_{11}$$

$$\Gamma_{\text{oct}} = \Gamma_{\alpha=0}$$

$$\Gamma_{\text{cusp}} = \Gamma_{\alpha=\pi/4}$$

$$\Gamma_{\text{hex}} = \Gamma_{\alpha=\pi/3}$$

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$$\Gamma_{\text{hex}} = \Gamma_{\alpha=\pi/3}$$

Constant uses determinant

$$D(\alpha) = \log\det\left(1+\mathbb{K}(\alpha)\right)$$

$$C_0 = D(\pi/4) - D(\pi/3) - \frac{1}{2}D(0) - \frac{\zeta_2}{2}\Gamma_{\text{cusp}}$$

Very efficient at weak coupling

Matrix truncates $\mathbb{K}_{ij} = O(g^{i+j})$

$$\frac{1}{1+\mathbb{K}(\alpha)} = 1 - \mathbb{K}(\alpha) + \mathbb{K}(\alpha)^2 - \dots$$

Easy to get perturbative expansion

$$\Gamma_{\alpha} = 4g^2 - 16\zeta_2 \cos^2\alpha\, g^4 + 32\zeta_4 \cos^2\alpha(3 + 5\cos^2\alpha)\, g^6 + \dots$$

* Same structure as for cusp (ζ of increasing transcendentality)

** Coefficients dressed with even powers of $\cos\alpha$

Agreement with perturbation theory through 7 loops!

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Numerically (large order behaviour)

* Radius of convergence is independent of angle: $g_c = \pm i/4$

** Nature of branch point depends on it

e.g. looks like $\Gamma(g,\alpha) - \Gamma(g_c,\alpha) \propto (g_c - g)^{2\alpha/\pi}$

Special point - "Octagon"

When $\alpha = 0$ the kernel becomes block diagonal

$$\mathbb{K}(0) = \left[\begin{array}{cc} 2\mathbb{K}_{\circ\circ} & 0 \\ 0 & 2\mathbb{K}_{\bullet\bullet} \end{array} \right]$$

Odd zeta's go away, left with integer powers of π^2

The sums can be taken exactly

[Belitsky,Korchemsky'19]

$$\Gamma_{\rm oct} = \Gamma_{\alpha=0} = \frac{2}{\pi^2} \log \cosh\left(2\pi g\right)$$

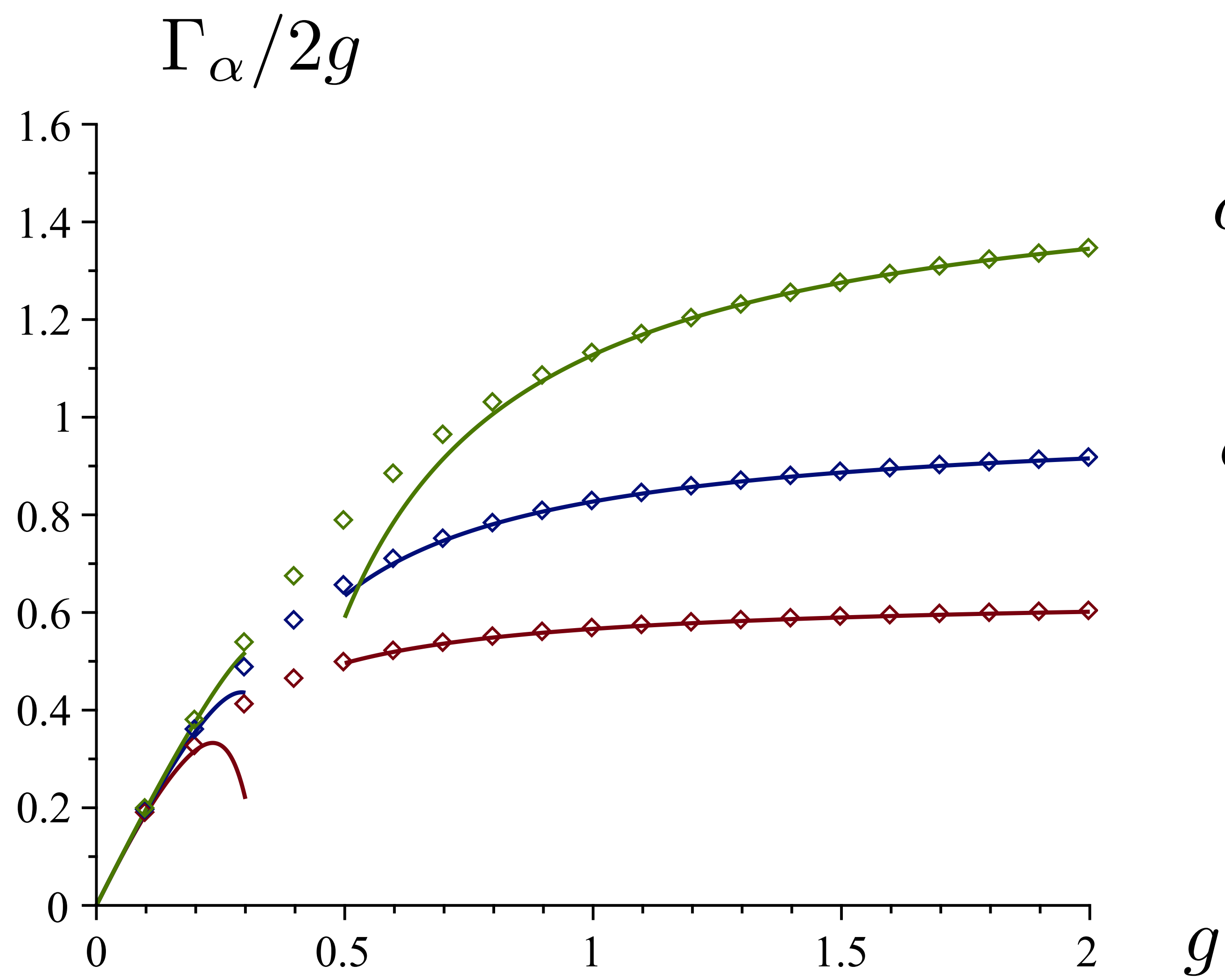
Similarly for the log of the determinant

$$D(0) = \frac{1}{4} \log \left[\frac{\sinh\left(4\pi g\right)}{4\pi g} \right]$$

Numerics I

Truncate matrix at large order and invert it numerically

Same strategy as for cusp
[Benna,Benvenuti,Klebanov,Scardicchio'06]



$\alpha = \frac{\pi}{3}$ (hex)

$\alpha = \frac{\pi}{4}$ (cusp)

$\alpha = 0$ (oct)

Solid lines: weak and strong coupling expansion

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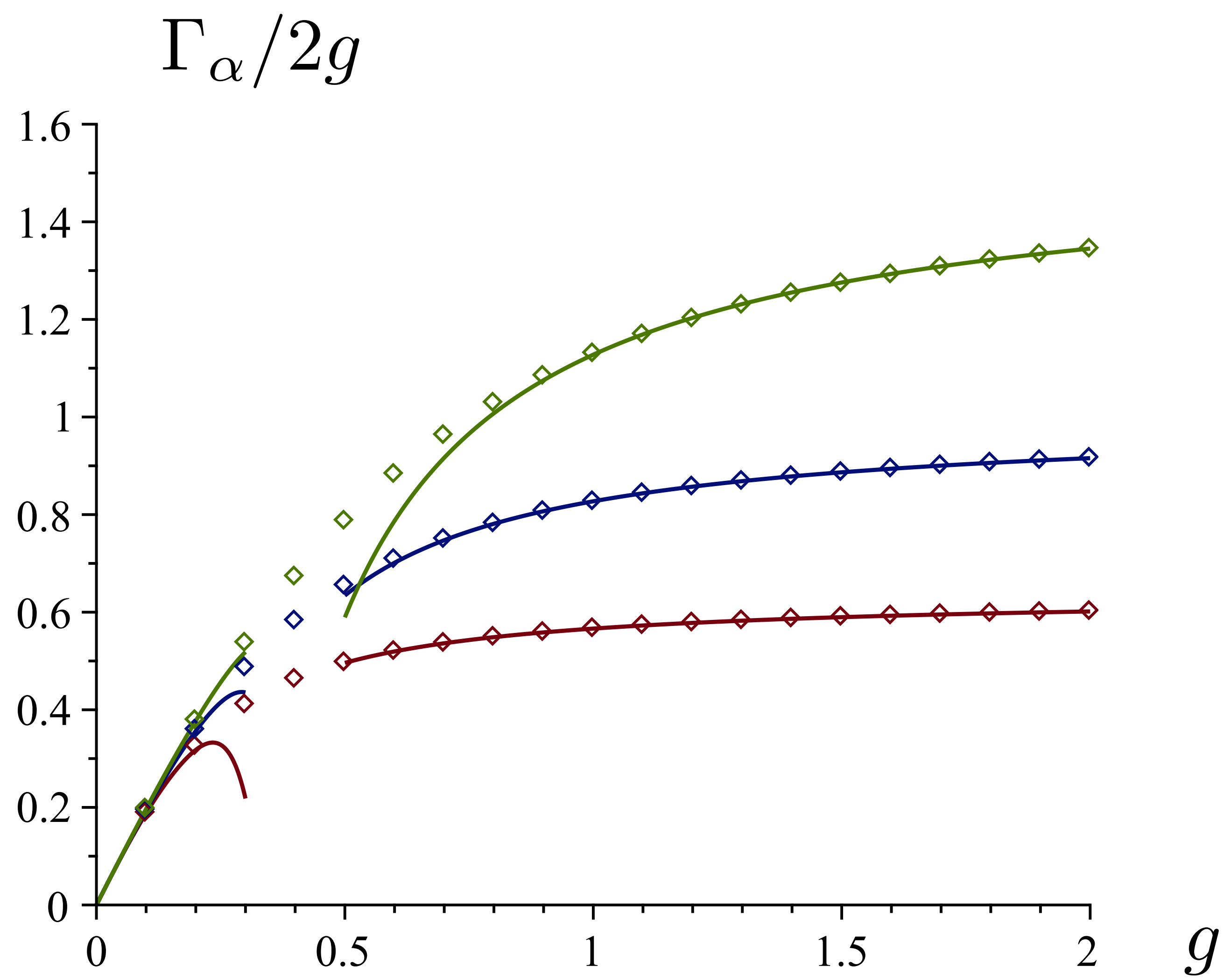


$$\frac{\Gamma_{\text{hex}}}{2g} \to \frac{8}{3\sqrt{3}} \approx 1.54$$

$$\frac{\Gamma_{\text{cusp}}}{2g} \to 1$$

$$\frac{\Gamma_{\text{oct}}}{2g} \to \frac{2}{\pi} \approx 0.63$$

$$\Gamma_\alpha \approx \frac{8\alpha g}{\pi \sin(2\alpha)}$$

Solid lines: weak and strong coupling expansion

Strong coupling expansion

Systematic analysis for the anomalous dimension

$$\Gamma_\alpha \approx \frac{8\alpha g}{\pi \sin(2\alpha)} \left[1 - \frac{s_1}{8\pi g} - \frac{\alpha s_2}{64\pi^3 g^2} - \dots\right]$$

Transcendental numbers $s_{k+1} = -\psi_k(\tfrac{1}{2} + \tfrac{\alpha}{\pi}) - \psi_k(\tfrac{1}{2} - \tfrac{\alpha}{\pi}) + (1 + (-1)^k)\psi_k(1)$

Series is divergent and non-Borel summable

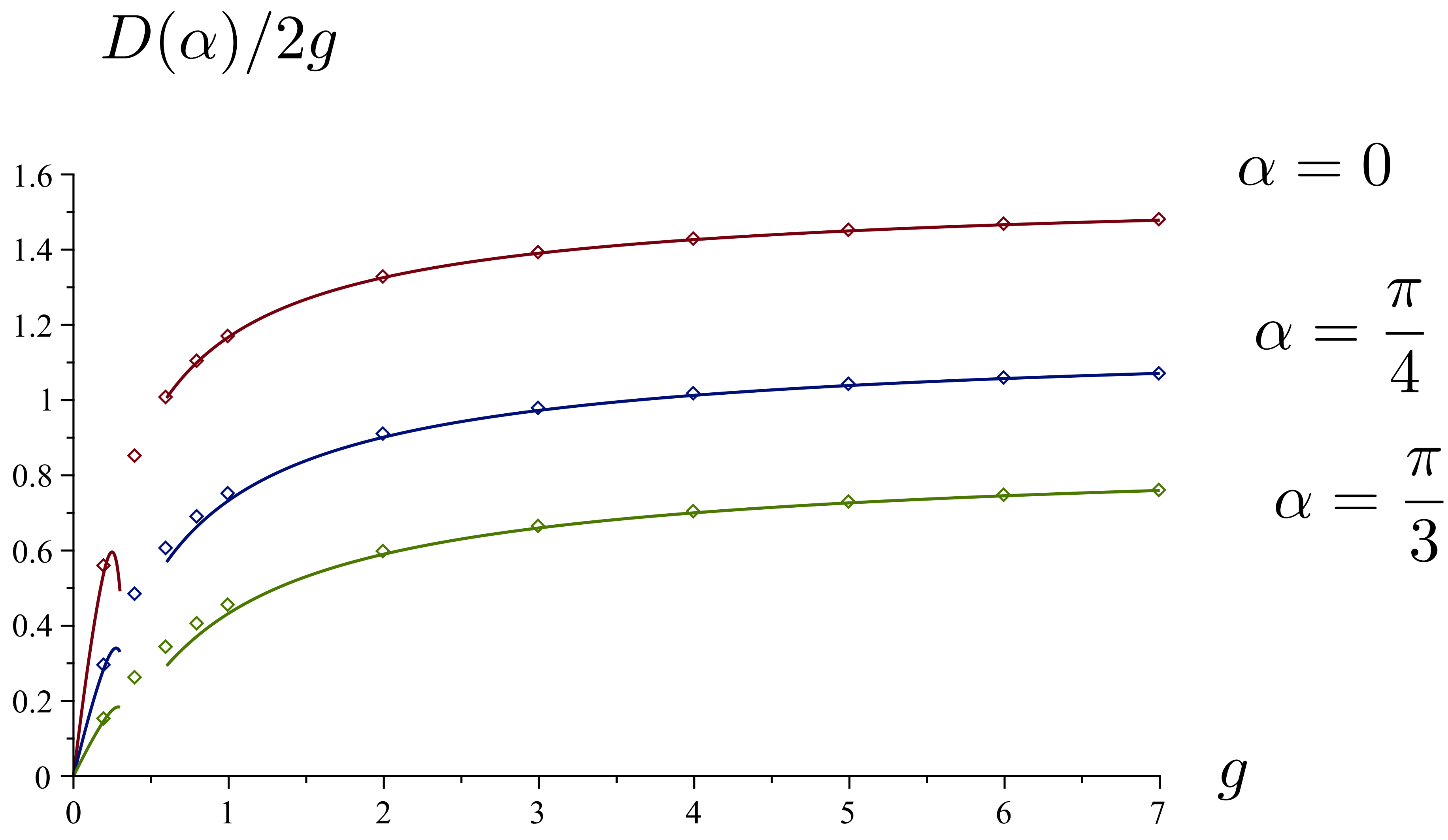
Borel ambiguity generates exponentially small scale $\Lambda^2 \sim g^{\alpha/\pi} e^{-4(\pi - 2\alpha)g}$

LO analysis for constant

$$D(\alpha) = 4\pi g \left[\frac{1}{4} - \frac{\alpha^2}{\pi^2}\right] + O(\log g)$$

Not clear how to expand it systematically

Numerics II



$D(\alpha)/2g$
$\alpha = 0$
$\alpha = \frac{\pi}{4}$
$\alpha = \frac{\pi}{3}$
g
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7

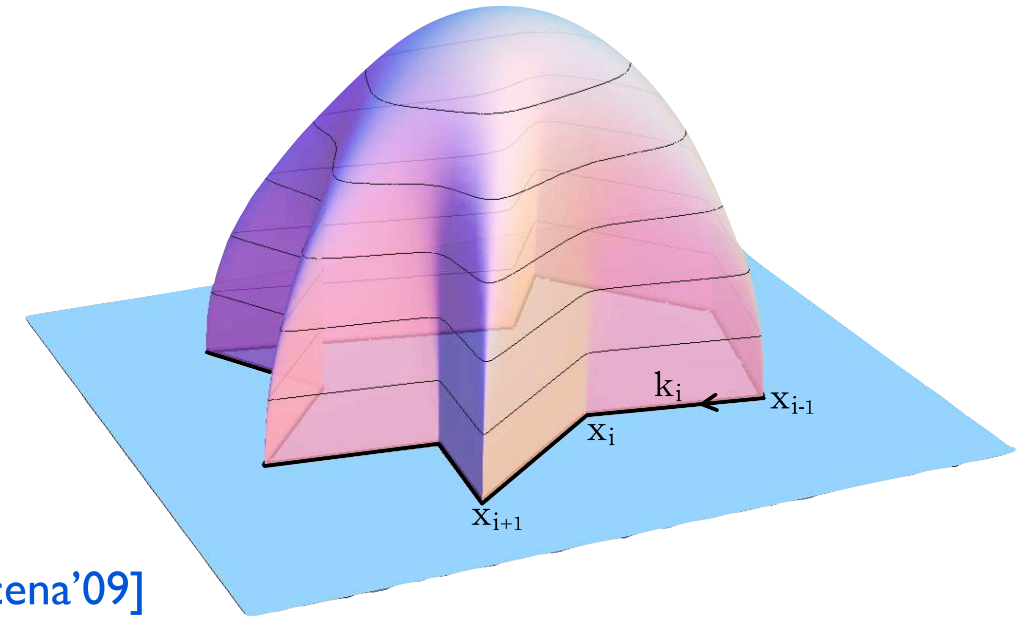
Stringy theory

$$\sqrt{\lambda} = 4\pi g \to \infty$$

String ending on WL contour at boundary of AdS

$$\log \mathcal{W}_6 \approx -\frac{\sqrt{\lambda}}{2\pi} A_6$$

area



Minimal area can be found using **TBA** equations

[Alday,Gaiotto,Maldacena'09]

Two main regimes

"IR" = collinear limit (iterative solution controlled by fluctuations of GKP string)

[Frolov,Tseytlin'02]

"UV" = symmetric polygon, but also near origin behaviour

[Alday,Gaiotto,Maldacena'09]
[Ito,Satoh,Suzuki'18]

Origin from TBA

1) Exact solution on the diagonal $u_{1,2,3} = u$ [Alday,Gaiotto,Maldacena'09]

$$\log \mathcal{E} \sim -\frac{3g}{2\pi} \log^2 u$$ at the origin

It agrees with formula for $\Gamma_{\text{oct}} \sim 4g/\pi$

2) Problem can also be solved off the diagonal if close enough to origin [Ito,Satoh,Suzuki'18]

Simplification: **TBA** equations linearize

$$f(\theta) = \frac{\varphi - \tau\sqrt{2}\cosh\theta}{\cosh(2\theta)} + \int \frac{d\theta'}{2\pi\cosh(\theta-\theta')} f(\theta')$$

It confirms the prediction $\Gamma_{\text{hex}} \sim 16g/3\sqrt{3}$

Conclusion

Six-gluon MHV amplitude exponentiates and simplifies at origin of kinematical space

Why? Field theory argument?

Lots of intriguing relations (starting from the double-logarithmic behaviour itself)

New factorisation?

New relations between correlators and WLs?

Meaning / universality of tilted BES kernel and associated angle

Hidden geometry of amplitudes?

What about higher multiplicity? Double-logs observed through 4 loops for heptagon

(Lance's talk)

How general is that?

Thank you!