Sequential Discontinuities of Scattering Amplitudes

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Motivation for Studying Discontinuities of Amplitudes

- Discontinuities of amplitudes related to cuts of corresponding Feynman diagram.
  - Cut computation often easier.
  - Reconstruct $\mathcal{M}$ from its discontinuities using a basis of functions for Feynman integrals.

$$\text{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} = -\sum \text{cut}\mathcal{M}$$

- What can we learn from studying **sequential discontinuities** of $\mathcal{M}$ in different kinematic regions?
- How do we relate sequential discontinuities of $\mathcal{M}$ to **cuts**?
- What do we gain from a **systematic treatment** of computing discontinuities?

Outline

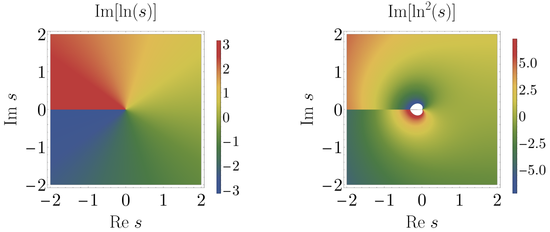
1. Discontinuities
   - Need more powerful tools than $\pm i\varepsilon$ to define sequential discontinuities.
   - Define discontinuities in **channels** using **monodromies**.
   - **Kinematic regions** important for definitions.
2. Relations between **discontinuities** and **cuts**
   - Use **Time-Ordered Perturbation Theory (TOPT)** to prove results.
   - Previous work: Sequential discontinuities in **different channels**.
   - **New results** for relations between sequential discontinuities in the **same channel** and multiple cuts.
   - **New proof** of the **Steinmann** relations.

Problems with $i\varepsilon$ Definition of Discontinuity

$\text{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon}$ **only** defined on the **branch cut**:

$$\text{Disc}_s \ln s = \ln(s + i\varepsilon) - \ln(s - i\varepsilon) = 2\pi i\theta(-s)$$

$$\text{Disc}_s \ln^2 s = \ln^2(s + i\varepsilon) - \ln^2(s - i\varepsilon) = 4\pi i\theta(-s)\ln|s|$$

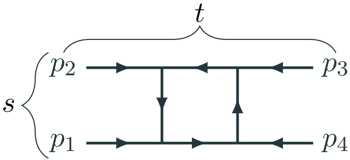


- $\text{Disc}_s \ln^n(s)$ only defined on the **negative real axis**, not on the complex plane.
- What is the $i\varepsilon$ prescription of $\text{Disc}_s\mathcal{M}$?

Need a better definition of Disc to take sequential discontinuities.

Problems with $i\varepsilon$ Definition of Discontinuity

Want to study Disc$\mathcal{M}$ in **each Mandelstam separately**



$$= \mathcal{M}\left(p_j^2, s, t, u\right)$$

Intiuitvely: Define **discontinuity in a channel** s as

$$\mathrm{Disc}_s\mathcal{M} = \mathcal{M}\left(p_j^2, s+i\varepsilon, t, u\right) - \mathcal{M}\left(p_j^2, s-i\varepsilon, t, u\right)$$

- Agrees with cuts in only s?
- **Problem:** Mandelstams are not all independent:

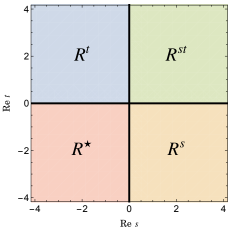
$$s + t + u = \sum p_j^2$$

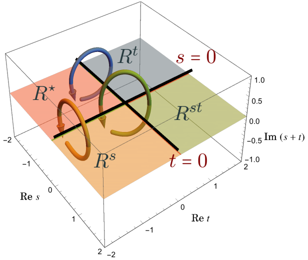
Disc$_s$ should be invariant under rewriting $\mathcal{M}$.

Definition of Discontinuity

Resolution: Abandon the $\pm i\varepsilon$ notation, take **monodromies**.

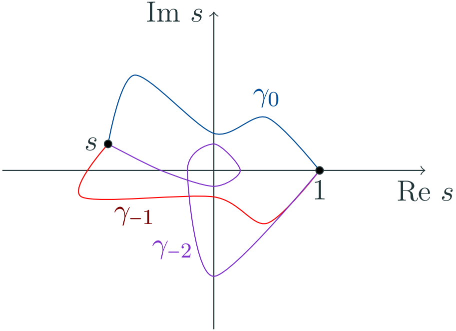
Definition: $\mathrm{Disc}_s\mathcal{M}$ is the **monodromy** of $\mathcal{M}$ around $s = 0$, starting in $\boldsymbol{R^s}$.





- **Monodromy:** How a function changes when analytically continuing around a singularity.
- $\boldsymbol{R^s}$: Region in space of Mandelstams where $s > 0$, all other Mandelstams $s_{i,j,\ldots} < 0$.

Example: Definition of Discontinuity of ln



$$\ln(s+i\varepsilon) = \ln_{\gamma_0} s = \int_{\gamma_0} \frac{dx}{x} \qquad \ln(s-i\varepsilon) = \ln_{\gamma_{-1}} s = \int_{\gamma_{-1}} \frac{dx}{x}$$

Sequential discontinuities natural:

$$\mathrm{Disc}_s \ln_{\gamma_0} s = \ln_{\gamma_0} s - \ln_{\gamma_{-1}} s \qquad \mathrm{Disc}_s \ln_{\gamma_{-1}} s = \ln_{\gamma_{-1}} s - \ln_{\gamma_{-2}} s$$

$$\mathrm{Disc}_s^2 \ln_{\gamma_0} s = (\ln_{\gamma_0} s - \ln_{\gamma_{-1}} s) - (\ln_{\gamma_{-1}} s - \ln_{\gamma_{-2}} s)$$

Definition of Discontinuity in a Channel

$\mathrm{Disc}_s\mathcal{M}$ is the **monodromy** of $\mathcal{M}$ around $s = 0$, starting in R^s.

- Agrees with the $i\varepsilon$ definition in R^s:

$$[\mathrm{Disc}_s\mathcal{M}]_{R^s} = [\mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon}]_{R^s}$$

- Agrees with **cuts** in s, since all other cuts in R^s vanish:

$$[\mathrm{Disc}_s\mathcal{M}]_{R^s} = -\sum \mathrm{Cut}_s\mathcal{M}$$

  - Cuts computed by putting particles **on-shell** with **positive energy flow** across cut.
- Results in a **function on complex space**.
- Machinery: **monodromy operator**.

$$[\mathrm{Disc}_s\mathcal{M}]_{R^s} = \left[\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft^s_0}\right)\mathcal{M}\right]_{R^s}$$

  - Calculation of monodromies becomes algebraic.

Example: Discontinuities, Monodromies and Cuts

$\mathcal{M} =$ 

$\propto -\frac{i}{16\pi^2}\log\left(-p^2 - i\varepsilon\right)$

$$\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^{p^2}}\right)\mathcal{M} \propto -\frac{i}{16\pi^2}\left(-2\pi i\right) = -\frac{1}{8\pi}$$

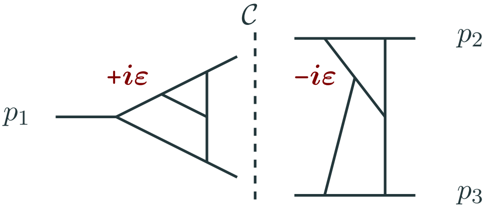
$$\left[\mathrm{Disc}\mathcal{M}\right]_{R^{p^2}} \propto -\frac{i}{16\pi^2}\left(-2\pi i\right)\Theta(p^0) = -\frac{1}{8\pi}\Theta(p^0)$$

$\mathrm{Cut}\mathcal{M} \propto$ 

$= \frac{1}{8\pi}\Theta(p^0)$

Traditional Cutting Rules

$$\text{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} = -\sum \text{Cut}\mathcal{M}$$



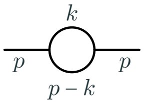
L.h.s. of cut has $+i\varepsilon$, r.h.s. of cut has $-i\varepsilon$.

Proofs:

- Cutkosky, using the Landau equations.
- t'Hooft and Veltman, using the largest time equation.
- Time-ordered perturbation theory (TOPT).
  - **Most transparent and easily generalizable.**

Review of Time-Ordered Perturbation Theory (TOPT)

Sum of $v!$ TOPT diagrams = Feynman diagram



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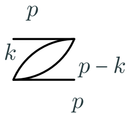
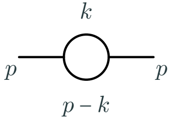
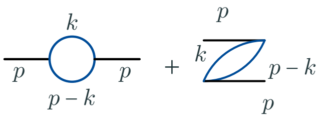
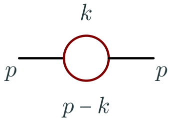




TOPT diagrams	**Feynman diagrams**
• Time passes from left to right	• Vertices are not ordered
• All particles on-shell: $E^2 = \vec{p}^2 + m^2$	• Internal particles virtual: $E^2 \neq \vec{p}^2 + m^2$
• $\vec{p}$ conservation at each vertex	• $\vec{p}$ conservation at each vertex
• **Not E conservation** at each vertex	• E conservation at each vertex
• Overall E & $\vec{p}$ conservation	• Overall E & $\vec{p}$ conservation
• Individual diagrams not Lorentz invariant	• Manifestly Lorentz-invariant
• Good for proofs & intuition	• Good for calculations

Sum of $v!$ TOPT diagrams = Feynman diagram





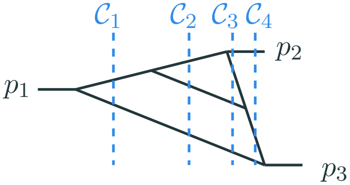
$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \frac{1}{2\omega_{p-k}} \left[\frac{1}{E_p - (\omega_k + \omega_{p-k}) + i\varepsilon} + \frac{1}{E_p - (\omega_k + \omega_{p-k} + 2\omega_p) + i\varepsilon} \right]$$

$$= \int \frac{d^4k}{i(2\pi)^4} \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(p-k)^2 - m_2^2 + i\varepsilon}$$

Cutting Rules in TOPT

Advantages to TOPT:

- Energies are **independent**, Mandelstams are not.
- **One delta** function for **each cut**.
  - Various numbers of on-shell Feynman propagators for each cut through a Feynman diagram.



$$\mathcal{M}|_{+i\varepsilon} \propto \int \frac{1}{E_1 - \omega_1 + i\varepsilon} \frac{1}{E_1 - \omega_2 + i\varepsilon} \frac{1}{E_1 - E_2 - \omega_3 + i\varepsilon} \frac{1}{E_1 - E_2 - \omega_4 + i\varepsilon}$$

Relate Disc$\mathcal{M}$ to cuts using $\frac{1}{E_i + i\varepsilon} - \frac{1}{E_i - i\varepsilon} = -2\pi i \delta\left(E_i\right)$

Results Derived using TOPT

Same channel sequential discontinuities: Equal to a sum of diagrams cut multiple times with a **combinatorial** factor.

$$[\mathrm{Disc}_s^m \mathcal{M}]_{R^s} = (\mathbb{1} - \mathscr{M}_{\circlearrowleft^s_0})^m \mathcal{M}$$
$$= \sum_{k=m} \left\{ \sum_{\ell=1}^{m} (-1)^\ell \binom{m}{\ell} \ell^k \right\} [\mathcal{M}^+_{k\text{-cuts}}]_{R^s}$$

Different channel sequential discontinuities: Equal to a sum of diagrams cut multiple times in a **region** R^{st} where both cuts can be computed.

$$[\mathrm{Disc}_t \mathrm{Disc}_s \mathcal{M}]_{R^{st}} = (\mathbb{1} - \mathscr{M}_{\circlearrowleft^t_{st}})(\mathbb{1} - \mathscr{M}_{\circlearrowleft^s_{st}})\mathcal{M}$$
$$= \left[\sum_{k=1} \sum_{\ell=1} (-1)^{k+\ell} \mathcal{M}^+_{k - s\text{-cuts and } \ell - t\text{-cuts}} \right]_{R^{st}}$$

$\mathcal{M}^+$: $\mathcal{M}$ computed with all $+i\varepsilon$.

Previous work: Discontinuities in Different Channels

Formula presented in Abreu et al. [1]:

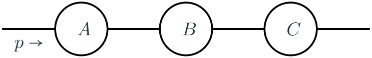
$$\mathrm{Cut}_{s_1,\ldots,s_k}\mathcal{M} = (-1)^k \,\mathrm{Disc}_{s_1,\ldots,s_k}\mathcal{M}$$

- $\mathrm{Cut}_{s_1,\ldots,s_k}$ computed by putting particles on-shell with generalized $+i\varepsilon$ rules in the region

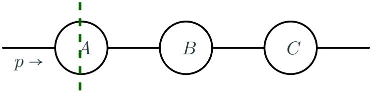
$$R^{s_1,\ldots,s_k} = \{s_1,\ldots,s_k > 0, s_{k+1},\ldots,s_m < 0\}$$

- $\mathrm{Disc}_{s_1,\ldots,s_k}$ computed by taking Disc_{s_1} in R^{s_1}, then Disc_{s_2} in R^{s_1,s_2} etc.
- Checked in multiple examples in [1]: One loop triangle, two loop triangle, boxes.
  - Can check our expressions using examples in [1].

[1] Abreu, S., Britto, R., Duhr, C. and Gardi, E., JHEP 10, 125 (2014), arxiv:1401.3546 [hep-th]

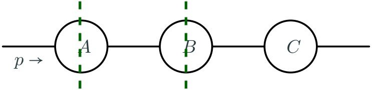
$\mathcal{M} =$ 

$\propto \ln^3\left(-p^2 - i\varepsilon\right)$

$\mathrm{Disc}\mathcal{M} =$ 

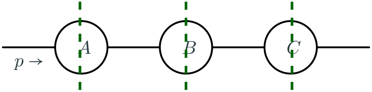
$+ \cdots$

$$\propto 3\,(-2\pi i)\ln^2\left(-p^2 - i\varepsilon\right) - 3(-2\pi i)^2 \ln\left(-p^2 - i\varepsilon\right) + (-2\pi i)^3$$

$\mathrm{Disc}^2\mathcal{M} = 2$ 

$+ \cdots$

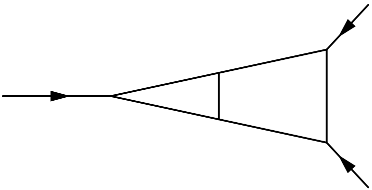
$$\propto 6\,(-2\pi i)^2 \ln\left(-p^2 - i\varepsilon\right) - 6(-2\pi i)^3$$

$\mathrm{Disc}^3\mathcal{M} = 6$ 

$\propto 6\,(-2\pi i)^3$

Discontinuities computed using monodromy matrices
= sum of multiple cut diagrams with a combinatorial factor

Sequential Discontinuities - Two-loop Triangle

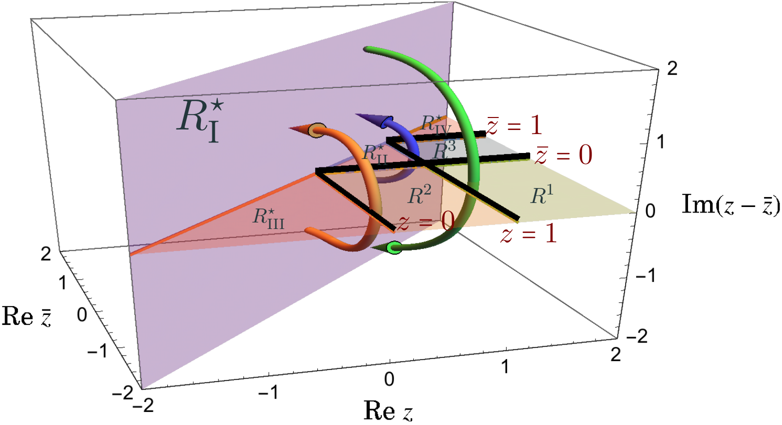


$$\mathcal{M} \propto 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\ln(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\ln^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})]$$

with $z\bar{z} = p_2^2/p_1^2, \quad (1-z)(1-\bar{z}) = p_3^2/p_1^2$

Energy rotations in z, $\bar{z}$ plane

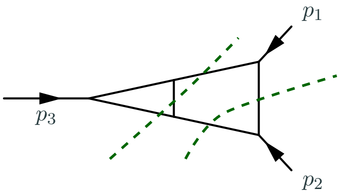
$$\mathcal{M} \propto 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\ln(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\ln^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})]$$



Same Channel Sequential Disc - Two-loop Triangle

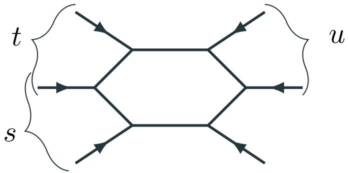
$$\mathcal{M} \propto 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\ln(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\ln^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})]$$

$$\left[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M}(z,\bar{z})\right]_{R^2} \propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})$$

$\left[\mathcal{M}^{\mathrm{cut}}\right]_{R^2} =$  $\propto \frac{1}{2}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\right)$

$$\boxed{\left[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M}(z,\bar{z})\right]_{R^2} = 2\left[\mathcal{M}^{\mathrm{cut}}\right]_{R^2}}$$

$\mathcal{M}$ cannot have sequential discontinuities in partially overlapping channels

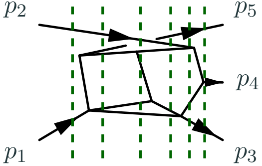


$\mathcal{M}$ cannot contain $\ln(s)\ln(t)$ but can contain $\ln(s)\ln(u)$.

- Important for bootstrapping amplitudes.
- Old proof in S-matrix theory (Steinmann 1960).
  - Non-perturbative, used unitarity.
- Our new proof in TOPT.
  - Applies to individual Feynman integrals.

Proof of Steinmann Relations in TOPT

- Each TOPT diagram has denominators with a sequence of energies, corresponding to Mandelstams.



$$-E_5,\ \ E_1-E_5,\ \ E_1-E_5,\ \ E_1-E_5,\ \ E_1-E_5-E_3,\ \ E_1-E_5-E_3+E_2$$

$$p_5^2,\ \ (p_1-p_5)^2,\ \ (p_1-p_5-p_3)^2,\ \ (p_1-p_5-p_3+p_2)^2$$

- Each energy is a subset of the sequential ones.

Cannot have sequential discontinuities in partially overlapping channels when corresponding regions exist.

Regions may not exist when some particles are massless.

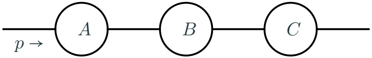
Does not constrain branch points at infinity.

Results

- Discontinuities defined as **monodromies** around singularities.
  - Start in **kinematic region** where cut can be performed.
  - Monodromy matrices make calculations of monodromies algebraic.
- TOPT used to prove:
  1. **Same channel discontinuities:** Equal to a sum of diagrams cut multiple times with a **combinatorial** factor.
  2. **Different channel discontinuities:** Equal to a sum of diagrams cut multiple times in a **kinematic region** where all cuts can be computed.
  3. **Steinmann Relations:** $\mathcal{M}$ cannot have sequential discontinuities in partially overlapping channels.

Backup Slides

Sequential Discontinuities - $\ln^3(s)$

$\mathcal{M} =$  $\propto \ln^3(s)$

Calculate discontinuity and sequential discontinuities using **monodromy matrices**:

- Collect total differentials into a vector.

$$d\left(\frac{\ln^n s}{n!}\right) = \left(\frac{\ln^{n-1} s}{(n-1)!}\right)\frac{ds}{s}\,,$$

$$\mathcal{V} \equiv \begin{pmatrix} 1 & \ln s & \frac{1}{2}\ln^2 s & \frac{1}{3!}\ln^3 s \end{pmatrix}$$

- Solve differential equation.

$$d\mathcal{V} = \mathcal{V}\cdot\omega$$

with

$$\omega = \begin{pmatrix} 0 & \frac{ds}{s} & 0 & 0 \\ 0 & 0 & \frac{ds}{s} & 0 \\ 0 & 0 & 0 & \frac{ds}{s} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Sequential Discontinuities - $\ln^3(s)$

- Collect solutions in a normalized matrix.

$$\mathscr{M}_{\gamma_0} = \begin{pmatrix} 1 & \ln s & \frac{1}{2}\ln^2 s & \frac{1}{3!}\ln^3 s \\ 0 & 1 & \ln s & \frac{1}{2}\ln^2 s \\ 0 & 0 & 1 & \ln s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $d\mathscr{M}_{\gamma_0} = \mathscr{M}_{\gamma_0} \cdot \omega$.

- Calculate monodromies around $s = 0$.

$$\mathscr{M}_{\circlearrowleft} = \begin{pmatrix} 1 & 2\pi i & \frac{1}{2}(2\pi i)^2 & \frac{1}{3!}(2\pi i)^3 \\ 0 & 1 & 2\pi i & \frac{1}{2}(2\pi i)^2 \\ 0 & 0 & 1 & 2\pi i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Sequential Discontinuities - $\ln^3(s)$

- Compute any sequence of discontinuities by multiplying matrices.

$$(\mathbb{1} - \mathscr{M}_{\circlearrowleft}) \cdot \mathscr{M}_{\gamma_0}(s)$$

$$= \begin{pmatrix} 0 & 2\pi i & 2\pi i \ln s + \frac{(2\pi i)^2}{2} & \frac{2\pi i}{2} \ln^2 s + \frac{(2\pi i)^2}{2} \ln s + \frac{(2\pi i)^3}{3!} \\ 0 & 0 & 2\pi i & 2\pi i \ln s + \frac{(2\pi i)^2}{2} \\ 0 & 0 & 0 & 2\pi i \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Disc}_s \frac{\ln^3(s)}{3!} = \frac{2\pi i}{2} \ln^2 s + \frac{(2\pi i)^2}{2} \ln s + \frac{(2\pi i)^3}{3!}$$