Sequential Discontinuities of Scattering Amplitudes

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May 11, 2020

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Motivation for Studying Discontinuities of Amplitudes

- Discontinuities of amplitudes related to cuts of corresponding Feynman diagram.
  - Cut computation often easier.
  - Reconstruct $\mathcal{M}$ from its discontinuities using a basis of functions for Feynman integrals.

$$\text{Disc}\mathcal{M} = \mathcal{M}|_{+i\epsilon} - \mathcal{M}|_{-i\epsilon} = -\sum \text{cut}\mathcal{M}$$

- What can we learn from studying sequential discontinuities of $\mathcal{M}$ in different kinematic regions?
- How do we relate sequential discontinuities of $\mathcal{M}$ to cuts?
- What do we gain from a systematic treatment of computing discontinuities?
Outline

1. Discontinuities
   - Need more powerful tools than $\pm i\varepsilon$ to define sequential discontinuities.
   - Define discontinuities in channels using monodromies.
   - Kinematic regions important for definitions.

2. Relations between discontinuities and cuts
   - Use Time-Ordered Perturbation Theory (TOPT) to prove results.
   - Previous work: Sequential discontinuities in different channels.
   - New results for relations between sequential discontinuities in the same channel and multiple cuts.
   - New proof of the Steinmann relations.
Problems with $i\varepsilon$ Definition of Discontinuity

\[
\text{Disc} \mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} \text{ only defined on the branch cut:}
\]

\[
\text{Disc}_s \ln s = \ln(s + i\varepsilon) - \ln(s - i\varepsilon) = 2\pi i\theta(-s)
\]

\[
\text{Disc}_s \ln^2 s = \ln^2(s + i\varepsilon) - \ln^2(s - i\varepsilon) = 4\pi i\theta(-s) \ln |s|
\]

- $\text{Disc}_s \ln^n(s)$ only defined on the negative real axis, not on the complex plane.
- What is the $i\varepsilon$ prescription of $\text{Disc}_s \mathcal{M}$?

**Need a better definition of Disc to take sequential discontinuities.**
Problems with $i\varepsilon$ Definition of Discontinuity

Want to study $\text{Disc}\mathcal{M}$ in each Mandelstam separately

Intuitively: Define \textbf{discontinuity in a channel} $s$ as

$$\text{Disc}_s\mathcal{M} = \mathcal{M}\left(p_j^2, s + i\varepsilon, t, u\right) - \mathcal{M}\left(p_j^2, s - i\varepsilon, t, u\right)$$

- Agrees with cuts in only $s$?
- \textbf{Problem:} Mandelstams are not all independent:

$$s + t + u = \sum p_j^2$$

\textbf{Disc}_s \text{ should be invariant under rewriting } \mathcal{M}.$
Definition of Discontinuity

Resolution: Abandon the $\pm i\varepsilon$ notation, take monodromies.
Definition: $\text{Disc}_s \mathcal{M}$ is the monodromy of $\mathcal{M}$ around $s = 0$, starting in $R^s$.

- **Monodromy**: How a function changes when analytically continuing around a singularity.
- $R^s$: Region in space of Mandelstams where $s > 0$, all other Mandelstams $s_{i,j,...} < 0$. 
Example: Definition of Discontinuity of $\ln$

\[
\ln(s + i\varepsilon) = \ln_{\gamma_0} s = \int_{\gamma_0}^{\gamma_0} \frac{dx}{x} \quad \ln(s - i\varepsilon) = \ln_{\gamma_{-1}} s = \int_{\gamma_{-1}}^{\gamma_{-1}} \frac{dx}{x}
\]

Sequential discontinuities natural:

\[
\text{Disc}_s \ln_{\gamma_0} s = \ln_{\gamma_0} s - \ln_{\gamma_{-1}} s \quad \text{Disc}_s \ln_{\gamma_{-1}} s = \ln_{\gamma_{-1}} s - \ln_{\gamma_{-2}} s
\]

\[
\text{Disc}_s^2 \ln_{\gamma_0} s = (\ln_{\gamma_0} s - \ln_{\gamma_{-1}} s) - (\ln_{\gamma_{-1}} s - \ln_{\gamma_{-2}} s)
\]
**Definition of Discontinuity in a Channel**

\( \text{Disc}_s \mathcal{M} \) is the **monodromy** of \( \mathcal{M} \) around \( s = 0 \), starting in \( R^s \).

- Agrees with the \( i\varepsilon \) definition in \( R^s \):
  \[
  [\text{Disc}_s \mathcal{M}]_{R^s} = [\mathcal{M}|_{s+i\varepsilon} - \mathcal{M}|_{s-i\varepsilon}]_{R^s}
  \]

- Agrees with cuts in \( s \), since all other cuts in \( R^s \) vanish:
  \[
  [\text{Disc}_s \mathcal{M}]_{R^s} = - \sum \text{Cut}_s \mathcal{M}
  \]
  - Cuts computed by putting particles **on-shell** with **positive energy flow** across cut.

- Results in a **function on complex space**.

- Machinery: **monodromy operator**.
  \[
  [\text{Disc}_s \mathcal{M}]_{R^s} = \left[ (1 - \mathcal{M} \mathcal{O}_0^s) \mathcal{M} \right]_{R^s}
  \]
  - Calculation of monodromies becomes algebraic.
Example: Discontinuities, Monodromies and Cuts

\[ \mathcal{M} = \frac{p}{16\pi^2} \log \left( -p^2 - i\varepsilon \right) \]

\[ (1 - \mathcal{M} \circ_{p^2}^0) \mathcal{M} \propto -\frac{i}{16\pi^2} (-2\pi i) = -\frac{1}{8\pi} \]

\[ [\text{Disc}\mathcal{M}]_{Rp^2} \propto -\frac{i}{16\pi^2} (-2\pi i) \Theta(p^0) = -\frac{1}{8\pi} \Theta(p^0) \]

\[ \text{Cut}\mathcal{M} \propto \frac{p}{8\pi} \Theta(p^0) \]
Traditional Cutting Rules

\[
\text{Disc} \mathcal{M} = \mathcal{M}|_{+i\epsilon} - \mathcal{M}|_{-i\epsilon} = - \sum \text{Cut} \mathcal{M}
\]

**Proofs:**
- Cutkosky, using the Landau equations.
- \(t\)'Hooft and Veltman, using the largest time equation.
- Time-ordered perturbation theory (TOPT).
  - Most transparent and easily generalizable.
Review of Time-Ordered Perturbation Theory (TOPT)

**Sum of \( v! \) TOPT diagrams = Feynman diagram**

\[
\begin{align*}
\text{TOPT diagrams} & \quad \text{Feynman diagrams} \\
\hline
\text{● Time passes from left to right} & \text{● Vertices are not ordered} \\
\text{● All particles on-shell:} & \text{● Internal particles virtual:} \\
\quad E^2 = \vec{p}^2 + m^2 & \quad E^2 \neq \vec{p}^2 + m^2 \\
\text{● } \vec{p} \text{ conservation at each vertex} & \text{● } \vec{p} \text{ conservation at each vertex} \\
\text{● Not } E \text{ conservation at each vertex} & \text{● } E \text{ conservation at each vertex} \\
\text{● Overall } E \text{ & } \vec{p} \text{ conservation} & \text{● Overall } E \text{ & } \vec{p} \text{ conservation} \\
\text{● Individual diagrams not Lorentz invariant} & \text{● Manifestly Lorentz-invariant} \\
\text{● Good for proofs & intuition} & \text{● Good for calculations}
\end{align*}
\]
Sum of $v!$ TOPT diagrams = Feynman diagram

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} \frac{1}{2\omega_{p-k}} \left[ \frac{1}{E_p - (\omega_k + \omega_{p-k}) + i\varepsilon} + \frac{1}{E_p - (\omega_k + \omega_{p-k} + 2\omega_p) + i\varepsilon} \right]$$

$$= \int \frac{d^4 k}{i(2\pi)^4} \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(p-k)^2 - m_2^2 + i\varepsilon}$$
Advantages to TOPT:

- Energies are **independent**, Mandelstams are not.
- **One delta function for each cut.**
  - Various numbers of on-shell Feynman propagators for each cut through a Feynman diagram.

\[
\mathcal{M}|_{+i\varepsilon} \propto \int \frac{1}{E_1 - \omega_1 + i\varepsilon} \frac{1}{E_1 - \omega_2 + i\varepsilon} \frac{1}{E_1 - E_2 - \omega_3 + i\varepsilon} \frac{1}{E_1 - E_2 - \omega_4 + i\varepsilon}
\]

Relate Disc\(\mathcal{M}\) to cuts using \(\frac{1}{E_i + i\varepsilon} - \frac{1}{E_i - i\varepsilon} = -2\pi i\delta(E_i)\)
Results Derived using TOPT

**Same channel sequential discontinuities:** Equal to a sum of diagrams cut multiple times with a **combinatorial** factor.

\[
\left[ \text{Disc}_s^m \mathcal{M} \right]_{R^s} = (1 - \mathcal{M} \cup_{0}^s)^m \mathcal{M} \\
= \sum_{k=m} \left\{ \sum_{\ell=1} (-1)^\ell \binom{m}{\ell} \ell^k \right\} [\mathcal{M}^+_k \text{-cuts}]_{R^s}
\]

**Different channel sequential discontinuities:** Equal to a sum of diagrams cut multiple times in a **region** \( R^{st} \) where both cuts can be computed.

\[
\left[ \text{Disc}_t \text{Disc}_s \mathcal{M} \right]_{R^{st}} = (1 - \mathcal{M} \cup_{st}^t)(1 - \mathcal{M} \cup_{st}^s)\mathcal{M} \\
= \left[ \sum_{k=1} \sum_{\ell=1} (-1)^{k+\ell} \mathcal{M}^+_{k \text{- } s\text{-cuts and } \ell \text{- } t\text{-cuts}} \right]_{R^{st}}
\]

\( \mathcal{M}^+ \): \( \mathcal{M} \) computed with all \(+i\varepsilon\).
Formula presented in Abreu et al. [1]:
\[ \text{Cut}_{s_1, \ldots, s_k} \mathcal{M} = (-1)^k \text{Disc}_{s_1, \ldots, s_k} \mathcal{M} \]

- \text{Cut}_{s_1, \ldots, s_k} computed by putting particles on-shell with generalized \( +i\varepsilon \) rules in the region
\[ \mathcal{R}_{s_1, \ldots, s_k} = \{ s_1, \ldots, s_k > 0, s_{k+1}, \ldots, s_m < 0 \} \]

- \text{Disc}_{s_1, \ldots, s_k} computed by taking Disc_{s_1} in \( \mathcal{R}_{s_1} \), then Disc_{s_2} in \( \mathcal{R}_{s_1,s_2} \) etc.

- Checked in multiple examples in [1]: One loop triangle, two loop triangle, boxes.
  - Can check our expressions using examples in [1].

Sequential Discontinuities - Chain of Bubbles

\[ \mathcal{M} = \frac{A}{p \to} \frac{B}{C} \propto \ln^3 (-p^2 - i\varepsilon) \]

\[ \text{Disc}\mathcal{M} = \frac{A}{p \to} \frac{B}{C} + \ldots \]
\[ \propto 3 (-2\pi i) \ln^2 (-p^2 - i\varepsilon) - 3(-2\pi i)^2 \ln (-p^2 - i\varepsilon) + (-2\pi i)^3 \]

\[ \text{Disc}^2\mathcal{M} = 2 \frac{A}{p \to} \frac{B}{C} + \ldots \]
\[ \propto 6 (-2\pi i)^2 \ln (-p^2 - i\varepsilon) - 6(-2\pi i)^3 \]

\[ \text{Disc}^3\mathcal{M} = 6 \frac{A}{p \to} \frac{B}{C} \]
\[ \propto 6 (-2\pi i)^3 \]

Discontinuities computed using monodromy matrices = sum of multiple cut diagrams with a combinatorial factor
Sequential Discontinuities - Two-loop Triangle

\[ M \propto 6[\text{Li}_4(z) - \text{Li}_4(\bar{z})] - 3 \ln(z\bar{z})[\text{Li}_3(z) - \text{Li}_3(\bar{z})] + \frac{1}{2} \ln^2(z\bar{z})[\text{Li}_2(z) - \text{Li}_2(\bar{z})] \]

with \( z\bar{z} = \frac{p_2^2}{p_1^2}, \quad (1 - z)(1 - \bar{z}) = \frac{p_3^2}{p_1^2} \)
Energy rotations in $z$, $\bar{z}$ plane

\[
\mathcal{M} \propto 6[\text{Li}_4(z) - \text{Li}_4(\bar{z})] - 3 \ln(z\bar{z})[\text{Li}_3(z) - \text{Li}_3(\bar{z})]
+ \frac{1}{2} \ln^2(z\bar{z})[\text{Li}_2(z) - \text{Li}_2(\bar{z})]
\]
\[ \mathcal{M} \propto 6[\text{Li}_4(z) - \text{Li}_4(\bar{z})] - 3 \ln(z\bar{z})[\text{Li}_3(z) - \text{Li}_3(\bar{z})] \]
\[ + \frac{1}{2} \ln^2(z\bar{z})[\text{Li}_2(z) - \text{Li}_2(\bar{z})] \]

\[ \left[ \text{Disc}_{p_2^2} \text{Disc}_{p_2^2} \mathcal{M}(z, \bar{z}) \right]_{R^2} \propto \text{Li}_2(z) - \text{Li}_2(\bar{z}) \]

\[ \left[ \mathcal{M}^{\text{cut}} \right]_{R^2} = \propto \frac{1}{2} (\text{Li}_2(z) - \text{Li}_2(\bar{z})) \]

\[ \left[ \text{Disc}_{p_2^2} \text{Disc}_{p_2^2} \mathcal{M}(z, \bar{z}) \right]_{R^2} = 2 \left[ \mathcal{M}^{\text{cut}} \right]_{R^2} \]
\( \mathcal{M} \) cannot have sequential discontinuities in partially overlapping channels

\[ \mathcal{M} \text{ cannot contain } \ln(s) \ln(t) \text{ but can contain } \ln(s) \ln(u). \]

- Important for bootstrapping amplitudes.
- Old proof in \( S \)-matrix theory (Steinmann 1960).
  - Non-perturbative, used unitarity.
- Our new proof in TOPT.
  - Applies to individual Feynman integrals.
Proof of Steinmann Relations in TOPT

- Each TOPT diagram has denominators with a sequence of energies, corresponding to Mandelstams.

\[-E_5, \ E_1-E_5, \ E_1-E_5, \ E_1-E_5, \ E_1-E_5-E_3, \ E_1-E_5-E_3+E_2\]

\[p_5^2, \ (p_1-p_5)^2, \ (p_1-p_5-p_3)^2, \ (p_1-p_5-p_3+p_2)^2\]

- Each energy is a subset of the sequential ones.

**Cannot have sequential discontinuities in partially overlapping channels when corresponding regions exist.**

Regions may not exist when some particles are massless.

Does not constrain branch points at infinity.
Results

- Discontinuities defined as \textit{monodromies} around singularities.
  - Start in \textit{kinematic region} where cut can be performed.
  - Monodromy matrices make calculations of monodromies algebraic.

- TOPT used to prove:
  1. \textbf{Same channel discontinuities}: Equal to a sum of diagrams cut multiple times with a \textit{combinatorial} factor.
  2. \textbf{Different channel discontinuities}: Equal to a sum of diagrams cut multiple times in a \textit{kinematic region} where all cuts can be computed.
  3. \textbf{Steinmann Relations}: $M$ cannot have sequential discontinuities in partially overlapping channels.
Backup Slides
**Sequential Discontinuities - \( \ln^3(s) \)**

\[
\mathcal{M} = \begin{array}{c}
p \\ \rightarrow \\ A \\ B \\ C \\ \infty \ln^3(s)
\end{array}
\]

Calculate discontinuity and sequential discontinuities using **monodromy matrices**:

- Collect total differentials into a vector.

\[
d\left( \frac{\ln^n s}{n!} \right) = \left( \frac{\ln^{n-1} s}{(n-1)!} \right) \frac{ds}{s},
\]

\[
\mathcal{V} \equiv \begin{pmatrix}
1 \\
\ln s \\
\frac{1}{2} \ln^2 s \\
\frac{1}{3!} \ln^3 s
\end{pmatrix}
\]

- Solve differential equation.

\[
d\mathcal{V} = \mathcal{V} \cdot \omega
\]

with

\[
\omega = \begin{pmatrix}
0 & \frac{ds}{s} & 0 & 0 \\
0 & 0 & \frac{ds}{s} & 0 \\
0 & 0 & 0 & \frac{ds}{s} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

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Sequential Discontinuities - $\ln^3(s)$

- Collect solutions in a normalized matrix.

$$M_{\gamma_0} = \begin{pmatrix}
1 & \ln s & \frac{1}{2} \ln^2 s & \frac{1}{3!} \ln^3 s \\
0 & 1 & \ln s & \frac{1}{2} \ln^2 s \\
0 & 0 & 1 & \ln s \\
0 & 0 & 0 & 1
\end{pmatrix}$$

with $dM_{\gamma_0} = M_{\gamma_0} \cdot \omega$.

- Calculate monodromies around $s = 0$.

$$M_{\gamma} = \begin{pmatrix}
1 & 2\pi i & \frac{1}{2} (2\pi i)^2 & \frac{1}{3!} (2\pi i)^3 \\
0 & 1 & 2\pi i & \frac{1}{2} (2\pi i)^2 \\
0 & 0 & 1 & 2\pi i \\
0 & 0 & 0 & 1
\end{pmatrix}$$
Sequential Discontinuities - $\ln^3(s)$

- Compute any sequence of discontinuities by multiplying matrices.

\[
(1 - \mathcal{M}_\otimes) \cdot \mathcal{M}_{\gamma_0}(s) = \begin{pmatrix} 0 & 2\pi i & 2\pi i \ln s + \frac{(2\pi i)^2}{2} & \frac{2\pi i}{2} \ln^2 s + \frac{(2\pi i)^2}{2} \ln s + \frac{(2\pi i)^3}{3!} \\ 0 & 0 & 2\pi i & 2\pi i \ln s + \frac{(2\pi i)^2}{2} \\ 0 & 0 & 0 & 2\pi i \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\text{Disc}_s \frac{\ln^3(s)}{3!} = \frac{2\pi i}{2} \ln^2 s + \frac{(2\pi i)^2}{2} \ln s + \frac{(2\pi i)^3}{3!}
\]