SAGEX

Scattering Amplitudes:
from Geometry to Experiment

Niels Bohr Institutet

The Niels Bohr International Academy

AMPLITUDES 2021

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Niels Bohr International Academy -- Niels Bohr Institute

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Organizing committee:

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SAGEX

Scattering Amplitudes:
from Geometry to Experiment

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Einstein Gravity from Scattering Amplitudes

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Amplitudes 2020

Work together with

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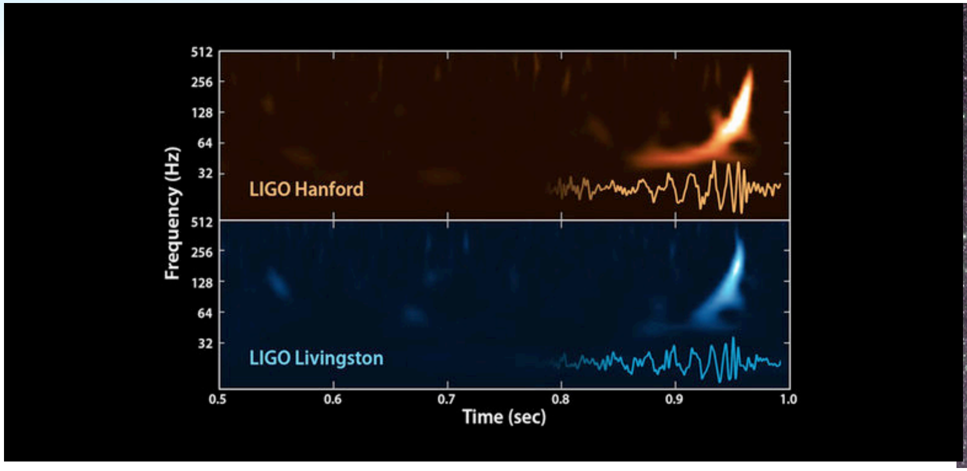
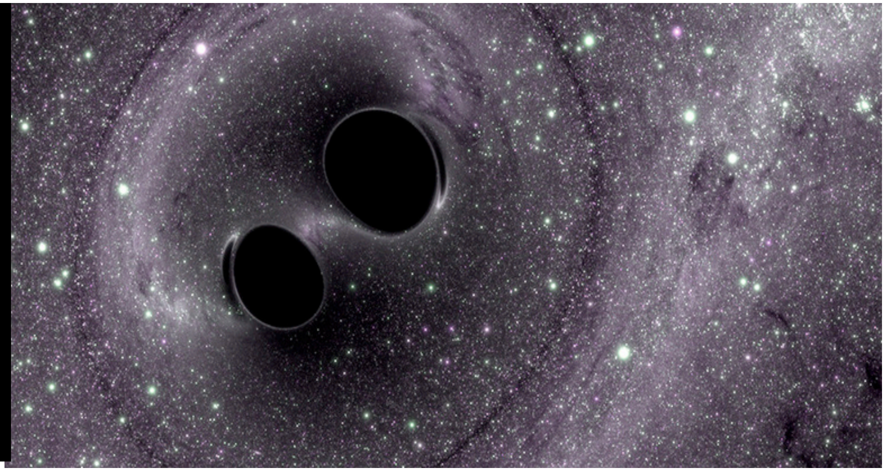
CARLSBERGFONDET

Outline

- General Relativity as a perturbative effective field theory
- New on-shell toolboxes for computations
- New applications: computation of observables in general relativity
  - Scattering angles
  - Energy relation
- Outlook

Exciting motivations to study gravity





- First direct observation of a binary merger of black holes
- Direct access to gravitational interactions in the most extreme regimes
  - Conventional brute-force analysis is becoming inadequate.
  - Need to catch up with observational progress
  - Develop gravity phenomenology

Exciting motivations to study gravity from amplitudes

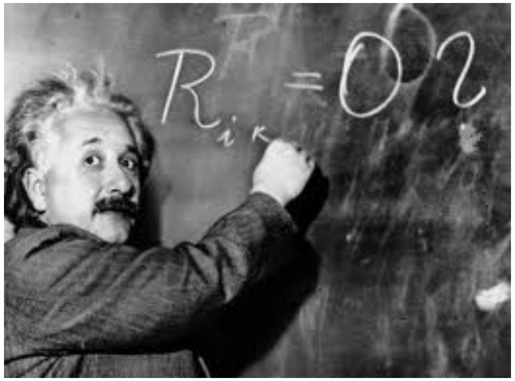
- **Surprise:** Classical physics from a relativistic quantized theory of gravitons seems more efficient than directly solving Einstein's field equations!
- Today we will hear many talks on this topic, e.g. (**Julio, Michele, Radu, Chia-Hsien)**
- **One key question:** Can we formulate a precise extraction of classical gravitational physics from on-shell amplitudes? (with least amount of work)

Goal: Increased potential for discovery of new physics with a faster and more accurate theoretical breakdown of gravitational wave events!

General Relativity as an effective field theory

General Relativity as a quantum field theory

- Known for a long time that a particle version of General Relativity can be derived from the Einstein Hilbert Lagrangian



- Expand Einstein-Hilbert Lagrangian :

$$\mathcal{L}_{\mathrm{E}H} = \int d^4x \left[\sqrt{-g} R\right] \qquad g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

- Derive vertices as in a particle theory - computations using Feynman diagrams!

Gravity as a quantum field theory

- Viewpoint: Gravity as a non-abelian gauge field theory with self-interactions
- Non-renormalisable theory! ('t Hooft and Veltman)

Dimensionful coupling: $G_N = 1/M^2_{planck}$

- Traditional belief : – no known symmetry can remove all UV-divergences

String theory can by introducing new length scales

Quantum gravity as an effective field theory

- (Weinberg) proposed to view the quantization of general relativity as that of an effective field theory

$$\mathcal{L} = \sqrt{-g}\left[\frac{2R}{\kappa^2} + \mathcal{L}_{\text{matter}}\right]$$

$$\Downarrow$$

$$\mathcal{L} = \sqrt{-g}\left\{\frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \ldots\right\}$$

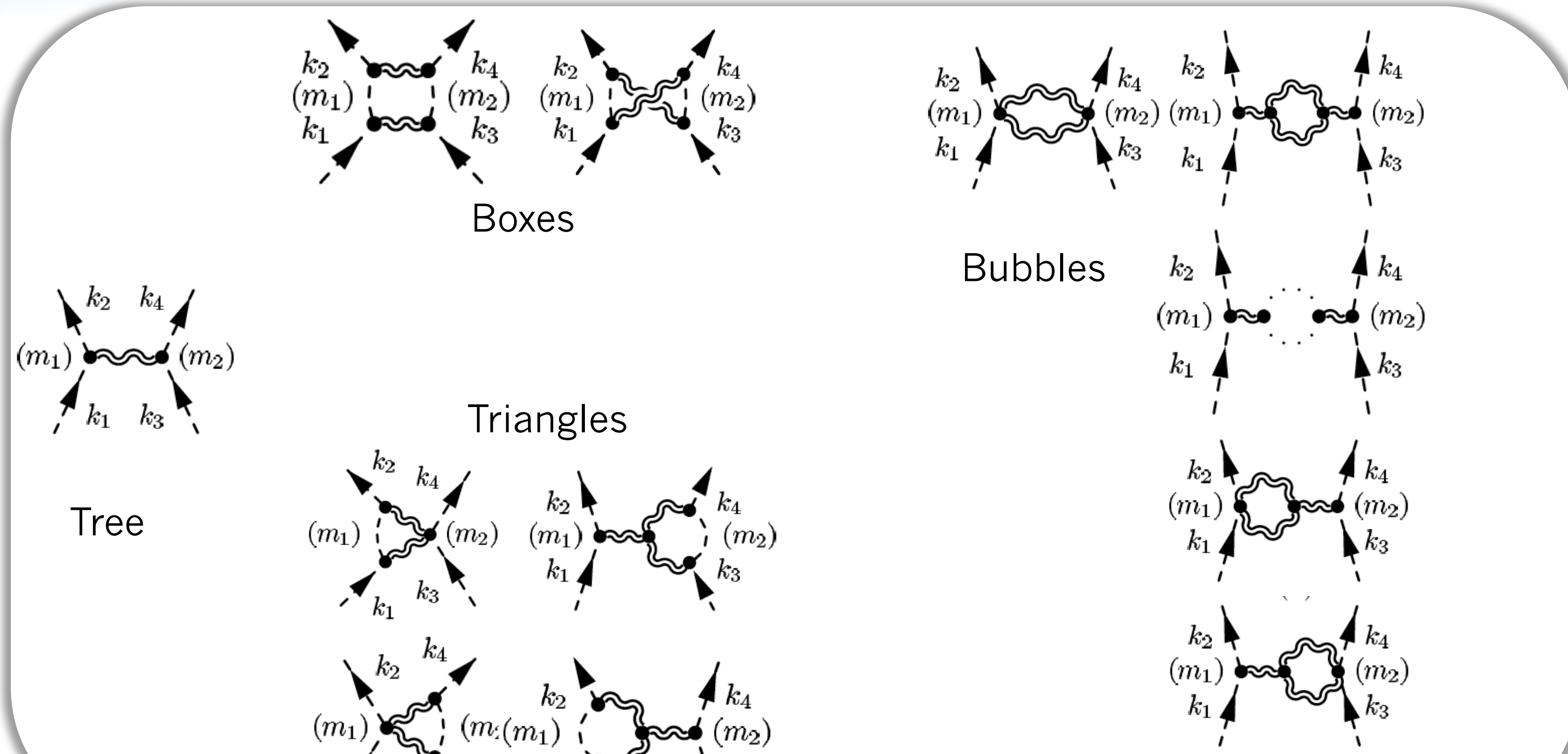
Practical quantum gravity at low energies

- Consistent quantum theory:
  - Quantum gravity at low energies (Donoghue)
  - Direct connection to low energy dynamics of string and super-gravity theories
    - Suggest Einstein's theory augmented by higher derivative operators – the most general modified theory

Classical physics from quantum theory! (Iwasaki; Donoghue, Holstein; Kosower, Maybee, O'Connell)

NB: Contact with Einstein's theory require care..!

One-loop (off-shell) gravity computation



(NEJB, Donoghue, Holstein 2001)

One-loop result for gravity

- Four point one-loop amplitude can be deduced to take the form

$$\mathcal{M} \sim \left(A + Bq^2 + \ldots + \alpha\kappa^4 \frac{1}{q^2} + \beta_1 \kappa^4 \ln(-q^2) + \beta_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \ldots\right)$$

Short range behavior

Focus on deriving these ~>
Long-range behavior
(no higher derivative contributions)

One-loop result for gravity

- Four point one-loop amplitude can be deduced to take the form

$$\mathcal{M} \sim \left(A + Bq^2 + \ldots + \alpha\kappa^4 \frac{1}{q^2} + \beta_1 \kappa^4 \ln(-q^2) + \beta_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \ldots\right)$$

$$\Downarrow$$

Lesson I: We do not have to compute full amplitudes:
Only long-range contributions needed for most applications.

Lesson II: We get unique quantum terms even with effective field theory action (one-loop).

Lesson III: Classical contributions arise from loop diagrams.

One-loop result for gravity

- The result for the amplitude (in coordinate space) after summing all diagrams is:

(NEJB, Donoghue, Holstein)

$$-\frac{Gm_1m_2}{r}\left[1+3\frac{G(m_1+m_2)}{r}+\frac{41}{10\pi}\frac{G\hbar}{r^2}\right]$$

$\Downarrow$ Post-Newtonian term

$\Downarrow$ New quantum term

Post-Newtonian term in complete accordance with general relativity*: (Iwasaki; Holstein and Ross; Neill and Rothstein, NEJB, Damgaard, Festuccia, Plante, Vanhove)

Einstein-Infeld-Hoffman Potential

- *In order to see this. Solve for potential in non-relativistic limit, (Born subtraction)

$$
\begin{aligned}
i\langle f|T|i\rangle &= -2\pi i\delta(E-E') \\
&\times \left[\langle f|\tilde{V}_{bs}(\mathbf{q})|i\rangle + \sum_n \frac{\langle f|\tilde{V}_{bs}(\mathbf{q})|n\rangle\langle n|\tilde{V}_{bs}(\mathbf{q})|i\rangle}{E-E_n+i\epsilon} + \ldots\right]
\end{aligned}
$$

$$
\langle f|\tilde{V}_{bs}(\mathbf{q})|i\rangle = -\frac{Gm_1m_2}{r}\left[1+3\frac{G(m_1+m_2)}{r}\right]
$$

- Contact with Einstein-Infeld-Hoffmann Hamiltonian

$$
\tilde{V}_{bs}(r) = V(r) + \frac{7Gm_1m_2(m_1+m_2)}{2c^2r^2}
$$

Post-Newtonian interaction potentials

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_4^{\,2}}{2m_2} - \frac{\vec{p}_1^{\,4}}{8m_1^3} - \frac{\vec{p}_4^{\,4}}{8m_2^3}$$

$$- \frac{Gm_1m_2}{r} - \frac{G^2m_1m_2(m_1+m_2)}{2r^2}$$

$$- \frac{Gm_1m_2}{2r}\left(\frac{3\vec{p}_1^{\,2}}{m_1^2} + \frac{3\vec{p}_4^{\,2}}{m_2^2} - \frac{7\vec{p}_1\cdot\vec{p}_4}{m_1m_2} - \frac{(\vec{p}_1\cdot\vec{r})(\vec{p}_4\cdot\vec{r})}{m_1m_2r^2}\right)$$

(Einstein-Infeld-Hoffman, Iwasaki)

Crucial subtraction of Born term to in order to get the correct PN potential

(3 – 7/2 -> -1/2)

Gravity as an EFT

- Treating general relativity as an effective field avoid many of the usual complications and confusions in quantizing gravity

- The new results are unique consequences of an underlying more fundamental theory.

- Classical GR has a huge validity for normal energies, but GR-EFT provides attractive alternative for low-energy quantum treatments
  - Ideal perturbative setup for theoretical analysis of black hole binary mergers

New on-shell toolbox for computations

Off-shell computation of amplitudes

- Expand Lagrangian, laborious and tedious process....
- Vertices: 3pt, 4pt, 5pt,..n-pt
- Complicated off-shell expressions

$$
\begin{aligned}
V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[&-\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\
&+ \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\
&- P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\
&+ 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big],
\end{aligned}
$$

45 terms + sym

(DeWitt;Sannan)

Much more complicated than Yang-Mills theory but still many useful applications..

Concrete computation gravity

Off-shell computations feature a number of unpleasant features

- Numerous double contractions
- Factorial growth in number of legs
- Feynman diagram topologies: no ordering!
- Gauge choices
- Ghost terms
- Complicated tensor loop-integrals

(Donoghue; NEJB, Donoghue, Holstein; Cheung, Solon)

Simplifications from Spinor-Helicity

Huge simplifications

$$s_{ij} = -\langle \lambda, \mu \rangle [\tilde{\lambda}, \tilde{\mu}]$$

$$\begin{aligned}
V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\, \mathrm{sym}\Big[& -\frac{1}{2}P_3(k_1\cdot k_2\, \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\
& + \frac{1}{2}P_3(k_1\cdot k_2\, \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\, \eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\
& - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\
& + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\, \eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big],
\end{aligned}$$

One term!

Vanish in spinor helicity formalism

Contractions

$$\varepsilon^-_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \qquad \tilde{\varepsilon}^+_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}$$

Gravity:

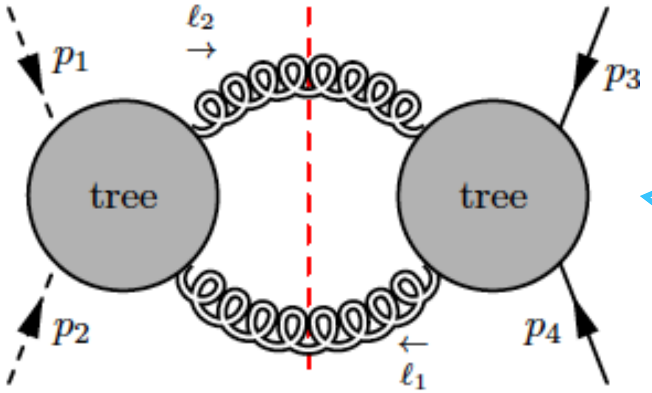
$\varepsilon^- \; \varepsilon^-$

$\tilde{\varepsilon}^+ \; \tilde{\varepsilon}^+$

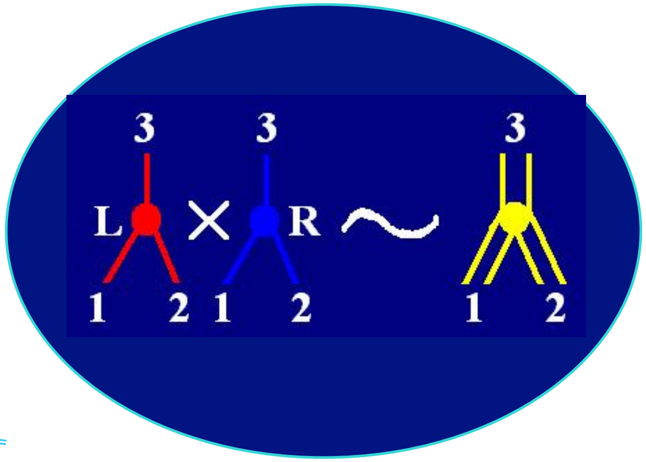
One-loop and the cut

- It is in fact much simpler to capture the long-range behavior from unitarity and on-shell tree amplitudes

$$C_{i,\ldots,j} = \mathrm{Im}_{K_{i,\ldots,j}>0} M^{1\text{-loop}}$$



- No ghosts!



KLT + on-shell 4D input trees recycled from Yang-Mills

(Badger et al; Forde Kosower)

in D-dimensions (from CHY)

(NEJB, Cristofoli, Damgaard, Gomez)

(Neill,Rothstein; NEJB, Donoghue, Vanhove)

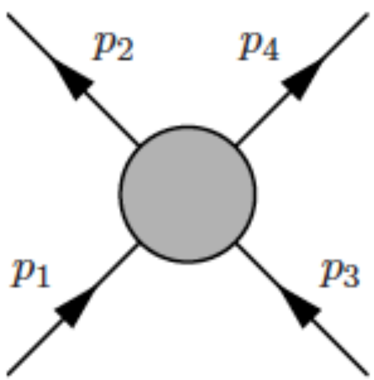
Example: Massive scalar-scalar scattering

- Will consider scalar-scalar scattering amplitudes mediated through graviton field theory interaction

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G_N} + \frac{1}{2} \sum_a \left(g^{\mu\nu} \partial_\mu \phi_a \, \partial_\nu \phi_a - m_a^2 \phi_a^2 \right) \right]$$

$$\mathcal{M}^{\mathrm{L-loop}} \sim O(G_N^{L+1})$$

$\mathcal{M} =$



$= \sum_{L=0}^{+\infty} \mathcal{M}^{\mathrm{L-loop}}$

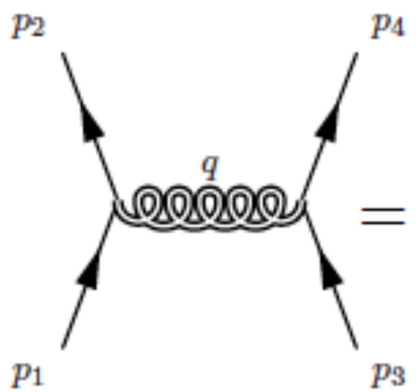
$$p_1^\mu = (E_a, \vec{p}) \,, \quad p_2^\mu = (E_a, \vec{p}\,') \,,$$

$$p_3^\mu = (E_b, -\vec{p}) \,, \; p_4^\mu = (E_b, -\vec{p}\,')$$

$$|\vec{p}| = |\vec{p}\,'| \qquad q^\mu = p_1^\mu - p_2^\mu$$

Tree level

$$\mathcal{M}^{\text{tree}} =$$



$$= -\frac{4\pi G_N}{E_a E_b} \frac{[2(p_1 \cdot p_3)^2 - m_a^2 m_b^2 - |\vec{q}\,|^2 (p_1 \cdot p_3)]}{|\vec{q}\,|^2}$$

$$p_1 \cdot p_3 = E_a(p) E_b(p) + |\vec{p}\,|^2$$

Newton's law through Fourier transform

$$V(r) = -\frac{G m_1 m_2}{r}$$

Unitarity method trees

- Starting from Yang-Mills trees we have $\kappa_{(4)}^2 = 32\pi G_N$

$$iM_s^{\rm tree}(p_1,p_2,k_1,k_2) = \kappa_{(4)}^2\,(p_1\cdot k_1)\,A_s^{\rm tree}(p_1,p_2,k_2,k_1)A_0^{\rm tree}(p_1,k_2,p_2,k_1)$$

s = 0, ½, 1,

- The color striped YM amplitude satisfies

$$A_s^{\rm tree}(p_1,p_2,k_2,k_1) = \frac{p_1\cdot k_2}{k_1\cdot k_2}\,A_s^{\rm tree}(p_1,k_2,p_2,k_1)$$

$$\boxed{iM_s^{\rm tree}(p_1,p_2,k_1,k_2) = \frac{\kappa_{(4)}^2}{e^2}\frac{(p_1\cdot k_1)\,p_1\cdot k_2}{k_1\cdot k_2}\,A_s^{\rm tree}(p_1,k_2,p_2,k_1)A_0^{\rm tree}(p_1,k_2,p_2,k_1)}$$

(NEJB, Donoghue, Vanhove)

Massive KLT squaring

In all generality we have

$$iM^{\text{tree}} = \sum_{\sigma,\gamma\in\mathfrak{S}_{n-3}} \mathcal{S}[\sigma(2,\cdots,n-2)|\gamma(2,\cdots,n-2)]|_{k_1} \times$$
$$\times A^{\text{tree}}(1,\sigma(2,\cdots,n-2),n-1,n)A^{\text{tree}}(n,n-1,\gamma(2,\cdots,n-2),1)$$

Where

$$\boxed{S[i_1,\ldots,i_r|j_1,\ldots,j_r]|_p = \prod_{t=1}^{r}(p\cdot k_{i_r} + \sum_{s>t}^{r}\theta(i_r,i_s)\,k_{i_r}\cdot k_{i_s})}$$

(NEJB, Damgaard, Feng, Søndergaard; NEJB, Damgaard, Sondergaard, Vanhove)

Unitary cut

- One has

$$A_0^{\text{tree}}(p_1, k_2^+, p_2, k_1^+) = -\frac{m^2\,[k_1\,k_2]^2}{4(p_1\cdot k_1)\,(p_1\cdot k_2)}$$

$$A_0^{\text{tree}}(p_1, k_2^-, p_2, k_1^+) = \frac{\langle k_2|p_1|k_1]^2}{4\,(k_1\cdot p_1)(p_1\cdot k_2)}$$

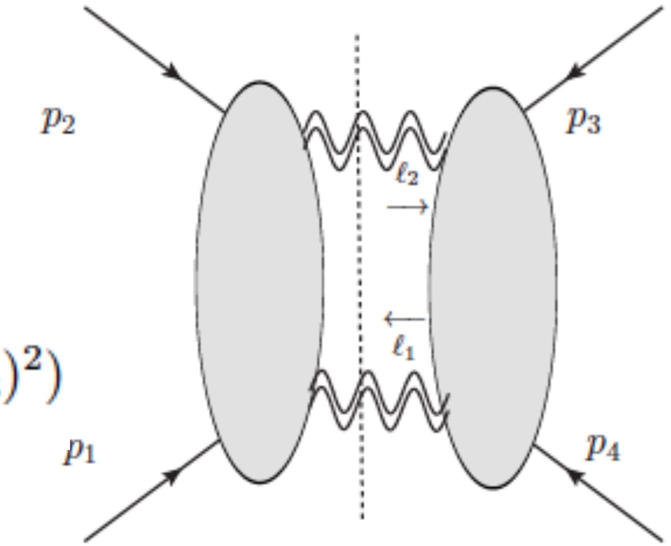
- This yields

$$\boxed{iM_0^{\text{tree}}(p_1, p_2, k_1^+, k_2^+) = \frac{\kappa_{(4)}^2}{16}\,\frac{1}{(k_1\cdot k_2)}\,\frac{m^4\,[k_1\,k_2]^4}{(k_1\cdot p_1)(k_1\cdot p_2)}}$$

$$\boxed{iM_0^{\text{tree}}(p_1, p_2, k_1^-, k_2^+) = \frac{\kappa_{(4)}^2}{16}\,\frac{1}{(k_1\cdot k_2)}\,\frac{\langle k_1|p_1|k_2]^2\,\langle k_1|p_2|k_2]^2}{(k_1\cdot p_1)(k_1\cdot p_2)}}$$

Compact double cut

- In the cut we have



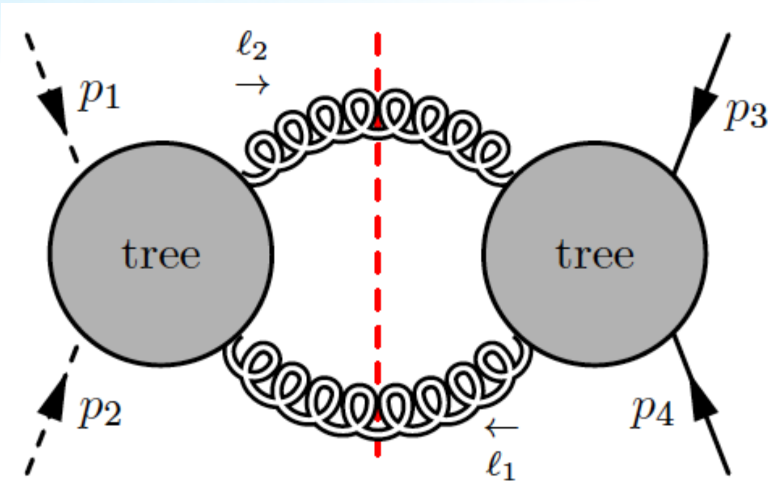
$$\mathcal{N}^{\text{singlet}} = m_1^2 m_2^2 s^2$$

$$\mathcal{N}^{\text{non-singlet}} = \frac{1}{2}(\text{tr}_-(\ell_2 p_1 \ell_1 p_3)^2 + \text{tr}_+(\ell_2 p_1 \ell_1 p_3)^2)$$

$$iM^{1-\text{loop}}\big|_{disc} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\sum_{\lambda_1,\lambda_2} M^{\text{tree}}_{\lambda_1\lambda_2}(p_1, p_2, -\ell_2^{\lambda_2}, \ell_1^{\lambda_1})(M^{\text{tree}}_{\lambda_1\lambda_2}(p_3, p_4, \ell_2^{\lambda_2}, -\ell_1^{\lambda_1}))^*}{\ell_1^2 \ell_2^2}\Bigg|_{cut}$$

$$\boxed{iM^{1-\text{loop}}\big|_{disc} = \frac{e^4}{16} \int \frac{d^D\ell}{(2\pi)^D} \frac{\mathcal{N}}{\ell_1^2 \ell_2^2 \prod_{i=1}^4 (p_i \cdot \ell_1)}}$$

Result for the one-loop amplitude



1) Expand out traces
2) Reduce to scalar basis of integrals
3) Isolate coefficients

(NEJB, Donoghue, Vanhove)

(See also Cachazo and Guevara;
(Bern, Cheung, Roiban, Shen, Solon, Zeng)

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_{\Box}\mathcal{I}_{\Box} + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots \right)$$

Integrals in the one-loop amplitude

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_\Box \mathcal{I}_\Box + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

$$\mathcal{I}_\Box = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_a^2 + i\varepsilon)((\ell-p_3)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}$$

$$\mathcal{I}_{\bowtie} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_a^2 + i\varepsilon)((\ell+p_4)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}$$

$$\mathcal{I}_{\triangleright} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+p_1)^2 - m_a^2 + i\varepsilon)}$$

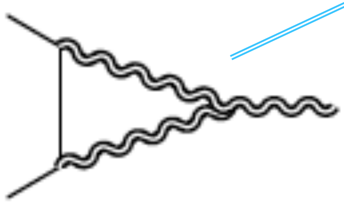
$$\mathcal{I}_{\triangleleft} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell-q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell-p_3)^2 - m_b^2 + i\varepsilon)}$$

Classical pieces in loops

Surprise: loop diagrams has classical physics

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell+q)^2 + i\epsilon} \frac{1}{(\ell+p_1)^2 - m_1^2 + i\epsilon}$$

$$(\ell + p_1)^2 - m_1^2 = \ell^2 + 2\ell \cdot p_1 \simeq 2m_1 \ell_0$$



(NEJB, Damgaard, Festuccia, Plante, Vanhove)

$$\frac{1}{2m_1} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell+q)^2 + i\epsilon} \frac{1}{\ell_0 + i\epsilon}$$

Classical pieces in loops

$$\frac{1}{2m_1}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\epsilon}\,\frac{1}{(\ell+q)^2+i\epsilon}\,\frac{1}{\ell_0+i\epsilon}$$

Close contour

$$\int_{|\vec{\ell}|\ll m}\frac{d^3\vec{\ell}}{(2\pi)^3}\,\frac{i}{4m}\,\frac{1}{\vec{\ell}^2}\,\frac{1}{(\vec{\ell}+q)^2} = -\frac{i}{32m|\vec{q}|}$$

Putting it all together

Imaginary phase

Ignore quantum pieces

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_{\Box}\mathcal{I}_{\Box} + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

$$\mathcal{I}_{\Box} = -\frac{i}{16\pi^2|\vec{q}|^2}\left(-\frac{1}{m_a m_b} + \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2} + \frac{i\pi}{|p|E_p}\right)\left(\frac{2}{3-d} - \log|\vec{q}|^2\right) + \cdots$$

$$\mathcal{I}_{\bowtie} = -\frac{i}{16\pi^2|\vec{q}|^2}\left(\frac{1}{m_a m_b} - \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2}\right)\left(\frac{2}{3-d} - \log|\vec{q}|^2\right) + \cdots$$

$$\mathcal{I}_{\triangleright} = -\frac{i}{32m_a}\frac{1}{|\vec{q}|} + \cdots$$

$$\mathcal{I}_{\triangleleft} = -\frac{i}{32m_b}\frac{1}{|\vec{q}|} + \cdots$$

$$c_{\Box} = c_{\bowtie} = 4\left(m_a^2 m_b^2 - 2(p_1\cdot p_3)^2\right)^2$$

$$c_{\triangleright} = 3m_a^2\left(m_a^2 m_b^2 - 5(p_1\cdot p_3)^2\right)$$

$$c_{\triangleleft} = 3m_b^2\left(m_a^2 m_b^2 - 5(p_1\cdot p_3)^2\right)$$

Relation to a PM potential

- We use the language of old-fashioned time-ordered perturbation theory
- In particular we eliminate by hand
    - Annihilation channels
    - Back-tracking diagrams (no intermediate multiparticle states)
    - Anti-particle intermediate states

We will also assume (classical) long-distance scattering

(Cristofoli, Bjerrum-Bohr, Damgaard, Vanhove)

Relation to a PM potential

- One-loop amplitude after summing all contributions

$$\mathcal{M}^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^2 \xi}\left[\frac{1}{2|\vec{q}|}\left(\frac{c_{\triangleright}}{m_a} + \frac{c_{\triangleleft}}{m_b}\right) + \frac{i}{E_p}\frac{c_{\Box}}{|\vec{p}|}\frac{\left(\frac{2}{3-d} - \log|\vec{q}|^2\right)}{\pi|\vec{q}|^2}\right]$$

Imaginary super-classical/ singular

- How to relate to a classical potential?
  - Choice of coordinates
  - Born subtraction/Lippmann-Schwinger

Relation to a relativistic PM potential

- Amplitude defined via perturbative expansion around a flat Minkowskian metric

- Now we need to relate the Scattering Amplitude to the potential for a bound state problem – alternative to matching (Cheung, Solon, Rothstein; Bern, Cheung, Roiban, Shen, Solon, Zeng)

- Starting point: the Hamiltonian of the relativistic Salpeter equation

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}, \qquad \hat{\mathcal{H}}_0 = \sqrt{\hat{k}^2 + m_a^2} + \sqrt{\hat{k}^2 + m_b^2}$$

Relation to a potential

- Analysis involves solution of the Lippmann-Schwinger recursive equation:

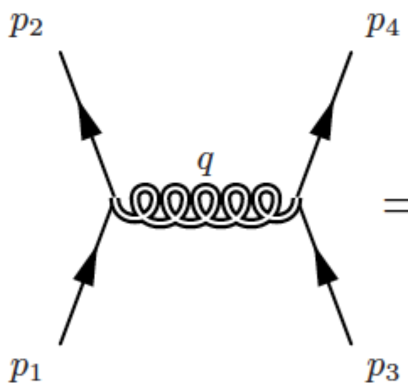
$$\mathcal{M}(p,p') = \langle p|V|p'\rangle + \int \frac{d^3k}{(2\pi)^3} \frac{\langle p|V|k\rangle \mathcal{M}(k,p')}{E_p - E_k + i\epsilon}$$

$$\langle p|V|p'\rangle = \mathcal{M}(p,p') - \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{M}(p,k)\mathcal{M}(k,p')}{E_p - E_k + i\epsilon} + \cdots$$

$$V(p,r) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} V(p,q)$$

Tree level

$$\mathcal{M}^{\text{tree}} =$$



$$= -\frac{4\pi G_N}{E_a E_b} \frac{[2(p_1 \cdot p_3)^2 - m_a^2 m_b^2 - |\vec{q}\,|^2 (p_1 \cdot p_3)]}{|\vec{q}\,|^2}$$

$$p_1 \cdot p_3 = E_a(p) E_b(p) + |\vec{p}\,|^2$$

Same result as from matching (Cheung, Solon, Rothstein; Bern, Cheung, Roiban, Shen, Solon, Zeng)

$$V_{1PM}(p,r) = \frac{1}{E_p^2 \xi} \frac{G_N c_1(p^2)}{r} + \cdots$$

$$c_1(p^2) \equiv m_a^2 m_b^2 - 2(p_1 \cdot p_3)^2 \quad , \quad \xi \equiv \frac{E_a E_b}{E_p^2}$$

One-loop

$$\mathcal{M}^{\text{Iterated}} = -\frac{16\pi^2 G_N^2}{E_a(p^2)E_b(p^2)} \int \frac{d^d k}{(2\pi)^d} \frac{A(\vec{p},\vec{k})}{|\vec{p}-\vec{k}|^2} \frac{A(\vec{k},\vec{p}\,')}{|\vec{p}\,'-\vec{k}|^2} \frac{\mathcal{G}(p^2,k^2)}{E_a(k^2)E_b(k^2)}$$

$$\mathcal{G}(p^2,k^2) = \frac{1}{E_p - E_k + i\epsilon}$$

Expanded

$$\mathcal{M}^{\text{Iterated}} = \frac{32\pi^2 G_N^2}{E_p^3 \xi} c_1^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{|\vec{p}-\vec{k}|^2 |\vec{p}\,'-\vec{k}|^2 (k^2 - p^2)}$$
$$- \frac{16\pi^2 G_N^2}{E_p^3 \xi^2} \left(\frac{c_1^2 (1-\xi)}{2 E_p^2 \xi} + 4 c_1 p_1 \cdot p_3 \right) \int \frac{d^d k}{(2\pi)^d} \frac{1}{|\vec{p}-\vec{k}|^2 |\vec{p}\,'-\vec{k}|^2} + \cdots$$

One-loop

$$\mathcal{M}^{\text{Iterated}} = \frac{i\pi G_N^2}{E_p^3 \xi} \frac{4c_1^2}{|\vec{p}|} \frac{(\log|\vec{q}|^2 - \frac{2}{3-d})}{|\vec{q}|^2} + \frac{2\pi^2 G_N^2}{E_p^3 \xi^2 |\vec{q}|} \left(\frac{c_1^2(\xi - 1)}{2E_p^2 \xi} - 4c_1 p_1 \cdot p_3 \right)$$

$$\mathcal{M}^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^2 \xi} \left[\frac{1}{2|\vec{q}|} \left(\frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b} \right) + \frac{i}{E_p} \frac{c_\Box}{|\vec{p}|} \frac{(\frac{2}{3-d} - \log|\vec{q}|^2)}{\pi |\vec{q}|^2} \right]$$

$$V_{2\text{PM}}(p, q) = \mathcal{M}^{1-\text{loop}} + \mathcal{M}^{\text{Iterated}}$$

$$V_{2\text{PM}}(p, q) = \frac{\pi^2 G_N^2}{E_p^2 \xi |\vec{q}|} \left[\frac{1}{2} \left(\frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b} \right) + \frac{2}{E_p \xi} \left(\frac{c_1^2(\xi - 1)}{2E_p^2 \xi} - 4c_1 p_1 \cdot p_3 \right) \right]$$

Again same result as from matching, no singular term

Effective potential

In fact we do not have to go through either matching procedure or solving Lippmann-Schwinger to derive observables such as the scattering angle

Energy relation makes everything simple:

$$p^2 = p_\infty^2 - 2E\xi\left[\widetilde{\mathcal{M}}^{cl.}_{tree}(p_\infty^2, r) + \widetilde{\mathcal{M}}^{cl.}_{1-loop}(p_\infty^2, r)\right]$$

(Damour; Bern, Cheung, Roiban, Shen, Solon, Zeng; Kalin, Porto; NEJB, Cristofoli, Damgaard; Cristofoli, Damgaard, Di Vecchia, Heissenberg)

Effective potential

Thus given the classical amplitude

$$\widetilde{\mathcal{M}}^{cl.}(p,r) \equiv -\frac{1}{2E\xi}\sum_{n=1}^{\infty}\frac{G_N^n \widetilde{c}_{(n-1)-loop}(p)}{r^n}$$

$$f_n(E) = \widetilde{c}_{(n-1)-loop}(p_\infty) \quad V_{\text{eff}}(r) \equiv -\sum_{n=1}^{\infty}\frac{G_N^n f_n(E)}{r^n}$$

Non-relativistic Hamiltonian
with effective potential

$$\hat{\mathcal{H}} = \hat{p}^2 + V_{\text{eff}}(r)$$

Scattering angle all orders

$$\chi = \sum_{k=1}^{\infty} \widetilde{\chi}_k(b) \,, \quad \widetilde{\chi}(b) \equiv \frac{2b}{k!} \int_0^{+\infty} du \left(\frac{d}{du^2} \right)^k \frac{V_{\mathit{eff}}^k(r)\, r^{2(k-1)}}{p_\infty^{2k}}$$

(Kalin, Porto; NEJB, Cristofoli,Damgaard)

$$p_r = \sqrt{p_\infty^2 - \frac{L^2}{r^2} - V_{\mathit{eff}}(r)}$$

$$\frac{\chi}{2} = -\int_{r_m}^{+\infty} dr \, \frac{\partial p_r}{\partial L} - \frac{\pi}{2}$$

Corrects naïve
light-bending
'Bohm's formula'
+ no reference
minimal distance

Any PM order given amplitude...

PM	$\chi^{\text{PM}}/\left(\frac{G_N}{p_\infty L}\right)^{\text{PM}}$
1	f_1
2	$\frac{1}{2}\pi p_\infty^2 f_2$
3	$2f_3 p_\infty^4 + f_1 f_2 p_\infty^2 - \frac{f_1^3}{12}$
4	$\frac{3}{8}\pi p_\infty^4\left(2f_4 p_\infty^2 + f_2^2 + 2f_1 f_3\right)$
5	$\frac{8}{3}f_5 p_\infty^8 + 4\left(f_2 f_3 + f_1 f_4\right)p_\infty^6 + f_1\left(f_2^2 + f_1 f_3\right)p_\infty^4 - \frac{1}{6}f_1^3 f_2 p_\infty^2 + \frac{f_1^5}{80}$
6	$\frac{5}{16}\pi p_\infty^6\left(3f_6 p_\infty^4 + 3\left(f_3^2 + 2f_2 f_4 + 2f_1 f_5\right)p_\infty^2 + f_2^3 + 6f_1 f_2 f_3 + 3f_1^2 f_4\right)$
7	$\frac{16}{5}f_7 p_\infty^{12} + 8\left(f_3 f_4 + f_2 f_5 + f_1 f_6\right)p_\infty^{10} + 6\left(f_3 f_2^2 + 2f_1 f_4 f_2 + f_1\left(f_3^2 + f_1 f_5\right)\right)p_\infty^8$ $+f_1\left(f_2^3 + 3f_1 f_3 f_2 + f_1^2 f_4\right)p_\infty^6 - \frac{1}{8}f_1^3\left(2f_2^2 + f_1 f_3\right)p_\infty^4 + \frac{3}{80}f_1^5 f_2 p_\infty^2 - \frac{f_1^7}{448}$
8	$\frac{35}{128}\pi p_\infty^8\left(4f_8 p_\infty^6 + 6\left(f_4^2 + 2\left(f_3 f_5 + f_2 f_6 + f_1 f_7\right)\right)p_\infty^4 + 12\left(f_4 f_2^2 + \left(f_3^2 + 2f_1 f_5\right)f_2\right.\right.$ $\left.\left.+f_1\left(2f_3 f_4 + f_1 f_6\right)\right)p_\infty^2 + f_2^4 + 6f_1^2 f_3^2 + 12 f_1 f_2^2 f_3 + 12 f_1^2 f_2 f_4 + 4 f_1^3 f_5\right)$

Confirmation of 3PM & 4PM

Bern, Cheung, Roiban, Shen, Solon, Zeng))

Comparison eikonal post-Minkowskian expansion

Connection to eikonal setup as for bending of light (extended to massive case):

$\vec{p}_1 = -\vec{p}_4$

b orthogonal and

$b \equiv |\vec{b}|$

Amplitude computed

$$M(\vec{b}) \equiv \int d^2\vec{q} e^{-i\vec{q}\cdot\vec{b}} M(\vec{q})$$

$$M(\vec{b}) = 4p(E_1 + E_2)(e^{i\chi(\vec{b})} - 1)$$

Eikonal phase

(NEJB, Damgaard, Festuccia, Plante, Vanhove)

Comparison eikonal post-Minkowskian expansion

Stationary phase condition (leading order in q)

$$2\sin(\theta/2) = \frac{-2M}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{\partial}{\partial b} \left(\chi_1(b) + \chi_2(b)\right)$$

$$\chi_1(b) = 2G \frac{\hat{M}^4 - 2m_1^2 m_2^2}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \left(\frac{1}{d-2} - \log\left(\frac{b}{2}\right) - \gamma_E \right)$$

$$\chi_2(b) = \frac{3\pi G^2}{8\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{m_1 + m_2}{b} (5\hat{M}^4 - 4m_1^2 m_2^2)$$

Comparison eikonal post-Minkowskian expansion

Final result becomes

$$2\sin\left(\frac{\theta}{2}\right) = \frac{4GM}{b}\Big(\frac{\hat{M}^4 - 2m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2} + \frac{3\pi}{16}\frac{G(m_1+m_2)}{b}\frac{5\hat{M}^4 - 4m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2}\Big)$$

Agrees with (Westpfahl)

(see also Cristofoli, Damgaard, Di Vecchia, Heissenberg); (Julio's talk)

Outlook

- Amplitude toolbox for computations already provided many new efficient methods for computation

  - Amplitude tools very useful for computations:
    - Double-copy and KLT
    - Recursion;
    - Unitarity
    - Spinor-helicity
    - CHY formalism
    - Low energy limits of string theory

Outlook

- A number of very impressive 3PM amplitude computations. (Bern, Cheung, Roiban, Shen, Solon, Zeng; Bern, Ita, Parra-Martinez, Ruf; Abreu, Febres Cordero, Ita, Jaquier, Page; Cheung, Solon; Parra-Martinez, Ruf, Zeng; Bern, Luna, Roiban, Shen, Zeng) + more to come.. (see Julio's talk)
- Endless tasks ahead
  - spin (see Michele's talk, see Justin's talk, see Radu's talk)
  - radiation
  - quantum terms and inclusion of high order curvature terms
- Clearly much more physics to learn....

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SAGEX

Scattering Amplitudes:
from Geometry to Experiment