Combinatorial Geometries for Scattering Amplitudes

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based on collaborations w. Nima Arkani-Hamed, Thomas Lam, Giulio Salvatori, Hugh Thomas

1912.08707, 1912.11764, 1912.12948, 2005.xxxx

+ w. Zhenjie Li, Prashanth Raman, Chi Zhang: 2005.xxxx

Zoomplitudes @ Brown

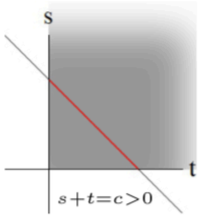
motivations

combinatorial geometries (w. factorizing bd.) underlying scattering amps in config. space & kinematic space

- moduli space $\mathscr{M}_{g,n}$ for conventional & (ambi-)twistor strings [Witten, Berkovits; Cachazo, SH, Yuan; Mason, Skinner; …]
- positive grassmannian $G_+(k,n)$, on-shell diagrams etc. for planar N=4 SYM [Postnikov; Arkani-Hamed et al]
- amplituhedron: map from $G_+(k,n) \to$ canonical forms in momentum twistor space [Hodges; Arkani-Hamed, Trnka + Thomas;…]
- helicity amps as differential form in momentum space [SH, Zhang]→momentum amplituhedron [talk by Ferro]
- kinematic associahedron (bi-adjoint ϕ^3 in Mandelstam space) & worldsheet assoc. [Arkani-Hamed, Bai, SH, Yan]
- cluster polytopes, ABCDs of ϕ^3 amps through 1-loop & beyond (clusterhedra?) [lectures & talk by Arkani-Hamed]
- gen. string amplitudes, cluster configuration space, binary geometries (old & new)… this talk!
- positive config. space/tropical grassmannian & N=4 SYM clusterology [talks by Cachazo, Drummond, Guevara, Lam, Spradlin …]

kinematic associahedra

associahedra (vertices ↔ planar cubic trees) in $\mathscr{K}_n$



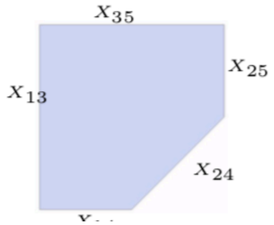
$e.g.\ \mathcal{A}_1 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$

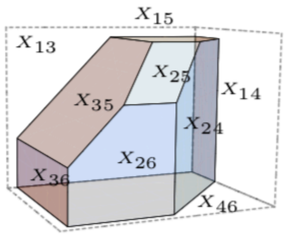
$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$

a family of (n-3)-dim subspaces: e.g. for $1 \leq i < j - 1 < n$

$$X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} = c_{i,j} > 0$$

$\mathscr{A}_{n-3}$: $\{X_{i,j} \geq 0\} \cap$ subspace [ABHY]





canonical form $\Omega(\mathscr{A}_{n-3})$=pullback of scattering form=bi-adjoint ϕ^3 tree amps

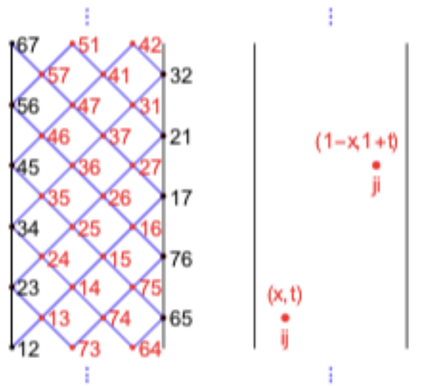
e.g. $\Omega(\mathscr{A}_1) = (\frac{ds}{s} - \frac{dt}{t})|_{s+t=c} = ds(\frac{1}{s} + \frac{1}{t})$

geometric picture: FD expansion = a special triangulation, others → new formula & recursion for ϕ^3 amps

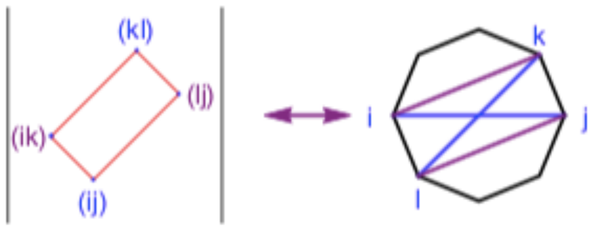
hidden symmetry of ϕ^3 amps (invisible in FD's), analog of dual conformal symmetry, manifest by geometry!

wave equation & causal diamonds

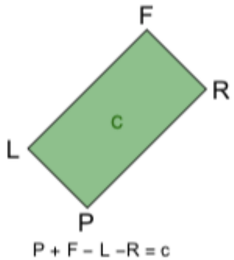
ABHY: discrete wave eq in (1+1)-d “kin. spacetime” (mesh): $\partial_u \partial_v X(u,v) = c(u,v)$



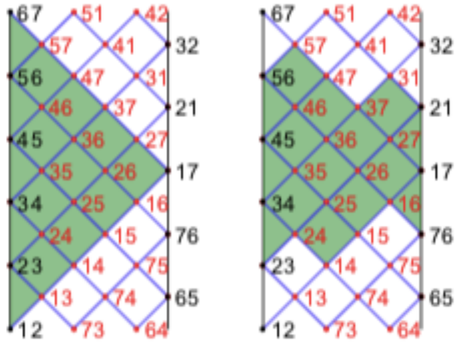
causal diamond ↔ (in-)compatible propagators/diagonals



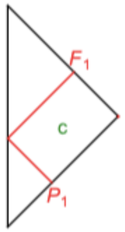
Gauss law:

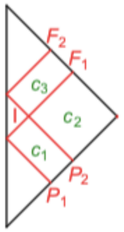


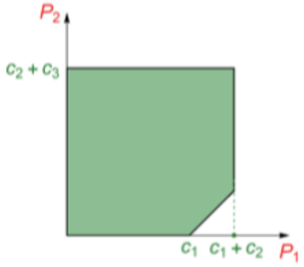
continuum limit of $\mathscr{A}_{n-3}$: space of positive sol. with positive source in a region

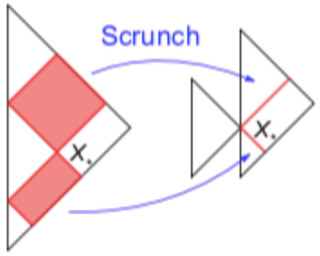


factorizing bd. $\partial_{(ij)} \mathscr{A}_{n-3} = \mathscr{A}_L \times \mathscr{A}_R$ ensured by causal structure!



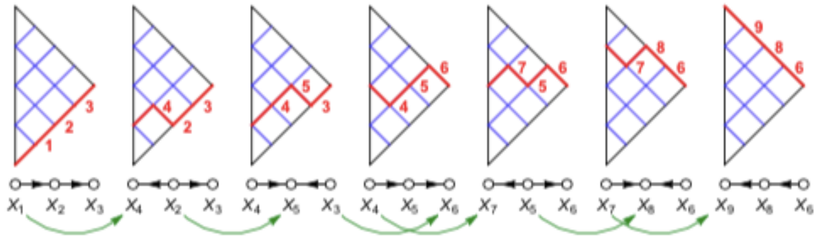






cluster polytopes & ϕ^3 amplitudes

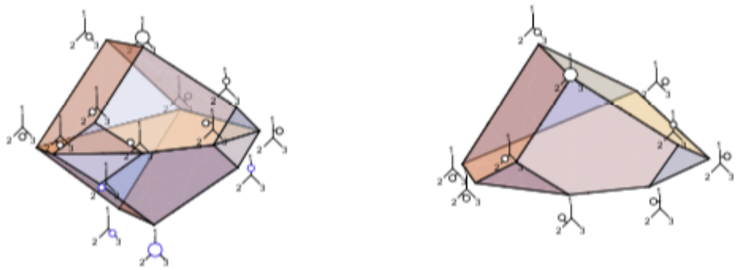
time evolution = "walk" on quiver (mutation on source) $v \to v' : X'_v + X_v - \sum_{w \leftarrow v} X_w = c_v$

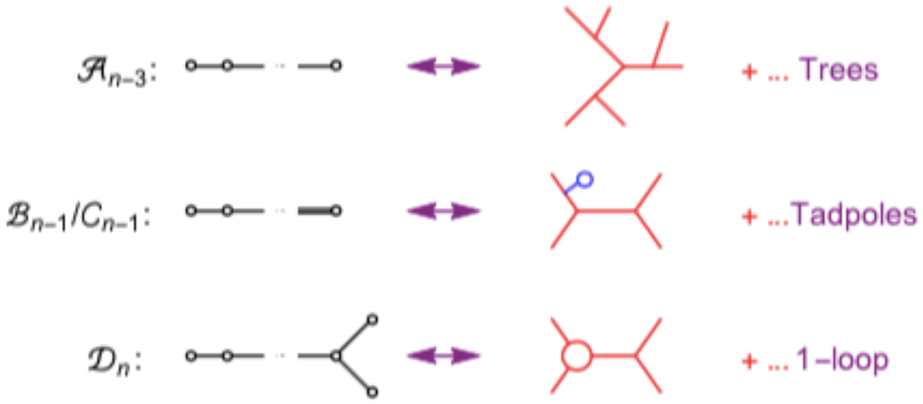


new facet $X'_v \geq 0$; new vector must lie outside convex hull, stop when span full space

only finite polytopes for Dynkin diagrams → gen. associahedra of finite type [Bazier-Matte et al] (cluster polytopes)

classical: bi-adjoint ϕ^3 through 1-loop; higher loops ↔ infinite type [talk by Arkani-Hamed]





1-loop bd.: $\partial\mathcal{D}_n = \mathcal{A} \times \mathcal{D}$ & $\mathcal{A}_{n-1}$ (tree x 1-loop & forward limit)

cut in halves → $\overline{\mathcal{D}}_n$; $\Omega(\overline{\mathcal{D}}_n)$ → new formulas for 1-loop ϕ^3 amps!

outline

- stringy canonical forms
- binary cluster polytopes & generalized string amplitudes
- (more) binary geometries from gen. permutohedra
- ~~grassmannian stringy integrals & config. spaces~~ [see also SH, Lecheng Ren, Yong Zhang, 2001.09603]

stringy canonical forms [w. N. Arkani-Hamed, T. Lam]

stringy canonical forms

vast generalizations of (open-)string amplitudes $\mathbf{I}_n(\{s\}) = (\alpha')^{n-3} \int_{\mathscr{M}_{0,n}^+} \frac{d^{n-3}z}{z_{1,2}\cdots z_{n,1}} \prod_{a<b} (z_{a,b})^{\alpha' s_{a,b}}$

positive parametrization, e.g. $z_3 = 1 + x_2\,, \quad z_4 = 1 + x_2 + x_3, \quad \ldots, \quad z_{n-1} = 1 + x_2 + \cdots + x_{n-2} \implies$

integral of the form w. positive polynomials: $\mathscr{I}_{\{p\}}(\mathbf{X}, \{c\}) = \alpha'^d \int_{\mathbb{R}^d_{>0}} \prod_i^d \frac{dx_i}{x_i}\, x_i^{\alpha' X_i} \prod_I p_I(\mathbf{x})^{-\alpha' c_I}$

long history ("Euler-Mellin"…) $\rightarrow$ $d^d\mathbf{X}\mathscr{I}\ (\mathbf{X})$ a diff. form exhibits geometries & remarkable properties

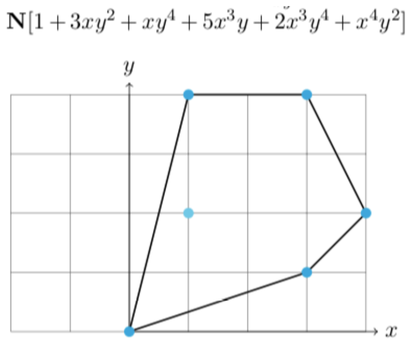
- leading order $(\alpha' \rightarrow 0)$, residues on "massless poles" (+ convergence): controlled by polytope P
- "stringy" properties: meromorphic with"massive poles", exponential UV softness, "channel duality"
- scattering (saddle-point) equations & twisted (co-)homology
- dual u variables, tropical compactifications, (complex) closed-stringy integrals …

Minkowski sum

single polynomial: $$\mathscr{I}_p(\mathbf{X}, c) = \alpha'^d \int_0^\infty \prod_{i=1}^d \frac{dx_i}{x_i}\, x_i^{\alpha' X_i}\, p(\mathbf{x})^{-\alpha' c}$$

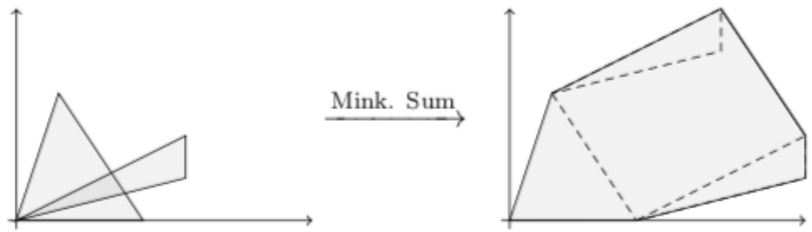
converges iff $\mathbf{X}$ is inside Newton polytope $c\,\mathbf{N}(p)$, $\lim_{\alpha'\to 0} \mathscr{I}_p(\mathbf{X}, c) = \underline{\Omega}(c\ \mathbf{N}[p(\mathbf{x})]; \mathbf{X})$

(also see [Nilsson, Passare; Berkesch et al])



trivial to gen. to $$\mathscr{I}_{\{p\}}(\mathbf{X}, \{c\}) = \alpha'^d \int_{\mathbb{R}^d_{>0}} \prod_{i=1}^d \frac{dx_i}{x_i}\, x_i^{\alpha' X_i} \prod_{I=1}^m p_I(\mathbf{x})^{-\alpha' c_I}$$

converges iff $\mathbf{X}$ is inside Minkowski sum $\mathscr{P} := \sum_I c_I \mathbf{N}[p_I(\mathbf{x})]$



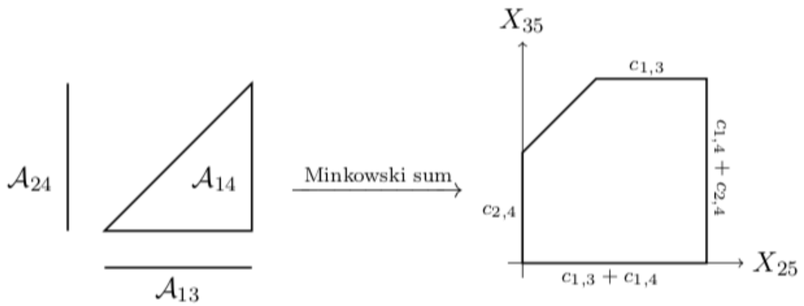
$\lim_{\alpha'\to 0} d^d\mathbf{X}\mathscr{I}_{\{p\}}(\mathbf{X}, \{c\}) = \Omega(\mathscr{P}; \mathbf{X})$ α'-deformations of canonical form of any (rational) polytope

equivalently: leading order = volume of dual polytope = tropical function (c.f. [Panzer] on Hepp bound)

string integrals from associahedra

ABHY assoc. is rigid: shape independent of $c_{i,j} > 0$; decompose as Minkowski sum $\mathscr{A}_{n-3} = \sum_{1 \le i < j-1 < n} c_{i,j}\, \mathscr{A}_{i,j}$

$X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} = c_{i,j}$



$$p_{13} = 1 + y_2, \qquad p_{24} = 1 + y_3, \qquad p_{14} = 1 + y_2 + y_2 y_3\,.$$

$$\mathbf{I}_5 = \int \frac{dy_2}{y_2}\frac{dy_3}{y_3} y_2^{\alpha' X_{25}} y_3^{\alpha' X_{35}} (1+y_2)^{-\alpha' c_{13}} (1+y_2+y_2y_3)^{-\alpha' c_{14}} (1+y_3)^{-\alpha' c_{24}}$$

Newton polynomial for $\mathscr{A}_{i,j}$: $p_{ij} = 1 + y_{i+1} + y_{i+1}y_{i+2} + \cdots + y_{i+1}\cdots y_{j-1} \implies$

$$\mathbf{I}_n = \mathscr{I}_{\mathscr{A}_{n-3}} = \int \prod_{i=2}^{n-2} \frac{dy_i}{y_i} y_i^{\alpha' X_{i,n}} \prod_{1 \le i < j-1 < n} p_{ij}(\mathbf{y})^{-\alpha' c_{i,j}}$$

re-discover open-string integrals & moduli space from ABHY!

two classes of examples

(1). stringy integral for gen. associahedra $\mathscr{P}(\Phi)$ of finite type: $\mathscr{I}_\Phi(\{S\}) = (\alpha')^d \int_{\mathbf{A}(\Phi)_+/T} (\omega/T) \prod_{\gamma\in\Gamma} x_\gamma^{\alpha' S_\gamma}$

leading order → ABHY cluster polytopes: $\lim_{\alpha'\to 0} d^d\mathbf{X}\mathscr{I}_\Phi(\mathbf{X},\{c\}) = \Omega(\mathscr{P}(\Phi^\vee, c);\mathbf{X})$

ABCD: α'-extension of ϕ^3 amps w. factorization at finite α' e.g. $\mathscr{I}_{D_n} \to \mathscr{I}_{A_m} \times \mathscr{I}_{D_{n-m-1}}$, $\mathscr{I}_{A_{n-3}} \times \mathscr{I}_{A_1} \times \mathscr{I}_{A_1}$, $\mathscr{I}_{A_{n-1}}$

(2). stringy integral over $G_+(k,n)/T$: $\mathscr{I}_{k,n}(\{S\}) := (\alpha')^d \int_{\mathrm{G}_+(k,n)/T} (\omega_{k,n}/T) \prod_I \Delta_I^{\alpha' s_I}$ $\quad d := (k-1)(n-k-1)$

$D = \binom{n}{k} - n$ independent S_I ; higher-k gen. of string integrals ($\mathscr{M}_{0,n}^+ \sim \mathrm{G}_+(2,n)/T$), non-trivial to compute

$\mathscr{P}_{k,n} = \sum_I S_I N[\Delta_I]$ [Arkani-Hamed, Lam, Spradlin] ↔ tropical $G_+(k,n)$ [Speyer, Williams] [talks by Cachazo, Drummond, Guevara]

generalized scattering equations

for any $\mathscr{I}$, "scattering eqs" (saddle points) $\Longrightarrow$ a diffeomorphism from int. domain to Mink. sum $\mathscr{P}$

$$\text{SE}: \; d\log\left(\prod_{i=1}^{d} x_i^{\alpha' X_i} \prod_{I=1}^{m} p_I(\mathbf{x})^{-\alpha' \mathbf{c_I}}\right) = 0 \quad \Longrightarrow \quad \text{map} \quad \mathbb{R}^d_{\geq 0} \to \mathscr{P}: \; \mathbf{X} = \sum_{I=1}^{m} c_I \frac{\partial \log p_I(\mathbf{x})}{\partial \log \mathbf{x}}$$

theorem: $y = f(x)$ diff. $\mathscr{X} \to \mathscr{Y} \quad \Longrightarrow \quad$ pushforward $\Omega(\mathscr{Y}; y) = \sum_{x = f^{-1}(y)} \Omega(\mathscr{X}; x)$ [Arkani-Hamed, Bai, Thomas]

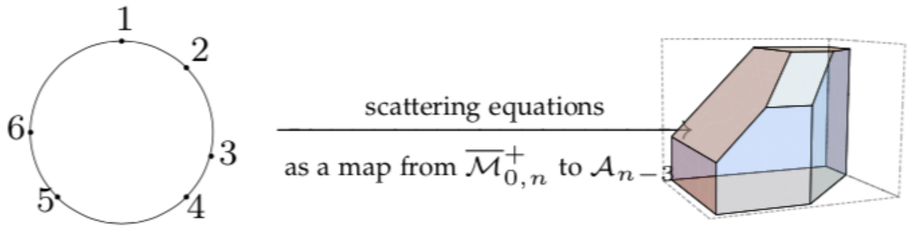
$\alpha' \to 0$ & $\alpha' \to \infty$ limits deeply connected: $\Omega(\mathscr{P}) = \lim_{\alpha' \to 0} d^d\mathbf{X}\, \mathscr{I} = \sum_{\text{sol.}} \prod_i \frac{dx_i}{x_i}$

gen. CHY formula: $\lim_{\alpha' \to 0} \mathscr{I} = \underline{\Omega}(\mathscr{P}) = \int \prod_{i=1}^{d} \frac{dx_i}{x_i}\, \delta\left(X_i - \sum_I c_I \frac{\partial \log p_I}{\partial \log x_i}\right)$

general phenomenon: field-theory limit = pushforward via saddle points @ high-energy limit [Gross, Mende]

apply to string integral → geometric origin of CHY formulas

pullback of $\{\sum_{b\neq a}\frac{s_{a,b}}{z_a-z_b}=0\}$: diffeomorphism $\mathscr{M}^+_{0,n}\to\mathscr{A}_{n-3}$



e.g. n=5, $X_{2,5}=x\left(\frac{c_{1,3}}{1+x}+\frac{c_{1,4}(1+y)}{1+x+xy}\right)$, $X_{3,5}=y\left(\frac{c_{2,4}}{1+y}+\frac{c_{1,4}x}{1+x+xy}\right)\implies\sum_{\text{sol.}}\frac{dxdy}{xy}=\Omega(\mathscr{A}_2)=d^2\mathbf{X}\ m_5$

string integrals very special: # of sol. (huge) reduction: $(n-3)!$ the smallest # for assoc. (CHY most efficient!)

theorem: # of saddle points=dim of space of integral functions: twisted homology (cycles) ↔ cohomology (forms)

$$\int\prod_{i=1}^{d}\frac{dx_i}{x_i}Q(\mathbf{x})\ \exp(\alpha' F(\mathbf{x})),\quad F(\mathbf{x}):=-\sum_I c_I\log p_I(\mathbf{x})$$

with $\alpha'\to\infty$, $\dim H^d((\mathbb{C}^\times)^d, dF\wedge)$= # of saddle points

e.g. cluster case: $\#(\mathscr{B}_n)=n^n$, $\#(\mathscr{C}_n)=(2n)!!/2$, $\#(\mathscr{G}_2)=13$, $\#(\mathscr{D}_4)=55$

Gr. case: #(3,6)=26, #(3,7)=1272, #(3,8)=188112 [Cachazo, Early, Guevara, Mizera; …] higher-k gen. of SE & bi-adjoint amps

all these closely related to intersection theory [Mizera; …] Feynman integrals? [c.f. Lee, Pomeransky; Mastrolia, Mizera; Frellesvig et al;…]

[also see talks by Britto, Caron-Huot, Duhr, …]

complex stringy integrals

closed strings? first mod-squared integrals: $\mathscr{I}^{|\cdot|^2}_{\{p\}}(\mathbf{X},\{c\}) = \alpha'^d \int_{\mathbb{C}^d} \prod_{i=1}^{d} \frac{dz_i d\bar{z}_i}{|z_i|^2} |z_i|^{2\alpha' X_i} \prod_I |p_I(\mathbf{z})|^{-2\alpha' c_I}$

for $0 < \alpha' < \epsilon$, convergence & leading order again controlled by $\mathscr{P} := \sum_I c_I \, \mathbf{N}[p_I]$

more general $\mathscr{I}^{|\cdot|^2}_{\{p\}}(\mathbf{X},\{c,n\}) := \cdots \prod_i \bar{z}_i^{n_i} \prod_I p_I(\bar{\mathbf{z}})^{-n_I}$ with non-negative integer shifts

Minkowski sum $\mathscr{P}$: unbounded polyhedron ($n/\alpha' \to \infty$), canonical form/dual volume still finite!

apply to closed & open string amps w. two orderings: $\mathbf{I}_n^{\text{closed}}(\alpha\,|\,\beta)$ & $\mathbf{I}_n^{\text{open}}(\alpha\,|\,\beta)$ [Carrasco, Mafra, Schlotterer;…]

unbounded $\mathscr{A}_{n-3}$ with facets @ infinity: (both) leading orders = $m(\alpha\,|\,\beta)$

similar unbounded polyhedron for bi-color ϕ^3 amps [Herderschee, SH, Teng, Zhang]

dual u variables

$\mathscr{P}$ bounded by N facets $W_a^J S_J \geq 0$ with $\mathbf{S} := (X_1, \ldots, X_d, -c_1, \ldots, -c_m) \in \mathbb{R}^{d+m} \to$ defines (d+m)-dim

big polyhedron $\mathscr{B}$; point inside $\mathbf{S} \in \mathscr{B}$ = positive comb. of vertices $\mathbf{V}^\alpha$: $S_J = \sum_\alpha V_J^\alpha F_\alpha$

$\mathscr{I}$ converges iff $F_\alpha > 0$, rewrite w. dual variables $u_\alpha := \prod_J p_J^{V_J^\alpha}$ ($p_J := (x_1, \cdots, x_d, p_1, \cdots, p_m)$)

$$\mathscr{I}(\mathbf{S}) = \int_{\mathbb{R}_+^d} \prod_{i=1}^{d} \frac{dx_i}{x_i} \prod_J^{d+m} p_J^{\alpha' S_J} = \int_{\mathbb{R}_+^d} \prod_{i=1}^{d} \frac{dx_i}{x_i} \prod_\alpha^{N_v} u_\alpha^{\alpha' F_\alpha} := \mathscr{I}(\mathbf{F})$$

special case: $\mathscr{B}$ = simplex ($N = N_v = d + m$), $V = W^{-1}$, u variables multiplicative independent

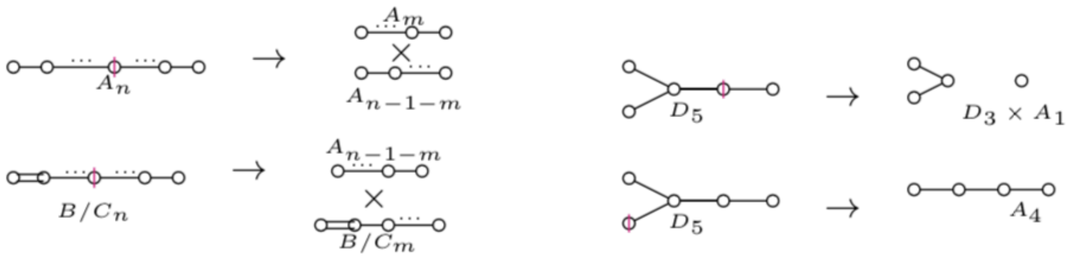
each facet $F_\alpha \to 0 \Longleftrightarrow u_\alpha \to 0$: true for $\mathscr{I}_\Phi$, not for $\mathscr{I}_{n,k>2}$ (c.f. $\mathscr{D}_4$ vs. $\mathscr{P}_{3,6}$)

configuration space $U^o, U, U_{\geq 0}$: $\{u_\alpha\}_{1 \leq \alpha \leq N}$ by u eqs, $U_{\geq 0} := \{u_\alpha \geq 0\}$ “curvy polytope” [talk by Lam]

binary cluster polytopes & generalized string amplitudes [w. N. Arkani-Hamed, T. Lam, H. Thomas]

cluster config. space & integrals

cluster polytopes [Chapton, Fomin, Zelevinsky] ↔ Dynkin diagram



ABHY → more rigid, canonical realization?

U space has same bd. as cluster polytope (algebraically)

binary geometry: $u_a \to 0$ all incompatible $u_b \to 1$ $(b||a > 0)$

$$u_a + \prod_{all\ b} u_b^{b||a} = 1, \forall a$$

gen. of $\mathscr{M}_{0,n}$ to finite-type cluster algebra

cluster stringy integrals over (positive & complex)

$$\mathscr{I}_\Phi(\mathbf{X}) = \alpha'^n \int_{U^+(\Phi)} \Omega(U^+(\Phi)) \prod_{a=1}^{N} u_a^{\alpha' X_a}, \text{ etc.}$$

binary geometries: gen. open & closed string amps

binary associahedra & string integrals

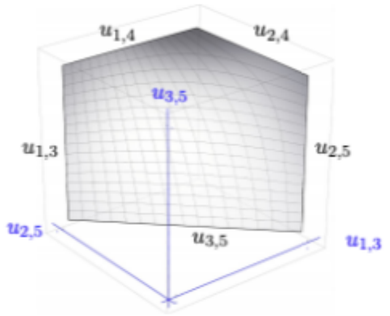
u eqs: $\{1 - u_{i,j} = \prod_{(k,l)\ \text{cross}\ (i,j)} u_{k,l}\}$ (deg.=0,1): (n-3)-dim solution space → configuration space of type A

$U_n^o = \mathscr{M}_{0,n} \subset U_n \subset \overline{\mathscr{M}}_{0,n}$ [see F. Brown]: algebraically same boundary structure as $\mathscr{A}_{n-3}$ (holds in $\mathbb{C}$)

any $u_{k,l} \to 0$, all incompatible $u_{i,j} \to 1$ (binary) $\Longrightarrow \partial_{u_{i,j}\to 0} U_n = U(i, i+1, \cdots, j) \times U(j, j+1, \cdots, i)$

ask $u_{i,j} > 0 \quad \Longrightarrow \quad 0 < u_{i,j} < 1$ defines positive part $U_n^+ \sim \mathscr{M}_{0,n}^+$

it has (curvy) shape of assoc. $\mathscr{A}_{n-3}$, e.g. U_5^+ is a pentagon



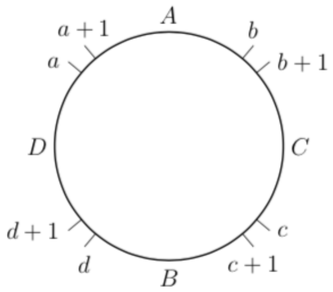
for U_n^+, open-string integrals (in u variables): $\mathbf{I}_n = \int_{U_n^+} \Omega(U_n^+) \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}, \quad \Omega(U_n^+) = \prod_{(i,j)}^{n-3} d\log \frac{u_{i,j}}{1 - u_{i,j}}$

here $(n-3)$ $u_{i,j}$ in a triangulation without internal triangle (acyclic quiver) $\Longrightarrow$ Parke-Taylor form!

worldsheet from U space

U_n : gauge-inv. description of $\mathscr{M}_{0,n}$ (u's cross-ratios of n points) → emergence of worldsheet!

extended u eqs [see also F. Brown]: $[a,b\,|\,c,d] + [b,c\,|\,d,a] = 1$, w . $[a,b\,|\,c,d] := \prod_{a\le i<b,\, c\le j<d} u_{i,j}$



(in total $\binom{n}{4}$ eqs, follow from u eqs, which are special case w. $u_{i,j} = [i, i+1\,|\,j, j+1]$)

& $[a,b\,|\,c,e][a,b\,|\,e,d] = [a,b\,|\,c,d]$: all constraints satisfied by cross-ratios $\implies [a,b\,|\,c,d] = \dfrac{(ad)(bc)}{(ac)(bd)}$

$U_n^+ :\ \ 0 < [a,b\,|\,c,d] < 1 \quad \implies$ n points ordered; how to see other orderings from u's? sign patterns

extended eqs $\prod u + \prod u = 1$ forbid sign patterns w. both terms <0: exactly $(n-1)!/2$ connected components of $U_n(\mathbb{R})$

identical extended u eqs by monomial transf. of u's → hidden S_n symmetry (automorphism for $U_n(\mathbb{C})$)

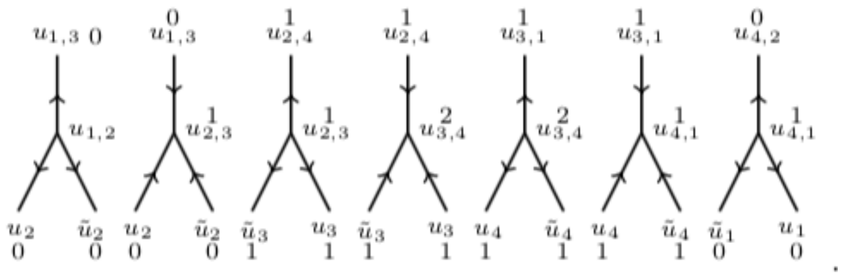
gen. u eqs from “walk”

gen. to any finite-type cluster algebra: only new ingredient compatibility degree (N by N matrix, beyond 0,1)

concise new def. from “walk” picture (each step $v \to v'$, $X'_v + X_v - \sum_{w \leftarrow v} n_w X_w = c_v$): $b||a := X_a$ integer “Green’s function”

for all a & a given source b in initial quiver (all initial $X = 0$, all $c = 0$ except $c_b = 1$)

$\mathscr{D}_4$



$$4 \text{ eqs}: \quad 1 - u_{1,2} = u_3 \tilde{u}_3 u_4 \tilde{u}_4 u_{3,4}^2 u_{2,3} u_{2,4} u_{4,1} u_{3,1}$$

$$4 \text{ eqs}: \quad 1 - u_{1,3} = u_4 \tilde{u}_4 u_{4,1} u_{4,2} u_{2,4} u_{3,4} \,,$$

$$8 \text{ eqs}: \quad 1 - u_1 = \tilde{u}_2 \tilde{u}_3 \tilde{u}_4 u_{2,3} u_{3,4} u_{2,4} \,.$$

non-simply laced also by folding: $\mathscr{A}_{2n-3} \to \mathscr{C}_{n-1}$, $\mathscr{D}_n \to \mathscr{B}_{n-1}$, $\mathscr{B}_3 \to \mathscr{G}_2$, $\mathscr{E}_6 \to \mathscr{F}_4$

u eqs equiv. to “local” eqs (Y system [Fomin, Zelevinsky]): each step $Y_v Y_{v'} = \prod_{w \leftarrow v} (1 + Y_w)^{n_w}$, $Y := \dfrac{u}{1-u}$ (N eqs).

cluster conf. space

$\{u_a + \prod_{all\ b} u_b^{b||a} = 1\}$ (N-n independent) → n-dim $U^o(\Phi)$ & $U(\Phi)$; algebraically same bd. structure as $\mathscr{P}(\Phi)$

binary geometries: any $u_b \to 0$, incompatible $u_a \to 1$ (in $\mathbb{C}$), factorize by removing node in Dynkin diagram

e.g. $U(\mathscr{B}/\mathscr{C}) \to U(\mathscr{A}) \times U(\mathscr{B}/\mathscr{C})$, $U(\mathscr{D}_n) \to U(\mathscr{A}_m) \times U(\mathscr{D}_{n-m-1})$, $U(\mathscr{D}_{n-3}) \times U(\mathscr{A}_1)^2$, & $U(\mathscr{A}_{n-1})$

ask all $u_a > 0$ cuts out curvy polytope $U^+(\Phi) \sim \mathscr{P}(\Phi)$, e.g. $U^+(\mathscr{B}/\mathscr{C})$ "curvy" cyclohedra

canonical form $\Omega(U^+) = \prod_{\alpha}^{n} d\log \frac{u_\alpha}{1-u_\alpha}$ for α in any acyclic quiver: rationally solve all N u's by $\{u_\alpha\}$

what about "orderings"? conjecture: connected comp. of $U(\mathbb{R}) \leftrightarrow$ sign patterns by extended eqs $U_I + V_I = 1$

1:1 with mutation relations: for $\mathscr{A}_{n-3}$, $I : (a,c) \to (b,d)$; $n(n-1)(n^2+4n-6)/6$ eqs for $\mathscr{D}_n$, etc.

monomial transf. & beyond: new shapes for other orderings, e.g. for $\mathscr{B}_2 = \mathscr{C}_2$, tiled by 4 hexagons + 12 pentagons

cluster string integrals

U space (type A) manifest factorization @ finite α': $\mathrm{Res}_{X_{a,b}=0}\mathbf{I}_n = \mathbf{I}(a, a+1, \cdots, b) \times \mathbf{I}(b, b+1, \cdots, a)$

as $X_{a,b} \to 0$, $u_{a,b} \to 0$: not only $\Omega \to \Omega_L \times \Omega_R$, $\prod u^{\alpha' X} \to \prod_L u^{\alpha' X} \times \prod_R u^{\alpha' X}$ (all incompatible u's decouple)

for $U_n(\mathbb{C})$, closed-string integrals (w. orderings α & β): $\mathbf{I}_n^{\mathrm{c}}(\alpha\,|\,\beta) = \int_{U_n(\mathbb{C})} \Omega(\alpha) \prod u^{\alpha' X} \left(\Omega(\beta) \prod u^{\alpha' X}\right)^*$

for any finite-type Φ: $\mathscr{I}^{\Phi}(\mathbf{X}) = \int_{U^+(\Phi)} \Omega(U^+(\Phi)) \prod_a u_a^{\alpha' X_a}, \quad \Omega(U^+(\Phi)) := \prod_\alpha d \log \frac{u_\alpha}{1 - u_\alpha}$

factorization by removing node of Dynkin diagram: $\Omega(U^+)$ & $\prod u^{\alpha' X}$ factorize nicely!

$\mathscr{B}_{n-1}/\mathscr{C}_{n-1}$ factorize as 1-loop tadpoles, $\mathscr{D}_n$ as 1-loop amp; exotic ones e.g. $\mathscr{E}_6 \to \mathscr{D}_5, \mathscr{A}_5, \mathscr{A}_4 \times \mathscr{A}_1$ & $\mathscr{A}_2^2 \times \mathscr{A}_1$

"closed stringy" integrals over $U^{\Phi}(\mathbb{C})$ well-defined for α & β related by monomial transf.

physical meaning of these integrals (contain string integrals)? e.g. $\mathscr{D}_n$ is α'-deformation of 1-loop integrand for ϕ^3

arithmetic geometry for U

count # of points in $U_n(\mathbb{F}_p)$: $N(p) = (p-2)(p-3)\cdots(p-n+2) \Longrightarrow$ topological prop. of $\mathscr{M}_{0,n}$

- $|N(-1)| =$ # of connected components/orderings: $(n-1)!/2$
- $|N(1)| = |\chi| =$ # of saddle points = dim of half-dim twisted cohomology: $(n-3)!$ [BCJ; CHY; Mizera …]
- coef. of $N(p) \leftrightarrow$ # of $d\log$ forms $\Longrightarrow$ $|N(0)| =$ # of $d\log$ top form: $(n-2)!$ [Kleiss, Kuijf]

$N(p) = (p-n+-1)^n$ for $\mathscr{B}_n$, $N(p) = (p-n-1)(p-3)(p-5)\cdots(p-2n+1)$ for $\mathscr{C}_n$ $\Longrightarrow$

	orderings	KK	BCJ/solutions
$\mathcal{B}_n$	$(n+2)^n$	$(n+1)^n$	n^n
$\mathcal{C}_n$	$(2n)!!\frac{n+2}{2}$	$(2n-1)!!(n+1)$	$\frac{(2n)!!}{2}$

(type $\mathscr{A}$ & $\mathscr{B}$: hyperplane arrangements, unknown for $\mathscr{C}$)

beyond: (conjecture) quasi-polynomial, e.g. for $\mathscr{G}_2$, $N(p) = (p-4)^2 + 4\delta_p$, $\delta_p = 0$ $(p = 2 \bmod 3)$ or 1 $(p = 1 \bmod 3)$

$\Longrightarrow$ 25 orderings & 13 saddle points; 547 orderings, 55 saddle points for $\mathscr{D}_4$; similarly for (some) $G_+(k,n)/T$

(more) binary geometries from gen. permutohedra [w. Z. Li, P. Raman, C. Zhang]

stringy integrals for gen. perm.

building set $\mathbf{B}$ = collection of subsets $I \subset [0,d]$ (w. conditions) [Postnikov] $\rightarrow$ d-dim $\mathscr{P}_{\mathbf{B}}$ as Minkowski sum of coordinate simplices Δ_I for $I \in \mathbf{B} \rightarrow$ beg for stringy canonical forms w. sum $x_I := \sum_{i \in I} x_i$ for each Δ_I

$$\mathscr{I}_{\mathbf{B}}(\mathbf{S}) := \int_{\mathbb{R}^d_+} \frac{1}{\mathrm{vol}\ GL(1)} \prod_{i=0}^{d} \frac{dx_i}{x_i} \prod_{I \in \mathbf{B}} x_I^{\alpha' S_I} = \int_{\mathbb{R}^d_+} \prod_{i=1}^{d} \frac{dx_i}{x_i}\ x_i^{\alpha' X_i} \prod_{I}^{m} x_I^{-\alpha' C_I}$$

projective: $\sum_{I \in \mathbf{B}} S_I = 0$ (fix e.g. $x_0 = 1$); leading order of $\mathscr{I}_{\mathbf{B}} = \underline{\Omega}(\mathscr{P}_{\mathbf{B}}; \mathbf{X})$, some examples:

- simplex: $\mathbf{B} =$ all singletons + [0,d], gen. beta function $\mathscr{I}_{\mathbf{B}} = \prod_i \Gamma(X_i)/\Gamma(\sum_i X_i)$
- associahedron: $\mathbf{B} = \{[i,j] \mid 0 \leq i \leq j \leq d\}$, open-string integral (type A_d)
- cyclohedron: $\mathbf{B} =$ all cyclic intervals, stringy integral of type B_d !
- permutohedron: $\mathbf{B} =$ all subsets, $\mathscr{I}_d :=$ (rigid) stringy integral for perm. $\mathscr{P}_d$!

remarkably, facets of $\mathscr{P}_{\mathbf{B}}$ labelled by subsets $I \neq [0,d]$ (factorize as $\mathscr{P}_{\mathbf{B}_I} \times \mathscr{P}_{\mathbf{B}_{\bar{I}}}$) $\implies N = d + m$

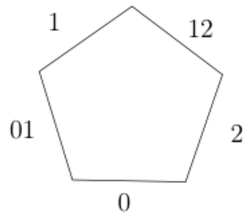
$$\prod_{i=1}^{d} x_i^{\alpha' X_i} \prod_{I}^{m} x_I^{-\alpha' C_I} = \prod_{I}^{N=d+m} u_I^{\alpha' X_I}$$

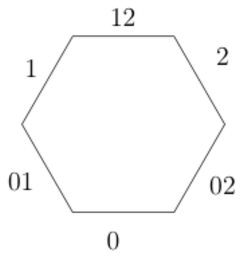
ABHY realization of $\mathscr{P}_{\mathbf{B}}$: $-C_J = V_J^I \, X_I$ (for J=1,…,m) $\leftrightarrow$ u variables: $u_I = \prod_J (x_J)^{V_J^I}$ (for I=1,…,N)

e.g. simplex Δ_d : $-C_{[0,d]} = \sum_i X_i \;\leftrightarrow\; u_i = \dfrac{x_i}{\sum_i x_i}$

permutohedron $\mathscr{P}_d$: $-C_J = \sum_{I \subset J} (-)^{|J|-|I|} X_I$ (alternating sum) $\leftrightarrow$ $u_I = \prod_{J \subset I} x_J^{(-)^{|J|-|I|}}$

for d=2 ($\mathscr{P}_2 = \mathscr{B}_2$): $u_0 = \dfrac{x_0 x_{012}}{x_{01} x_{02}}, u_1 = \dfrac{x_1 x_{012}}{x_{01} x_{12}}, u_2 = \dfrac{x_2 x_{012}}{x_{02} x_{12}}, u_{01} = \dfrac{x_{01}}{x_{012}}, u_{02} = \dfrac{x_{02}}{x_{012}}, u_{12} = \dfrac{x_{12}}{x_{012}}$





any $\mathscr{I}_{\mathbf{B}}$ & $\mathscr{P}_{\mathbf{B}}$ degeneration of $\mathscr{I}_d$ & $\mathscr{P}_d$ with some $C_I \to 0$, e.g. $\mathscr{P}_2|_{C_{02} \to 0} = \mathscr{A}_2$

binary geometries revisited

Q: what type of u-eqs needed for binary geometries, i.e. any $u_a \to 0$, all incompatible $u_b \to 1$?

trivial ex: u eqs for simplex Δ_d $\sum_i u_i = 1$; $u_i \to 0$, no $u \to 1$ (all compatible) → u eqs for Δ_{d-1} (facet)

d-dim polytope with N facets: binary geometry for d-dim solution space of u eqs of the form

$1 - u_a = p_a(\{u\}) \prod_b u_b^{b||a}, \quad \text{for } a = 1, \cdots, N,$ w. polynomial satisfying $p_a(\{u\})|_{u_a \to 0} = 1$

cluster polytopes (& products) : "perfect" u eqs with $p_a(\{u\}) \equiv 1$; more (non-trivial) binary geometries?

claim: configuration space of $\mathscr{I}_{\mathbf{B}}$ for any gen. permutohedron* (nestohdra & products) is binary!

(generally $p_a(\{u\}) \neq 1$, but infinite families with perfect u eqs)

config. space of $\mathscr{P}_d$

we prove for $\mathscr{P}_d$: as $u_I \to 0$, all incompatible $u_J \to 1$, remaining u's satisfy eqs for $\mathscr{P}_{|I|-1} \times \mathscr{P}_{d-|I|}$

proof: (recall u_I & u_J compatible iff $I \subset J$ or $J \subset I$) as $u_I \to 0$, all $x_i \to 0$ for $i \in I$, easy to show $u_J \to 1$

for (1). $I \cap J = \varnothing$ or (2). $I \cap J \neq \varnothing, I, J$; other u fall into either $\mathscr{P}_{|I|-1}$ ($J \subset I$) or $\mathscr{P}_{d-|I|}$ ($I \subset J$).

$\Longrightarrow$ stringy integral for $\mathscr{P}_d$ factorizes at finite α' : $\mathrm{Res}_{X_I \to 0} \mathscr{I}_d = \mathscr{I}_{|I|-1} \times \mathscr{I}_{d-|I|}$!

the proof can be extended to all gen. perm. considered: all binary, $\mathscr{I}_{\mathbf{B}}$ always factorizes at finite α'

the u eqs for $\mathscr{P}_d$ take the form: $1 - u_I = p_I(\{u\}) \prod_{J \not\supset I} u_J^{J||I}$, with $p_I(\{u\})|_{u_I \to 0} = 1$

"compatibility degree": $J||I = |J| - |I \cap J|$ for $I \not\subset J$ (0 for $I \subset J$)

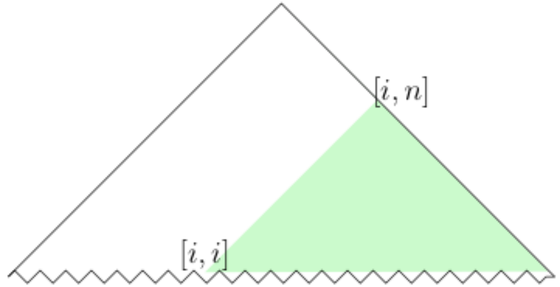
$p_I \neq 1$ for $|I| < d-1$, e.g. for d=3, $1 - u_0 = u_1 u_2 u_3 u_{12}^2 u_{23}^2 u_{13}^2 u_{123}^3 \left(1 + u_0 u_{023} u_{013} u_{012} u_{03} u_{01} u_{02}\right)$

perfect u eqs from $\mathscr{A}_d$ & $\mathscr{B}_d$

degenerations of $\mathscr{A}_d = \mathscr{M}^+_{0,d+3}$ with perfect u eqs: some $C \to 0$, new $\tilde{u} = \prod u$ (new eqs from old ones)

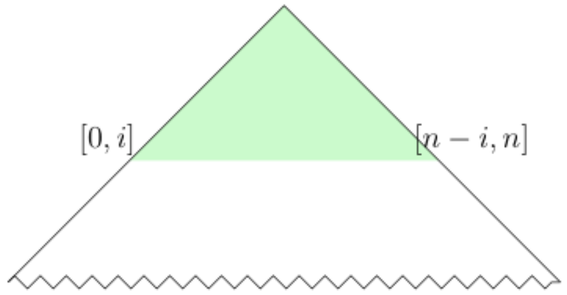
(1). $C_{k,l} = 0$ for $i \leq k < l \leq n$, new $\tilde{u}_{[k,k]} = \prod_{i \leq p \leq k \leq q \leq n} u_{[p,q]}$

$$1 - \tilde{u}_{[k,k]} = \prod_{j=0}^{i-1} \tilde{u}_{[j,k-1]}$$



(2). $C_{k,l} = 0$ for $l - k > i$, new $\tilde{u}_{[0,l]} = \prod_{j=0}^{l-i+1} u_{[j,l]}, \quad \tilde{u}_{[k,n]} = \prod_{j=k+i-1}^{n} u_{[k,j]}$

$$1 - \tilde{u}_{[0,l]} = \prod_{a=l-i+2}^{l+1} \tilde{u}_{[a,n]} \cdot \prod_{a=l-i+2}^{l+1} \prod_{b=l+1}^{a+i-2} \tilde{u}_{[a,b]}, \quad 1 - \tilde{u}_{[k,n]} = \prod_{b=k-1}^{k+i-2} \tilde{u}_{[0,b]} \cdot \prod_{b=k-1}^{k+i-2} \prod_{a=\max(b-i+2,1)}^{k-1} \tilde{u}_{[a,b]} .$$



include products of $\mathscr{A}$s; similar degenerations of $\mathscr{B}_d$ & more examples with $\mathscr{A} \times \mathscr{B}$ etc.

Q: binary geometries/perfect u eqs from degenerating type $\mathscr{D}$ etc. (no longer linear factors)?

outlook

- integral Qs: apply to Feynman integrals? recurrence & residues at “massive” poles? beyond polytopes?
- cluster algebra Qs: taming infinity (higher loops, $n \geq 8$)? U space for infinity? “kinematic spacetime”?
- binary geometry Qs: meaning of config. space & integrals? orderings & complex integrals? classifications?
- physics Qs: geometries for YM/NLSM forms, BCJ vs. projectivity & gravity, 4d amplitude forms, loops? ……

Thank you!