Combinatorial Geometries for Scattering Amplitudes

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based on collaborations w. Nima Arkani-Hamed, Thomas Lam, Giulio Salvatori, Hugh Thomas

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*Zoomplitudes @ Brown*
motivations

combinatorial geometries (w. factorizing bd.) underlying scattering amps in config. space & kinematic space

• moduli space $\mathcal{M}_{g,n}$ for conventional & (ambi-)twistor strings [Witten, Berkovits; Cachazo, SH, Yuan; Mason, Skinner; ...]

• positive grassmannian $G_+(k, n)$, on-shell diagrams etc. for planar N=4 SYM [Postnikov; Arkani-Hamed et al]

• amplituhedron: map from $G_+(k, n) \rightarrow$ canonical forms in momentum twistor space [Hodges; Arkani-Hamed, Trnka + Thomas; ...]

• helicity amps as differential form in momentum space [SH, Zhang]→momentum amplituhedron [talk by Ferro]

• kinematic associahedron (bi-adjoint $\phi^3$ in Mandelstam space) & worldsheet assoc. [Arkani-Hamed, Bai, SH, Yan]

• cluster polytopes, ABCDs of $\phi^3$ amps through 1-loop & beyond (clusterhedra?) [lectures & talk by Arkani-Hamed]

• gen. string amplitudes, cluster configuration space, binary geometries (old & new)... this talk!

• positive config. space/tropical grassmannian & N=4 SYM clusterology [talks by Cachazo, Drummond, Guevara, Lam, Spradlin ...]
kinematic associahedra

associahedra (vertices ↔ planar cubic trees) in \( \mathcal{H}_n \)
a family of \((n-3)\)-dim subspaces: e.g. for \(1 \leq i < j - 1 < n\)

\[
X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} = c_{i,j} > 0
\]

\( \mathcal{A}_{n-3} : \{ X_{i,j} \geq 0 \} \cap \text{subspace} \ [\text{ABHY}] \)

canonical form \( \Omega(\mathcal{A}_{n-3}) = \text{pullback of scattering form} = \text{bi-adjoint } \phi^3 \text{ tree amps} \)

e.g. \( \Omega(\mathcal{A}_1) = \left( \frac{ds}{s} - \frac{dt}{t} \right) \bigg|_{s+t=c} = ds \left( \frac{1}{s} + \frac{1}{t} \right) \)

geometric picture: FD expansion = a special triangulation, others → new formula & recursion for \( \phi^3 \) amps

hidden symmetry of \( \phi^3 \) amps (invisible in FD’s), analog of dual conformal symmetry, manifest by geometry!
wave equation & causal diamonds

ABHY: discrete wave eq in (1+1)-d “kin. spacetime” (mesh): \( \partial_u \partial_v X(u, v) = c(u, v) \)

causal diamond ⇄ (in-)compatible propagators/diagonals

continuum limit of \( \mathcal{A}_{n-3} \): space of positive sol. with positive source in a region

factorizing bd. \( \partial_{(ij)} \mathcal{A}_{n-3} = \mathcal{A}_L \times \mathcal{A}_R \) ensured by causal structure!
cluster polytopes & $\phi^3$ amplitudes

time evolution = “walk” on quiver (mutation on source) $v \to v' : X'_v + X_v - \sum_{w \leftarrow v} X_w = c_v$

new facet $X'_v \geq 0$; new vector must lie outside convex hull, stop when span full space

only finite polytopes for Dynkin diagrams $\to$ gen. associahedra of finite type [Bazier-Matte et al] (cluster polytopes)

classical: bi-adjoint $\phi^3$ through 1-loop; higher loops $\leftrightarrow$ infinite type [talk by Arkani-Hamed]

1-loop bd.: $\partial D_n = A \times D \& A_{n-1}$ (tree x 1-loop & forward limit)
cut in halves $\to D_n ; \Omega(D_n) \to$ new formulas for 1-loop $\phi^3$ amps!
outline

• stringy canonical forms

• binary cluster polytopes & generalized string amplitudes

• (more) binary geometries from gen. permutohedra

• grassmannian stringy integrals & config. spaces [see also SH, Lecheng Ren, Yong Zhang, 2001.09603]
stringy canonical forms  [w. N. Arkani-Hamed, T. Lam]
stringy canonical forms

vast generalizations of (open-)string amplitudes \( I_n(\{s\}) = (\alpha')^{n-3} \int_{\mathcal{M}_0^n} \frac{d^{n-3}z}{z_{1,2} \cdots z_{n,1}} \prod_{a < b} (z_{a,b})^{\alpha' s_{a,b}} \)

positive parametrization, e.g. \( z_3 = 1 + x_2, \ z_4 = 1 + x_2 + x_3, \ldots, \ z_{n-1} = 1 + x_2 + \cdots + x_{n-2} \Rightarrow \)

integral of the form w. positive polynomials: \( \mathcal{F}_{\{p\}}(X, \{c\}) = \alpha'^d \int_{\mathbb{R}^d_{>0}} \prod_i d x_i x_i^{\alpha' c_i} \prod_I p_I(x)^{-\alpha' c_I} \)

long history (“Euler-Mellin”…) \( \rightarrow \) \( d^d X \mathcal{F}(X) \) a diff. form exhibits geometries & remarkable properties

- leading order \( (\alpha' \rightarrow 0) \), residues on “massless poles” (+ convergence): controlled by polytope \( P \)
- “stringy” properties: meromorphic with “massive poles”, exponential UV softness, “channel duality”
- scattering (saddle-point) equations & twisted (co-)homology
- dual \( u \) variables, tropical compactifications, (complex) closed-stringy integrals …
**Minkowski sum**

single polynomial: \[ \mathcal{F}_p(X, c) = \alpha' \int_0^\infty \prod_{i=1}^d \frac{dx_i}{x_i} x_i^{\alpha' x_i} p(x)^{-\alpha' c} \]

converges iff \( X \) is inside **Newton polytope** \( c \, N(p) \), \( \lim_{\alpha' \to 0} \mathcal{F}_p(X, c) = \Omega(c \, N[p(x)]; X) \)

(also see [Nilsson, Passare; Berkesch et al])

trivial to gen. to \[ \mathcal{F}_{\{p\}}(X, \{c\}) = \alpha' \int_{\mathbb{R}_{>0}}^\infty \prod_{i=1}^d \frac{dx_i}{x_i} x_i^{\alpha' x_i} \prod_{I=1}^m p_I(x)^{-\alpha' c_I} \]

converges iff \( X \) is inside **Minkowski sum** \( \mathcal{P} := \sum_I c_I N[p_I(x)] \)

\[ \lim_{\alpha' \to 0} d^d X \mathcal{F}_{\{p\}}(X, \{c\}) = \Omega(\mathcal{P}; X) \quad \alpha'\text{-deformations} \text{ of canonical form of any (rational) polytope} \]

equivalently: leading order = **volume of dual polytope** = **tropical function** (c.f. [Panzer] on Hepp bound)
string integrals from associahedra

ABHY assoc. is rigid: shape independent of $c_{i,j} > 0$; decompose as Minkowski sum $\mathcal{A}_{n-3} = \sum_{1 \leq i < j - 1 < n} c_{i,j} \mathcal{A}_{i,j}$

$$X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} = c_{i,j}$$

Newton polynomial for $\mathcal{A}_{i,j}$: $p_{ij} = 1 + y_{i+1} + y_{i+1}y_{i+2} + \cdots + y_{i+1}\cdots y_{j-1}$

$$I_5 = \int \frac{dy_2}{y_2} \frac{dy_3}{y_3} \frac{\alpha'x_5 y_3}{\alpha'x_3}(1 + y_2)(1 + y_2 + y_2y_3)^{-\alpha'c_{13}}(1 + y_2 + y_2y_3)^{-\alpha'c_{14}}(1 + y_3)^{-\alpha'c_{24}}$$

re-discover open-string integrals & moduli space from ABHY!
two classes of examples

(1). stringy integral for gen. associahedra \( \mathcal{P}(\Phi) \) of finite type: 
\[
\mathcal{I}_\Phi(S) = (\alpha')^d \int_{\Lambda(\Phi)/T} (\omega/T) \prod_{\gamma \in \Gamma} \chi_\gamma^\alpha S_\gamma
\]
leading order \( \rightarrow \) ABHY cluster polytopes: 
\[
\lim_{\alpha' \to 0} d^d X \mathcal{I}_\Phi(X, \{c\}) = \Omega(\mathcal{P}(\Phi^v, c); X)
\]

ABCD: \( \alpha' \)-extension of \( \phi^3 \) amps w. factorization at finite \( \alpha' \) e.g. 
\[
\mathcal{I}_{D_n} \to \mathcal{I}_{A_n} \times \mathcal{I}_{D_n-m-1}, \quad \mathcal{I}_{A_{n-3}} \times \mathcal{I}_{A_1} \times \mathcal{I}_{A_1}, \quad \mathcal{I}_{A_{n-1}}
\]

(2). stringy integral over \( G_+(k, n)/T \): 
\[
\mathcal{I}_{k,n}(S) := (\alpha')^d \int_{G_+(k, n)/T} (\omega_{k,n}/T) \prod_l \Delta_l^{\alpha s_l} \quad d := (k-1)(n-k-1)
\]

\( D = \binom{n}{k} - n \) independent \( S_l \); higher-k gen. of string integrals (\( M^+_n \sim G_+(2, n)/T \), non-trivial to compute

\( \mathcal{P}_{k,n} = \sum_l S_l N[\Delta_l] \) [Arkani-Hamed, Lam, Spradlin] \( \leftrightarrow \) tropical \( G_+(k, n) \) [Speyer, Williams] [talks by Cachazo, Drummond, Guevara]
generalized scattering equations

for any $\mathcal{I}$, “scattering eqs” (saddle points) $\implies$ a diffeomorphism from int. domain to Mink. sum $\mathcal{P}$

$$\text{SE : } d \log \left( \prod_{i=1}^{d} x_i^{\alpha} \prod_{l=1}^{m} p_l(x)^{-\alpha c_l} \right) = 0 \implies \text{map } \mathbb{R}_d^+ \to \mathcal{P} : \mathbf{X} = \sum_{l=1}^{m} c_l \frac{\partial \log p_l(x)}{\partial \log x}$$

**theorem:** $y = f(x)$ diff. $\mathcal{X} \to \mathcal{Y} \implies$ pushforward $\Omega(\mathcal{Y}; y) = \sum_{x = f^{-1}(y)} \Omega(\mathcal{X}; x)$ [Arkani-Hamed, Bai, Thomas]

$\alpha' \to 0$ & $\alpha' \to \infty$ limits deeply connected: $\Omega(\mathcal{P}) = \lim_{\alpha' \to 0} d^d X \mathcal{I} = \sum_{\text{sol.}} \prod_{i} \frac{dx_i}{x_i}$

**gen. CHY formula:** $\lim_{\alpha' \to 0} \mathcal{I} = \Omega(\mathcal{P}) = \int \prod_{i=1}^{d} \frac{dx_i}{x_i} \delta \left( X_i - \sum_{l} c_l \frac{\partial \log p_l(x)}{\partial \log x_i} \right)$

**general phenomenon:** field-theory limit = pushforward via saddle points @ high-energy limit [Gross, Mende]
apply to string integral $\rightarrow$ geometric origin of CHY formulas

pullback of $\{ \sum_{b \neq a} \frac{S_{a,b}}{z_a - z_b} = 0 \}$: diffeomorphism $\mathcal{M}_{0,n}^+ \rightarrow \mathcal{A}_{n-3}$

e.g. $n=5$, $X_{2,5} = x \left( \frac{c_{1,3}}{1 + x} + \frac{c_{1,4}(1 + y)}{1 + x + xy} \right)$, $X_{3,5} = y \left( \frac{c_{2,4}}{1 + y} + \frac{c_{1,4}}{1 + x + xy} \right)$ $\Rightarrow$ $\sum_{\text{sol}} \frac{dxdy}{xy} = \Omega(\mathcal{A}_2) = d^2X_{m_5}$

string integrals very special: # of sol. (huge) reduction: $(n - 3)!$ the smallest # for assoc. (CHY most efficient!)

theorem: # of saddle points = dim of space of integral functions: twisted homology (cycles) $\leftrightarrow$ cohomology (forms)

\[
\int \prod_{i=1}^{d} \frac{dx_i}{x_i} Q(x) \exp(\alpha' F(x)), \quad F(x) := - \sum_{l} c_l \log p_l(x) \quad \text{with } \alpha' \to \infty, \dim H^d((\mathbb{C}^x)^d, dF) = \# \text{ of saddle points}
\]

e.g. cluster case: $\#(\mathcal{B}_n) = n^n$, $\#(\mathcal{C}_n) = (2n)!/2$, $\#(\mathcal{G}_2) = 13$, $\#(\mathcal{D}_4) = 55$

Gr. case: $(3,6)=26$, $(3,7)=1272$, $(3,8)=188112$ [Cachazo, Early, Guevara, Mizera; …] higher-k gen. of SE & bi-adjoint amps

all these closely related to intersection theory [Mizera; …] Feynman integrals? [c.f. Lee, Pomeransky; Mastrolia, Mizera; Frellesvig et al; …]
[also see talks by Britto, Caron-Huot, Duhr, …]
complex stringy integrals

closed strings? first mod-squared integrals: \( \mathcal{I}_{\{p\}}(X, \{c\}) = \alpha' d \int_{\mathbb{C}^d} \prod_{i=1}^{d} \frac{d\bar{z}_i d\bar{z}_i}{|z_i|^2} |z_i|^{2\alpha'} X_i \prod_I |p_I(z)|^{-2\alpha' c_i} \)

for \( 0 < \alpha' < \epsilon \), convergence & leading order again controlled by \( \mathcal{P} := \sum_I c_I \text{N}[p_I] \)

more general \( \mathcal{I}_{\{p\}}(X, \{c, n\}) := \cdots \prod_i \bar{z}_i^{n_i} \prod_I p_I(\bar{z})^{-n_I} \) with non-negative integer shifts

Minkowski sum \( \mathcal{P} \): unbounded polyhedron \( (n/\alpha' \to \infty) \), canonical form/dual volume still finite!

apply to closed & open string amps w. two orderings: \( I_n^{\text{closed}}(\alpha | \beta) \) & \( I_n^{\text{open}}(\alpha | \beta) \) [Carrasco, Mafra, Schlotterer;…]

unbounded \( \mathcal{A}_{n-3} \) with facets @ infinity: (both) leading orders = \( m(\alpha | \beta) \)

similar unbounded polyhedron for bi-color \( \phi^3 \) amps [Herderschee, SH, Teng, Zhang]
dual u variables

\( \mathcal{P} \) bounded by N facets \( W^J_S J \geq 0 \) with \( S := (X_1, \ldots, X_d, -c_1, \ldots, -c_m) \in \mathbb{R}^{d+m} \rightarrow \) defines (d+m)-dim big polyhedron \( \mathcal{B} \); point inside \( S \in \mathcal{B} = \) positive comb. of vertices \( V^\alpha : S_J = \sum_\alpha V^\alpha J F^\alpha \)

\( \mathcal{J} \) converges iff \( F^\alpha > 0 \), rewrite w. dual variables \( u^\alpha := \prod_J p^J^\alpha (p^J := (x_1, \cdots, x_d, p_1, \cdots, p_m)) \)

\[
\mathcal{J}(S) = \int_{\mathbb{R}^d} \prod_{i=1}^d \frac{dx_i}{x_i} \prod_J p^J^\alpha S_J = \int_{\mathbb{R}^d} \prod_{i=1}^d \frac{dx_i}{x_i} \prod_\alpha u^\alpha F^\alpha := \mathcal{J}(F)
\]

special case: \( \mathcal{B} = \) simplex ( \( N = N_v = d + m \) ), \( V = W^{-1} \), u variables multiplicative independent

each facet \( F^\alpha \rightarrow 0 \iff u^\alpha \rightarrow 0 \): true for \( \mathcal{J}_\Phi \), not for \( \mathcal{J}_{n,k>2} \) (c.f. \( \mathcal{D}_4 \) vs. \( \mathcal{P}_{3,6} \))

configuration space \( U^o, U, U_{\geq 0} : \{u^\alpha\}_{1 \leq \alpha \leq N} \) by u eqs, \( U_{\geq 0} := \{u^\alpha \geq 0\} \) “curvy polytope” [talk by Lam]
binary cluster polytopes & generalized string amplitudes

[w. N. Arkani-Hamed, T. Lam, H. Thomas]
cluster config. space & integrals

cluster polytopes [Chapton, Fomin, Zelevinsky] ↔ Dynkin diagram

ABHY \rightarrow more rigid, canonical realization?

U space has same bd. as cluster polytope (algebraically)

binary geometry: \( u_a \rightarrow 0 \) all incompatible \( u_b \rightarrow 1 \) (\( b \parallel a > 0 \))

gen. of \( \mathcal{M}_{0,n} \) to finite-type cluster algebra

cluster stringy integrals over (positive & complex)

binary geometries: gen. open \& closed string amps
binary associahedra & string integrals

u eqs: \( 1 - u_{i,j} = \prod_{(k,l) \text{ cross } (i,j)} u_{k,l} \) (deg.=0,1): (n-3)-dim solution space \( \rightarrow \) configuration space of type A

\( U^0_n = M_{0,n} \subset U_n \subset \overline{M}_{0,n} \) [see F. Brown]: algebraically same boundary structure as \( A_{n-3} \) (holds in \( \mathbb{C} \))

any \( u_{k,l} \to 0 \), all incompatible \( u_{i,j} \to 1 \) (binary) \( \Longrightarrow \) \( \partial_{u_{i,j} \to 0} U_n = U(i, i + 1, \ldots, j) \times U(j, j + 1, \ldots, i) \)

ask \( u_{i,j} > 0 \) \( \Longrightarrow \) \( 0 < u_{i,j} < 1 \) defines positive part \( U^+_n \sim M^+_{0,n} \)

it has (curvy) shape of assoc. \( A_{n-3} \), e.g. \( U^+_5 \) is a pentagon

for \( U^+_n \), open-string integrals (in \( u \) variables): \( I_n = \int_{U^+_n} \Omega(U^+_n) \prod_{(a,b)} u_{a,b} \alpha^a X_{a,b} \), \( \Omega(U^+_n) = \prod_{(i,j)} d \log \frac{u_{i,j}}{1 - u_{i,j}} \)

here \( (n - 3) \) \( u_{i,j} \) in a triangulation without internal triangle (acyclic quiver) \( \Longrightarrow \) Parke-Taylor form!
worldsheet from U space

\( U_n : \) gauge-inv. description of \( M_{0,n} \) (u’s cross-ratios of n points) \( \rightarrow \) emergence of worldsheet!

extended u eqs [see also F. Brown]: \( [a, b \mid c, d] + [b, c \mid d, a] = 1 \), w. \( [a, b \mid c, d] := \prod_{a \leq i < b, c \leq j < d} u_{i,j} \)

(in total \( \binom{n}{4} \) eqs, follow from u eqs, which are special case w. \( u_{i,j} = [i, i + 1 \mid j, j + 1] \))

& \( [a, b \mid c, e][a, b \mid e, d] = [a, b \mid c, d] \): all constraints satisfied by cross-ratios \( \implies [a, b \mid c, d] = \frac{(ad)(bc)}{(ac)(bd)} \)

\( U_n^+ : \) \( 0 < [a, b \mid c, d] < 1 \implies \) n points ordered; how to see other orderings from u’s? sign patterns

extended eqs \( \prod u + \prod u = 1 \) forbid sign patterns w. both terms <0: exactly \( (n - 1)!/2 \) connected components of \( U_n(\mathbb{R}) \)

identical extended u eqs by monomial transf. of u’s \( \rightarrow \) hidden \( S_n \) symmetry (automorphism for \( U_n(\mathbb{C}) \) )
gen. u eqs from “walk”

gen. to any finite-type cluster algebra: only new ingredient compatibility degree (N by N matrix, beyond 0,1)

concise new def. from “walk” picture (each step \( v \to v' \), \( X_{v'} + X_v - \sum_{w \leftarrow v} n_w X_w = c_v \)); \( b \mid a := X_a \) integer “Green’s function”

for all \( a \) & a given source \( b \) in initial quiver (all initial \( X = 0 \), all \( c = 0 \) except \( c_b = 1 \))

non-simply laced also by folding: \( A_{2n-3} \to C_{n-1} \), \( D_n \to B_{n-1} \), \( B_3 \to G_2 \), \( E_6 \to F_4 \)

u eqs equiv. to “local” eqs (Y system [Fomin, Zelevinsky]): each step \( Y_v Y_{v'} = \prod_{w \leftarrow v} (1 + Y_w)^{n_w} \), \( Y := \frac{u}{1 - u} \) (N eqs).
cluster conf. space

\{ u_a + \prod_{all \ b} u_b^{b||a} = 1 \} \ (N-n \ independent) \rightarrow n\text{-dim} \ U^o(\Phi) \ & \ U(\Phi); \ algebraically \ same \ bd. \ structure \ as \ \mathcal{P}(\Phi)

binary geometries: any \ u_b \rightarrow 0, \ incompatible \ u_a \rightarrow 1 \ (in \ \mathbb{C}), \ factorize \ by \ removing \ node \ in \ Dynkin \ diagram

e.g. \ U(\mathcal{B}/\mathcal{C}) \rightarrow U(\mathcal{A}) \times U(\mathcal{B}/\mathcal{C}), \ U(\mathcal{D}_n) \rightarrow U(\mathcal{A}_m) \times U(\mathcal{D}_{n-m-1}), \ U(\mathcal{D}_{n-3}) \times U(\mathcal{A}_1)^2, \ & \ U(\mathcal{A}_{n-1})

ask all \ u_a > 0 \ cuts \ out \ curvy \ polytope \ U^+(\Phi) \sim \mathcal{P}(\Phi), \ e.g. \ U^+(\mathcal{B}/\mathcal{C}) \ “curvy” \ cyclohedra

canonical form \ \Omega(U^+) = \prod_{\alpha} d \log \frac{u_\alpha}{1-u_\alpha} \ \text{for} \ \alpha \ \text{in any acyclic quiver: rationally solve all } N \ u’s \ by \ \{u_\alpha\}

what about “orderings”? conjecture: connected comp. of \ U(\mathbb{R}) \leftrightarrow \ sign \ patterns \ by \ extended \ eqs \ U_I + V_I = 1

1:1 \ with \ mutation \ relations: \ for \ \mathcal{A}_{n-3}, \ I : (a, c) \rightarrow (b, d) \ ; \ n(n-1)(n^2 + 4n - 6)/6 \ eqs \ for \ \mathcal{D}_n, \ etc.

monomial transf. & beyond: new shapes for other orderings, e.g. for \ \mathcal{B}_2 = \mathcal{C}_2, \ tiled \ by \ 4 \ hexagons + 12 \ pentagons
cluster string integrals

U space (type A) manifest factorization @ finite $\alpha'$: $\text{Res}_{X_{a,b}=0} I_n = I(a, a + 1, \ldots, b) \times I(b, b + 1, \ldots, a)$
as $X_{a,b} \to 0$, $u_{a,b} \to 0$: not only $\Omega \to \Omega_L \times \Omega_R$, $\prod u^{\alpha'X} \to \prod L u^{\alpha'X} \times \prod R u^{\alpha'X}$ (all incompatible u's decouple)

for $U_n(\mathbb{C})$, closed-string integrals (w. orderings $\alpha$ & $\beta$): $I_n^c(\alpha | \beta) = \int_{U_n(\mathbb{C})} \Omega(\alpha) \prod u^{\alpha'X} \left( \Omega(\beta) \prod u^{\alpha'X} \right)^*$

for any finite-type $\Phi$: $\mathcal{F}_\Phi(X) = \int_{U^+(\Phi)} \Omega(U^+(\Phi)) \prod u^{\alpha'X_a} \quad \Omega(U^+(\Phi)) := \prod_a d \log \frac{u_\alpha}{1 - u_\alpha}$

factorization by removing node of Dynkin diagram: $\Omega(U^+) & \prod u^{\alpha'X}$ factorize nicely!

$\mathcal{B}_{n-1}/\mathcal{E}_{n-1}$ factorize as 1-loop tadpoles, $\mathcal{D}_n$ as 1-loop amp; exotic ones e.g. $\mathcal{E}_6 \to \mathcal{D}_5$, $\mathcal{A}_5$, $\mathcal{A}_4 \times \mathcal{A}_1 & \mathcal{A}_2 \times \mathcal{A}_1$

“closed stringy” integrals over $U^\Phi(\mathbb{C})$ well-defined for $\alpha$ & $\beta$ related by monomial transf.

physical meaning of these integrals (contain string integrals)? e.g. $\mathcal{D}_n$ is $\alpha'$-deformation of 1-loop integrand for $\phi^3$
arithmetic geometry for $U$

count # of points in $U_n(𝔽_p)$: $N(p) = (p-2)(p-3)\cdots(p-n+2) \implies$ topological prop. of $ℳ_{0,n}$

- $\lvert N(-1) \rvert = \#$ of connected components/orderings: $(n-1)!/2$

- $\lvert N(1) \rvert = \lvert \chi \rvert = \#$ of saddle points = dim of half-dim twisted cohomology: $(n-3)!$ [BCJ; CHY; Mizera ...]

- coef. of $N(p) \leftrightarrow \#$ of $d$ log forms $\implies \lvert N(0) \rvert = \#$ of $d$ log top form: $(n-2)!$ [Kleiss, Kuijf]

$N(p) = (p-n+1)^n$ for $ℬ_n$, $N(p) = (p-n-1)(p-3)(p-5)\cdots(p-2n+1)$ for $ℂ_n$ $\implies$

<table>
<thead>
<tr>
<th>$ℬ_n$</th>
<th>orderings</th>
<th>$KK$</th>
<th>BCJ/solutions</th>
</tr>
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<tbody>
<tr>
<td>$(n+2)^n$</td>
<td>$(n+1)^n$</td>
<td>$n^n$</td>
<td>$\frac{(2n)!!}{2}$</td>
</tr>
</tbody>
</table>

| $ℂ_n$ | $(2n)!!(n+2)$ | $(2n-1)!!(n+1)$ |

beyond: (conjecture) quasi-polynomial, e.g. for $ℌ_2$, $N(p) = (p-4)^2 + 4\delta_p$, $\delta_p = 0$ $(p = 2 \mod 3)$ or 1 $(p = 1 \mod 3)$ $\implies$ 25 orderings & 13 saddle points; 547 orderings, 55 saddle points for $ℌ_4$; similarly for (some) $G_+(k, n)/T$
(more) binary geometries from gen. permutohedra

[w. Z. Li, P. Raman, C. Zhang]
stringy integrals for gen. perm.

building set $\mathcal{B} = \text{collection of subsets } I \subset [0,d] \text{ (w. conditions)}$ [Postnikov] $\rightarrow d$-dim $\mathcal{P}_B$ as Minkowski sum of coordinate simplices $\Delta_I$ for $I \in \mathcal{B} \rightarrow$ beg for stringy canonical forms w. sum $x_I := \sum_i x_i$ for each $\Delta_I$

$$J_B(S) := \int_{\mathbb{R}_+^d} \frac{1}{\text{vol GL}(1)} \prod_{i=0}^d \frac{dx_i}{x_i} \prod_{I \in \mathcal{B}} x_i^{\alpha S_i} = \int_{\mathbb{R}_+^d} \prod_{i=1}^d \frac{dx_i}{x_i} x_i^{\alpha x_i} \prod_I x_i^{-\alpha C_I}$$

projective: $\sum_{I \in \mathcal{B}} S_I = 0$ (fix e.g. $x_0 = 1$); leading order of $J_B = \Omega(\mathcal{P}_B; X)$, some examples:

- simplex: $\mathcal{B} = \text{all singletons } + [0,d]$, gen. beta function $J_B = \prod_i \Gamma(X_i)/\Gamma(\sum_i X_i)$

- associahedron: $\mathcal{B} = \{[i,j] \mid 0 \leq i \leq j \leq d\}$, open-string integral (type $A_d$)

- cyclohedron: $\mathcal{B} = \text{all cyclic intervals}$, stringy integral of type $B_d$

- permutohedron: $\mathcal{B} = \text{all subsets}$, $J_d := \text{(rigid) stringy integral for perm. } \mathcal{P}_d$
remarkably, facets of $\mathcal{P}_B$ labelled by subsets $I \neq [0,d]$ (factorize as $\mathcal{P}_{B_I} \times \mathcal{P}_{B_I}^c$) $\implies N = d + m$

$$\prod_{i=1}^{d} x_i^{\alpha X_i} \prod_{I} x_I^{-\alpha C_I} = \prod_{I} u_I^{\alpha X_I}$$

ABHY realization of $\mathcal{P}_B$: $-C_J = V^l_J X_I$ (for $J=1,\ldots,m$) $\leftrightarrow$ $u$ variables: $u_I = \prod_J (x_J)^{V^l_J}$ (for $I=1,\ldots,N$)

e.g. simplex $\Delta_d$: $-C_{[0,d]} = \sum_i X_i \leftrightarrow u_i = \frac{x_i}{\sum_i x_i}$

permutohedron $\mathcal{P}_d$: $-C_J = \sum_{I \subset J} (-)^{|J|-|I|} X_I$ (alternating sum) $\leftrightarrow$ $u_I = \prod_J x_J^{(-)^{|J|-|I|}}$

for $d=2$ ($\mathcal{P}_2 = \mathcal{B}_2$): $u_0 = \frac{x_0x_{012}}{x_{01}x_{02}}, u_1 = \frac{x_1x_{012}}{x_{01}x_{12}}, u_2 = \frac{x_2x_{012}}{x_{02}x_{12}}, u_{01} = \frac{x_{01}}{x_{012}}, u_{02} = \frac{x_{02}}{x_{012}}, u_{12} = \frac{x_{12}}{x_{012}}$

any $\mathcal{I}_B$ & $\mathcal{P}_B$ degeneration of $\mathcal{I}_d$ & $\mathcal{P}_d$ with some $C_I \rightarrow 0$, e.g. $\mathcal{P}_2|_{C_{02} \rightarrow 0} = \mathcal{A}_2$
binary geometries revisited

Q: what type of u-eqs needed for binary geometries, i.e. any \( u_a \to 0 \), all incompatible \( u_b \to 1 \) ?

trivial ex: u eqs for simplex \( \Delta_d \), \( \sum_i u_i = 1; u_i \to 0 \), no \( u \to 1 \) (all compatible) \( \to u \) eqs for \( \Delta_{d-1} \) (facet)

d-dim polytope with N facets: binary geometry for d-dim solution space of u eqs of the form

\[
1 - u_a = p_a(\{u\}) \prod_b u_b^{\delta_{|a|}} \quad \text{for } a = 1, \ldots, N, \quad \text{w. polynomial satisfying } p_a(\{u\}) |_{u_a \to 0} = 1
\]

cluster polytopes (& products) : “perfect” u eqs with \( p_a(\{u\}) \equiv 1 \); more (non-trivial) binary geometries?

claim: configuration space of \( \mathcal{F}_B \) for any gen. permutohedron* (nestohdra & products) is binary!

(generally \( p_a(\{u\}) \neq 1 \), but infinite families with perfect u eqs)
config. space of $\mathcal{P}_d$

we prove for $\mathcal{P}_d$: as $u_I \to 0$, all incompatible $u_J \to 1$, remaining u’s satisfy eqs for $\mathcal{P}_{|I|-1} \times \mathcal{P}_{d-|I|}$

proof: (recall $u_I$ & $u_J$ compatible iff $I \subset J$ or $J \subset I$) as $u_I \to 0$, all $x_i \to 0$ for $i \in I$, easy to show $u_J \to 1$

for (1). $I \cap J = \emptyset$ or (2). $I \cap J \neq \emptyset$, $I$, $J$; other u fall into either $\mathcal{P}_{|I|-1} (J \subset I)$ or $\mathcal{P}_{d-|I|} (I \subset J)$.

$\implies$ stringy integral for $\mathcal{P}_d$ factorizes at finite $\alpha'$: $\text{Res}_{X_I \to 0} \mathcal{I}_d = \mathcal{I}_{|I|-1} \times \mathcal{I}_{d-|I|}$!

the proof can be extended to all gen. perm. considered: all binary, $\mathcal{I}_B$ always factorizes at finite $\alpha'$

the u eqs for $\mathcal{P}_d$ take the form: $1 - u_I = p_I(\{u\}) \prod_{J \neq I} u_J^{p_J(I)}$, with $p_I(\{u\}) |_{u_I \to 0} = 1$

“compatibility degree”: $|J| - |I| - |I \cap J|$ for $I \notin J$ (0 for $I \subset J$)

$p_I \neq 1$ for $|I| < d - 1$, e.g. for $d=3$, $1 - u_0 = u_1 u_2 u_3 u_1^2 u_2 u_3 u_1 u_3 (1 + u_{023} u_{013} u_{012} u_{03} u_{01} u_{02})$
perfect u eqs from $\mathcal{A}_d \& \mathcal{B}_d$

degenerations of $\mathcal{A}_d = \mathcal{M}_0^{+d+3}$ with perfect u eqs: some $C \rightarrow 0$, new $\tilde{u} = \prod u$ (new eqs from old ones)

1. $C_{k,l} = 0$ for $i \leq k < l \leq n$, new $\tilde{u}_{[k,k]} = \prod_{i \leq p \leq k \leq q \leq n} u_{[p,q]}$

   $$1 - \tilde{u}_{[k,k]} = \prod_{j=0}^{i-1} \tilde{u}_{[j,k-1]}$$

2. $C_{k,l} = 0$ for $l - k > i$, new $\tilde{u}_{[0,l]} = \prod_{j=0}^{l-i+1} u_{[j,l]}$, $\tilde{u}_{[k,n]} = \prod_{j=k+i-1}^{n} u_{[k,j]}$

   $$1 - \tilde{u}_{[0,l]} = \prod_{a=l-i+2}^{l+1} \tilde{u}_{[a,n]} \cdot \prod_{a=l-i+2}^{l+1} \tilde{u}_{[a,b]} \cdot \prod_{b=l+1}^{a+i-2} \tilde{u}_{[a,b]}$$

   $$1 - \tilde{u}_{[k,n]} = \prod_{b=k-1}^{k+i-2} \tilde{u}_{[0,b]} \cdot \prod_{b=k-1}^{k+i-2} \tilde{u}_{[a,b]} \cdot \prod_{a=max(b-i+2,1)}^{k-1} \tilde{u}_{[a,b]}$$

include products of $\mathcal{A}$s; similar degenerations of $\mathcal{B}_d$ & more examples with $\mathcal{A} \times \mathcal{B}$ etc.

Q: binary geometries/perfect u eqs from degenerating type $\mathcal{D}$ etc. (no longer linear factors)?
outlook

• integral Qs: apply to Feynman integrals? recurrence & residues at “massive” poles? beyond polytopes?

• cluster algebra Qs: taming infinity (higher loops, $n \geq 8$)? U space for infinity? “kinematic spacetime”?

• binary geometry Qs: meaning of config. space & integrals? orderings & complex integrals? classifications?

• physics Qs: geometries for YM/NLSM forms, BCJ vs. projectivity & gravity, 4d amplitude forms, loops? ……

Thank you!