The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron

Lauren K. Williams, Harvard

Slides at

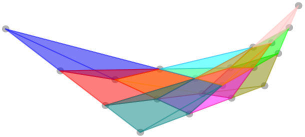
http://people.math.harvard.edu/~williams/Zoomplitudes.pdf

Based on:

- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Tomasz Lukowski and Matteo Parisi, arXiv:2002.06164
- "The positive Dressian equals the positive tropical Grassmannian," with David Speyer, arXiv:2003.10231

Overview of the talk

I. Amplituhedron '13
Arkani-Hamed–Trnka
$\mathcal{N} = 4$ SYM

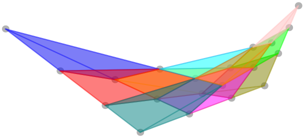


II. Hypersimplex and moment map '87
Gelfand-Goresky-MacPherson-Serganova
matroids, torus orbits on $Gr_{k,n}$

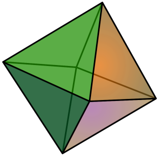
III. Positive tropical Grassmannian '05
Speyer–W.
associahedron, cluster algebras
connected to amplitudes, "pos. configuration space"

Overview of the talk

I. Amplituhedron '13
Arkani-Hamed–Trnka
$\mathcal{N} = 4$ SYM



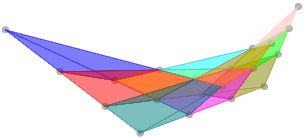
II. Hypersimplex and moment map '87
Gelfand-Goresky-MacPherson-Serganova
matroids, torus orbits on $Gr_{k,n}$



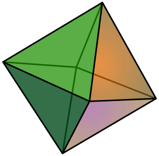
III. Positive tropical Grassmannian '05
Speyer–W.
associahedron, cluster algebras
connected to amplitudes, "pos. configuration space"

Overview of the talk

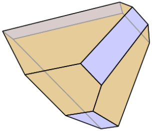
I. Amplituhedron '13
Arkani-Hamed–Trnka
$\mathcal{N} = 4$ SYM



II. Hypersimplex and moment map '87
Gelfand-Goresky-MacPherson-Serganova
matroids, torus orbits on $Gr_{k,n}$



III. Positive tropical Grassmannian '05
Speyer–W.
associahedron, cluster algebras
connected to amplitudes, "pos. configuration space"



Program

- Background on the Grassmannian and amplituhedron
-
-
-
-
-

Program

- Background on the Grassmannian and amplituhedron
-
-
-
-
-

Program

- Background on the Grassmannian and amplituhedron
- (Positroid) triangulations of the amplituhedron
-
-
-
-

Program

- Background on the Grassmannian and amplituhedron
- (Positroid) triangulations of the amplituhedron
- The (positive) tropical Grassmannian
-
-
-

Program

- Background on the Grassmannian and amplituhedron
- (Positroid) triangulations of the amplituhedron
- The (positive) tropical Grassmannian
- (Positroid) triangulations of the hypersimplex
-
-

Program

- Background on the Grassmannian and amplituhedron
- (Positroid) triangulations of the amplituhedron
- The (positive) tropical Grassmannian
- (Positroid) triangulations of the hypersimplex
- T-duality map connects amplituhedron triangulations and hypersimplex triangulations
-

Program

- Background on the Grassmannian and amplituhedron
- (Positroid) triangulations of the amplituhedron
- The (positive) tropical Grassmannian
- (Positroid) triangulations of the hypersimplex
- T-duality map connects amplituhedron triangulations and hypersimplex triangulations
- Conjecture, results, and summary

Background on the (TNN) Grassmannian

The **Grassmannian** $Gr_{k,n} = Gr_{k,n}(\mathbb{R}) := \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The **TNN (totally nonnegative) Grassmannian** $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where $p_I(A) \geq 0$.

Def due to Postnikov from early 2000's. Earlier Lusztig defined $(G/P)_{\geq 0}$. Not obvious that Lusztig's definition – in the case of $Gr_{k,n}$ – agrees with Postnikov's – but this is true (Rietsch 2007).

Background on the (TNN) Grassmannian

The **Grassmannian** $Gr_{k,n} = Gr_{k,n}(\mathbb{R}) := \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The **TNN (totally nonnegative) Grassmannian** $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where $p_I(A) \geq 0$.

Def due to Postnikov from early 2000's. Earlier Lusztig defined $(G/P)_{\geq 0}$. Not obvious that Lusztig's definition – in the case of $Gr_{k,n}$ – agrees with Postnikov's – but this is true (Rietsch 2007).

Background on the (TNN) Grassmannian

The **Grassmannian** $Gr_{k,n} = Gr_{k,n}(\mathbb{R}) := \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The **TNN (totally nonnegative) Grassmannian** $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where $p_I(A) \geq 0$.

Def due to Postnikov from early 2000's. Earlier Lusztig defined $(G/P)_{\geq 0}$. Not obvious that Lusztig's definition – in the case of $Gr_{k,n}$ – agrees with Postnikov's – but this is true (Rietsch 2007).

Background on the (TNN) Grassmannian

The **Grassmannian** $Gr_{k,n} = Gr_{k,n}(\mathbb{R}) := \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The **TNN (totally nonnegative) Grassmannian** $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where $p_I(A) \geq 0$.

Def due to Postnikov from early 2000's. Earlier Lusztig defined $(G/P)_{\geq 0}$. Not obvious that Lusztig's definition – in the case of $Gr_{k,n}$ – agrees with Postnikov's – but this is true (Rietsch 2007).

Background on the (TNN) Grassmannian

The **Grassmannian** $Gr_{k,n} = Gr_{k,n}(\mathbb{R}) := \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The **TNN (totally nonnegative) Grassmannian** $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where $p_I(A) \geq 0$.

Def due to Postnikov from early 2000's. Earlier Lusztig defined $(G/P)_{\geq 0}$. Not obvious that Lusztig's definition – in the case of $Gr_{k,n}$ – agrees with Postnikov's – but this is true (Rietsch 2007).

Background on the (TNN) Grassmannian

The **Grassmannian** $Gr_{k,n} = Gr_{k,n}(\mathbb{R}) := \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The **TNN (totally nonnegative) Grassmannian** $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where $p_I(A) \geq 0$.

Def due to Postnikov from early 2000's. Earlier Lusztig defined $(G/P)_{\geq 0}$. Not obvious that Lusztig's definition – in the case of $Gr_{k,n}$ – agrees with Postnikov's – but this is true (Rietsch 2007).

Background on the (TNN) Grassmannian

$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.

Background on the (TNN) Grassmannian

$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S^{tnn}_{\mathcal{M}} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S^{tnn}_{\mathcal{M}}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.

Background on the (TNN) Grassmannian

$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.

Background on the (TNN) Grassmannian

$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.

Background on the (TNN) Grassmannian

$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.

Background on the (TNN) Grassmannian

$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.

Background on the (TNN) Grassmannian

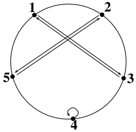
$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.



Background on the (TNN) Grassmannian

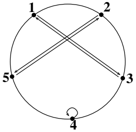
$(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ for all I.

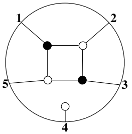
One can partition $(Gr_{k,n})_{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}}^{tnn} := \{A \in (Gr_{k,n})_{\geq 0} \mid p_I(A) > 0 \text{ iff } I \in \mathcal{M}\}$.

(Postnikov) If $S_{\mathcal{M}}^{tnn}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. Positroid cells S_π of $(Gr_{k,n})_{\geq 0}$ are in bijection with:

- Decorated permutations π on $[n]$ with k antiexcedances.
- other combinatorial objects such as on-shell diagrams.





Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k+m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\tilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \tilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\tilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \tilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k + m)$ matrix with maximal minors positive.
Let $\tilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \tilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k + m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k+m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

- The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka in 2013.
- $\mathcal{A}_{n,k,m}$ is the image of the TNN Grassmannian under a simple map.

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$.
Let Z be a $n \times (k + m)$ matrix with maximal minors positive.
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

- $\mathcal{A}_{n,k,m}$ has full dimension km inside $Gr_{k,k+m}$.
- When $m = 4$, its "volume" computes scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills theory.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.
- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$ (Arkani-Hamed – Trnka).
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.
- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$ (Arkani-Hamed – Trnka).
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^+_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.
- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$ (Arkani-Hamed – Trnka).
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^+_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.
- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$ (Arkani-Hamed – Trnka).
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

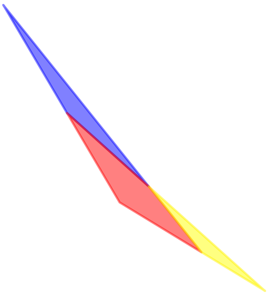
Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^+_{n,k+m}$ (max minors > 0).

Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.

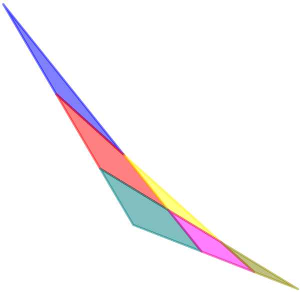
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

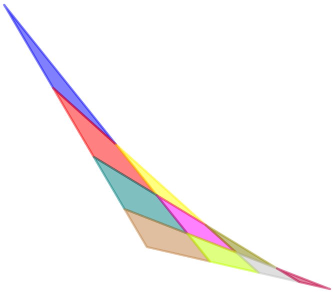
- If $m = n - k$, $\mathcal{A}_{n,k,m} = (Gr_{k,n})_{\geq 0}$.
- If $k = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{1,1+m}$ is equivalent to a cyclic polytope with n vertices in $\mathbb{P}^m$ (Arkani-Hamed – Trnka).
- If $m = 1$, $\mathcal{A}_{n,k,m} \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement (Karp–W.)



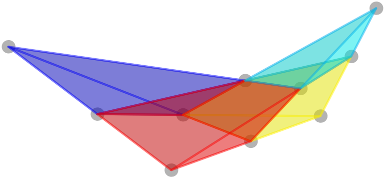
$\mathcal{A}_{4,2,1}$



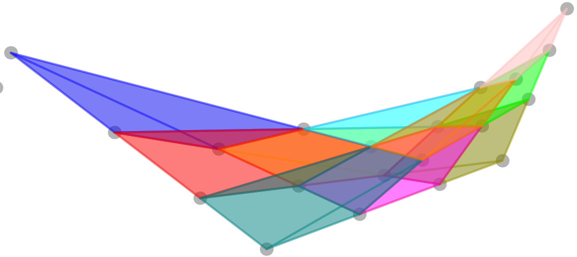
$\mathcal{A}_{5,2,1}$



$\mathcal{A}_{6,2,1}$



$\mathcal{A}_{5,3,1}$



$\mathcal{A}_{6,3,1}$

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^+_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^+_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Special cases

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$ (subject of today's talk):
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^{+}_{n,k+m}$.

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.

Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^{+}_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^{+}_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^{+}_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^{+}_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^{+}_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^+_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^{+}_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Background and Motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$. Let $Z \in \text{Mat}^+_{n,k+m}$.
Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

To understand its volume, find "triangulations" of the amplituhedron

- Have $\dim \mathcal{A}_{n,k,m} = km \leq \dim(Gr_{k,n})_{\geq 0}$, so $\widetilde{Z}$ generally not injective.
- Recall we have cell decomposition of $(Gr_{k,n})_{\geq 0}$ into positroid cells.
- Problem: Find collection of km-dimensional cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover (dense subset of) $\mathcal{A}_{n,k,m}$.
- Following the physicists, call this a **triangulation**.
- The BCFW recurrence can (conjecturally) be formulated as giving triangulations of $\mathcal{A}_{n,k,4}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall dim $\mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall $\dim \mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall $\dim \mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall $\dim \mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m=2, m=4, k=1$.
Remark: For $m=2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall $\dim \mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall dim $\mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall $\dim \mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m=2, m=4, k=1$.
Remark: For $m=2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall dim $\mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall dim $\mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.

Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Triangulating the amplituhedron

Triangulations of the amplituhedron

Have $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,m} \subset Gr_{k,k+m}$; recall $\dim \mathcal{A}_{n,k,m} = km$.
A **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images are disjoint and cover $\mathcal{A}_{n,k,m}$.

Wild conjecture (Karp-Zhang-W)

$$\text{Let } M(a,b,c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

be number of *plane partitions* contained in $a \times b \times c$ box. The number of cells in a triangulation of $\mathcal{A}_{n,k,m}$ for even m is $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results/conjectures for $m = 2, m = 4, k = 1$.
Remark: For $m = 2$, says there are $\binom{n-2}{k}$ cells in triangulation of $\mathcal{A}_{n,k,2}$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi(1)}, \ldots, S_{\pi(\ell)}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi(1)}, \ldots, Z_{\pi(\ell)}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi(i)} \cap Z_{\pi(j)}$ has codimension 1, it equals Z_{π}, the image of a cell S_{π} in the closure of both $S_{\pi(i)}$ and $S_{\pi(j)}$.

Data

\# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
\# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \dots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_{π}, the image of a cell S_{π} in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

\# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.

\# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi(1)}, \ldots, S_{\pi(\ell)}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi(1)}, \ldots, Z_{\pi(\ell)}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi(i)} \cap Z_{\pi(j)}$ has codimension 1, it equals Z_{π}, the image of a cell S_{π} in the closure of both $S_{\pi(i)}$ and $S_{\pi(j)}$.

Data

# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \dots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_{π}, the image of a cell S_{π} in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.

# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \ldots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \ldots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_π, the image of a cell S_π in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.

# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \dots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_{π}, the image of a cell S_{π} in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.

# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \dots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_{π}, the image of a cell S_{π} in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \dots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_{π}, the image of a cell S_{π} in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.

# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Good triangulations of the amplituhedron

Recall: a **triangulation** of $\mathcal{A}_{n,k,m}$ is a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of km-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that their images $\{Z_{\pi^{(1)}}, \dots, Z_{\pi^{(\ell)}}\}$ are disjoint and cover $\mathcal{A}_{n,k,m}$.

Remark: many of the known triangulations of the amplituhedron are "bad" in the sense that boundaries of images of cells overlap badly.

Definition (Lukowski–Parisi–W.)

Say that a triangulation is **good** if whenever $Z_{\pi^{(i)}} \cap Z_{\pi^{(j)}}$ has codimension 1, it equals Z_π, the image of a cell S_π in the closure of both $S_{\pi^{(i)}}$ and $S_{\pi^{(j)}}$.

Data

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Data on good triangulations of the amplituhedron

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
# of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
# of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
# of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
# of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

\# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
\# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that

\# of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
\# of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
# of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
# of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\tilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

\# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
\# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
\# of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
\# of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
# of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
# of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

$\#$ of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
$\#$ of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
$\#$ of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
$\#$ of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

$\#$ of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
$\#$ of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\mathrm{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
$\#$ of maximal cones in $\mathrm{Trop}^+ Gr_{2,n}$ is C_{n-2}.
$\#$ of maximal cones in $\mathrm{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\mathrm{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\mathrm{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

$\#$ of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
$\#$ of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
$\#$ of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
$\#$ of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

$\#$ of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
$\#$ of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
$\#$ of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
$\#$ of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$.
This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Data on good triangulations of the amplituhedron

\# of good triangulations of $\mathcal{A}_{n,1,2}$ is $C_{n-2} = \frac{1}{n-1}\binom{2n-2}{n-2}$.
\# of good triangulations of $\mathcal{A}_{n,2,2}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Theorem (Speyer – W. 2005)

The **positive tropical Grassmannian** $\text{Trop}^+ Gr_{k,n}$ is a polyhedral fan, such that
\# of maximal cones in $\text{Trop}^+ Gr_{2,n}$ is C_{n-2}.
\# of maximal cones in $\text{Trop}^+ Gr_{3,n}$ is $1, 5, 48, 693$ for $n = 4, 5, 6, 7$.

Why is $\mathcal{A}_{n,k,2}$ connected to $\text{Trop}^+ Gr_{k+1,n}$? This weird coincidence was starting point of joint work with Lukowski and Parisi.

Moreover: if we generalize the notion of *good triangulation* to *good dissection* (no longer require injectivity of $\widetilde{Z}$) and look at poset of good dissections of $\mathcal{A}_{7,2,2}$, we get f-vector $(693, 2163, 2583, 1463, 392, 42, 1)$. This is precisely the f-vector of $\text{Trop}^+ Gr_{3,7}$!

Trop$^+$ $Gr_{k,n}$ and physics

Before defining Trop$^+$ $Gr_{k,n}$, we note its recent appearances in physics, in the context of singularities of loop amplitudes in $\mathcal{N} = 4$ SYM and computing scattering amplitudes in (generalized) biadjoint scalar theories:

- Cachazo–Early–Guevara–Mizera, arXiv: 1903.08904
- Cachazo–Rojas, arXiv: 1906.05979
- Drummond–Foster–Gurdogan–Kalousios, arXiv:1907.01053
- Drummond–Foster–Gurdogan–Kalousios, arXiv:1912.08217
- Arkani-Hamed–Lam–Spradlin: arXiv:1912.08222
- Henke–Papathanasiou, arXiv:1912.08254
- Arkani-Hamed–He–Lam–Thomas: arXiv:1912.11764
- Early: arXiv:1912.13513
- More ...

Trop$^+$ $Gr_{k,n}$ and physics

Before defining Trop$^+$ $Gr_{k,n}$, we note its recent appearances in physics, in the context of singularities of loop amplitudes in $\mathcal{N} = 4$ SYM and computing scattering amplitudes in (generalized) biadjoint scalar theories:

- Cachazo–Early–Guevara–Mizera, arXiv: 1903.08904
- Cachazo–Rojas, arXiv: 1906.05979
- Drummond–Foster–Gurdogan–Kalousios, arXiv:1907.01053
- Drummond–Foster–Gurdogan–Kalousios, arXiv:1912.08217
- Arkani-Hamed–Lam–Spradlin: arXiv:1912.08222
- Henke–Papathanasiou, arXiv:1912.08254
- Arkani-Hamed–He–Lam–Thomas: arXiv:1912.11764
- Early: arXiv:1912.13513
- More ...

Trop$^+$ $Gr_{k,n}$ and physics

Before defining Trop$^+$ $Gr_{k,n}$, we note its recent appearances in physics, in the context of singularities of loop amplitudes in $\mathcal{N} = 4$ SYM and computing scattering amplitudes in (generalized) biadjoint scalar theories:

- Cachazo–Early–Guevara–Mizera, arXiv: 1903.08904
- Cachazo–Rojas, arXiv: 1906.05979
- Drummond–Foster–Gurdogan–Kalousios, arXiv:1907.01053
- Drummond–Foster–Gurdogan–Kalousios, arXiv:1912.08217
- Arkani-Hamed–Lam–Spradlin: arXiv:1912.08222
- Henke–Papathanasiou, arXiv:1912.08254
- Arkani-Hamed–He–Lam–Thomas: arXiv:1912.11764
- Early: arXiv:1912.13513
- More ...

The positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\mathrm{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\text{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\text{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\mathrm{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\mathrm{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\text{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr_{k,n}^+$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\text{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr_{k,n}^+$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\text{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\text{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian Trop$^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of Trop$^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian Trop$^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$

Speyer–W, 2005: introduced and gave several descriptions of $\text{Trop}^+ Gr_{k,n}$:

- image under valuation map of $Gr^+_{k,n}$ over Puisseux series
- the common refinement of fans associated to Plücker coordinates.
- dual fan to Minkowski sum of Newton polytopes of Plücker coords.

Simpler way to describe it (subset of $\mathbb{R}^{\binom{[n]}{k}}$ with fan structure):

A vector $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ is a **positive tropical Plücker vector** if for any $1 < a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$,

- $P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc}$ or
- $P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}$.

Theorem (Speyer–W. 2020, Arkani-Hamed–Lam–Spradlin 2020)

The positive tropical Grassmannian $\text{Trop}^+ Gr_{k,n}$ equals the set of **positive tropical Plücker vectors** (also called *positive Dressian*).

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}:(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}:(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}:(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}:(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}$:$(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus Trop$^+$ $Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}$:$(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, Trop$^+$ $Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of Trop$^+$ $Gr_{k+1,n}$. Why???
- How can we relate Trop$^+$ $Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}$:$(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?
- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus Trop$^+$ $Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}$:$(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, Trop$^+$ $Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of Trop$^+$ $Gr_{k+1,n}$. Why???
- How can we relate Trop$^+$ $Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}$:$(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}:(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

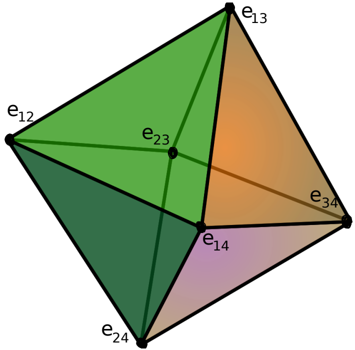
- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

Recap: $\mathcal{A}_{n,k,2}$ versus $\text{Trop}^+ Gr_{k+1,n}$

- Let $Z \in \text{Mat}^+_{n,k+2}$.
  $\widetilde{Z}$:$(Gr_{k,n})_{\geq 0} \to Gr_{k,k+2}$ sends matrix A to AZ.
  Set $\mathcal{A}_{n,k,2} := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+2}$.
- Meanwhile, $\text{Trop}^+ Gr_{k+1,n}$ is the set of **positive tropical Plücker vectors** $\{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k+1}}$.
- The good triangulations of $\mathcal{A}_{n,k,2}$ seem to be in bijection with maximal cones of $\text{Trop}^+ Gr_{k+1,n}$. Why???
- How can we relate $\text{Trop}^+ Gr_{k+1,n}$ to triangulations or subdivisions?

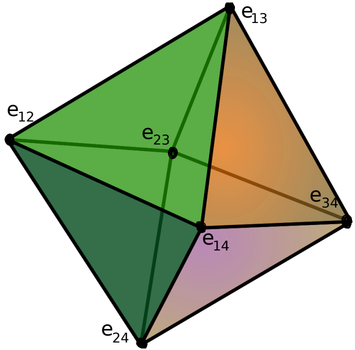
- Inspiration: Speyer '05 (see also Kapranov '93) related tropical Plücker vectors to nice subdivisions of hypersimplex $\Delta_{k,n}$.
- And $\Delta_{k,n}$ is the moment map image of the Grassmannian.
- We'll find positive analogue of this.

The hypersimplex $\Delta_{k,n}$



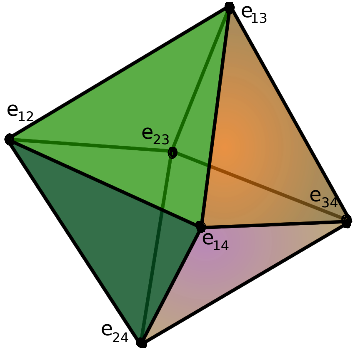
- Let $\{e_1, \ldots, e_n\}$ be standard basis of $\mathbb{R}^n$, and let $e_I := \sum_{i \in I} e_i$.
- The **hypersimplex** $\Delta_{k,n}$ is the convex hull $\text{Conv}\{e_I \;:\; |I| = k\}$.
- Equiv: it's the intersection of unit cube with hyperplane $\sum_i x_i = k$.
- Polytope of dim $n-1$.
- Our example is $\Delta_{2,4} \subset \mathbb{R}^4$.

The hypersimplex $\Delta_{k,n}$



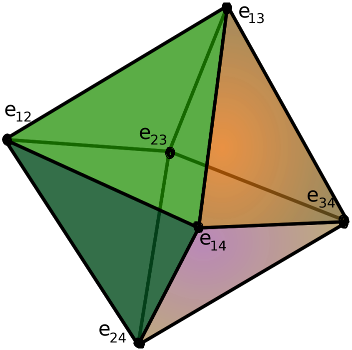
- Let $\{e_1, \ldots, e_n\}$ be standard basis of $\mathbb{R}^n$, and let $e_I := \sum_{i \in I} e_i$.
- The **hypersimplex** $\Delta_{k,n}$ is the convex hull $\text{Conv}\{e_I \ : \ |I| = k\}$.
- Equiv: it's the intersection of unit cube with hyperplane $\sum_i x_i = k$.
- Polytope of dim $n - 1$.
- Our example is $\Delta_{2,4} \subset \mathbb{R}^4$.

The hypersimplex $\Delta_{k,n}$



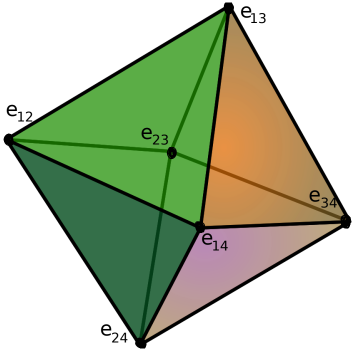
- Let $\{e_1, \ldots, e_n\}$ be standard basis of $\mathbb{R}^n$, and let $e_I := \sum_{i \in I} e_i$.
- The **hypersimplex** $\Delta_{k,n}$ is the convex hull $\text{Conv}\{e_I \ : \ |I| = k\}$.
- Equiv: it's the intersection of unit cube with hyperplane $\sum_i x_i = k$.
- Polytope of dim $n-1$.
- Our example is $\Delta_{2,4} \subset \mathbb{R}^4$.

The hypersimplex $\Delta_{k,n}$



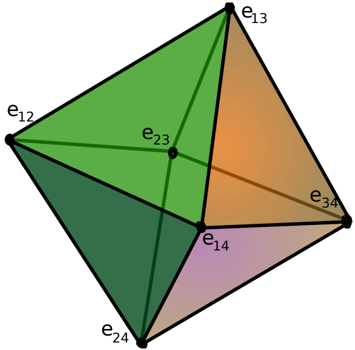
- Let $\{e_1, \ldots, e_n\}$ be standard basis of $\mathbb{R}^n$, and let $e_I := \sum_{i \in I} e_i$.
- The **hypersimplex** $\Delta_{k,n}$ is the convex hull $\text{Conv}\{e_I \ : \ |I| = k\}$.
- Equiv: it's the intersection of unit cube with hyperplane $\sum_i x_i = k$.
- Polytope of dim $n-1$.
- Our example is $\Delta_{2,4} \subset \mathbb{R}^4$.

The hypersimplex $\Delta_{k,n}$



- Let $\{e_1, \ldots, e_n\}$ be standard basis of $\mathbb{R}^n$, and let $e_I := \sum_{i \in I} e_i$.
- The **hypersimplex** $\Delta_{k,n}$ is the convex hull $\text{Conv}\{e_I \ : \ |I| = k\}$.
- Equiv: it's the intersection of unit cube with hyperplane $\sum_i x_i = k$.
- Polytope of dim $n-1$.
- Our example is $\Delta_{2,4} \subset \mathbb{R}^4$.

The hypersimplex $\Delta_{k,n}$



- Let $\{e_1, \ldots, e_n\}$ be standard basis of $\mathbb{R}^n$, and let $e_I := \sum_{i \in I} e_i$.
- The **hypersimplex** $\Delta_{k,n}$ is the convex hull $\mathrm{Conv}\{e_I \ : \ |I| = k\}$.
- Equiv: it's the intersection of unit cube with hyperplane $\sum_i x_i = k$.
- Polytope of dim $n - 1$.
- Our example is $\Delta_{2,4} \subset \mathbb{R}^4$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \dots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \text{Conv}\{e_I \; : \; |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi^{(1)}}, \dots, \Gamma_{\pi^{(\ell)}}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \ldots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \text{Conv}\{e_I \;:\; |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi(1)}, \ldots, S_{\pi(\ell)}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi(1)}, \ldots, \Gamma_{\pi(\ell)}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \ldots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \text{Conv}\{e_I \ : \ |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi^{(1)}}, \ldots, S_{\pi^{(\ell)}}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi^{(1)}}, \ldots, \Gamma_{\pi^{(\ell)}}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \dots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \mathrm{Conv}\{e_I \;:\; |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi^{(1)}}, \dots, \Gamma_{\pi^{(\ell)}}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \dots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \text{Conv}\{e_I \ : \ |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi(1)}, \dots, S_{\pi(\ell)}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi(1)}, \dots, \Gamma_{\pi(\ell)}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \ldots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \text{Conv}\{e_I \ : \ |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi(1)}, \ldots, S_{\pi(\ell)}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi(1)}, \ldots, \Gamma_{\pi(\ell)}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \ldots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \mathrm{Conv}\{e_I \ : \ |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi(1)}, \ldots, S_{\pi(\ell)}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi(1)}, \ldots, \Gamma_{\pi(\ell)}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \dots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \text{Conv}\{e_I \ : \ |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi^{(1)}}, \dots, \Gamma_{\pi^{(\ell)}}\}$ are disjoint and cover $\Delta_{k,n}$.

The moment map and triangulations of the hypersimplex

Recall $\{e_1, \dots, e_n\}$ is basis of $\mathbb{R}^n$, and $e_I := \sum_{i \in I} e_i$.
The *hypersimplex* $\Delta_{k,n} := \text{Conv}\{e_I \;:\; |I| = k\}$. Has dim $n-1$.

The **moment map** $\mu : Gr_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(A)|^2} \subset \mathbb{R}^n.$$

The images $\mu(Gr_{k,n}) = \mu((Gr_{k,n})_{\geq 0})$ are exactly $\Delta_{k,n}$.
Images of positroid cells S_π called **positroid polytopes** $\Gamma_\pi \subset \Delta_{k,n}$.

Define a (positroid) **triangulation** of $\Delta_{k,n}$ to be a collection $\{S_{\pi^{(1)}}, \dots, S_{\pi^{(\ell)}}\}$ of $(n-1)$-dim'l cells of $(Gr_{k,n})_{\geq 0}$ where μ is injective, such that their images $\{\Gamma_{\pi^{(1)}}, \dots, \Gamma_{\pi^{(\ell)}}\}$ are disjoint and cover $\Delta_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\text{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \text{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n}$ $\leftrightarrow$ the maximal cones of $\text{Trop}^+ Gr_{k,n}$.

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\text{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \text{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n}$ $\leftrightarrow$ the maximal cones of $\text{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n} \leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n} \leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\text{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \text{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n} \leftrightarrow$ the maximal cones of $\text{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\text{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \text{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n}$ $\leftrightarrow$ the maximal cones of $\text{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n} \leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n}$ $\leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\text{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \text{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n}$ $\leftrightarrow$ the maximal cones of $\text{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93).

Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n} \leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n}$ $\leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

Regular positroidal subdivisions of the hypersimplex

To construct regular subdivision of $\Delta_{k,n}$, choose some $P := \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$, thought of as *height function* on the vertices e_I of $\Delta_{k,n}$. Projecting "lower faces" of $\mathrm{Conv}\{(e_I, P_I)\}$ to $\Delta_{k,n}$ gives regular subdivision $\mathcal{D}_P$.

Theorem (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin 2020)

Let $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$. The following are equivalent.

- $P \in \mathrm{Trop}^+ Gr_{k,n}$, i.e. P is a positive tropical Plücker vector.
- Every face of $\mathcal{D}_P$ is a positroid polytope.

Rk: Theorem is pos. analogue of result of Speyer '05 (see also Kapranov '93). Theorem was anticipated/known by others including Speyer, Early, Rincon, Olarte.

Corollary

Regular positroid triangulations of $\Delta_{k,n}$ $\leftrightarrow$ the maximal cones of $\mathrm{Trop}^+ Gr_{k,n}$.

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \text{Mat}^{+}_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\text{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\text{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \mathrm{Mat}^{+}_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\mathrm{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\mathrm{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \text{Mat}^{+}_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\text{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\text{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \mathrm{Mat}^{+}_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\mathrm{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\mathrm{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \text{Mat}^+_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \mathrm{Mat}^{+}_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\mathrm{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\mathrm{Trop}^{+} Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \mathrm{Mat}^+_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\mathrm{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.

Thm: max'l cones of $\mathrm{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.

What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\tilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \text{Mat}^+_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\tilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Recap: two notions of positroid triangulation

- Have amplituhedron map $\widetilde{Z} : (Gr_{k,n})_{\geq 0} \to \mathcal{A}_{n,k,2}$, associated to matrix $Z \in \mathrm{Mat}^+_{n,k+2}$, sending matrix $A \mapsto AZ$.
- Have moment map $\mu : (Gr_{k+1,n})_{\geq 0} \to \Delta_{k+1,n}$, defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}, \text{ with } e_I := \sum_{i \in I} e_i.$$

(Positroid) triangulations for $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$

- A collection of $2k$-dim'l positroid cells of $(Gr_{k,n})_{\geq 0}$ where $\widetilde{Z}$ is injective, such that images are disjoint and cover $\mathcal{A}_{n,k,2}$.
- A collection of $(n-1)$-dim'l positroid cells of $(Gr_{k+1,n})_{\geq 0}$ where μ is injective, such that images are disjoint and cover $\Delta_{k+1,n}$.

For small n: max'l cones of $\mathrm{Trop}^+ \, Gr_{k+1,n} \leftrightarrow$ good triangs of $\mathcal{A}_{n,k,2}$.
Thm: max'l cones of $\mathrm{Trop}^+ \, Gr_{k+1,n} \leftrightarrow$ regular positroid triangs of $\Delta_{k+1,n}$.
What's going on??!!!

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations . . .

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations ...

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$,
while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations ...

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations . . .

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$,
while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations ...

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations ...

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations . . .

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$,
while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations ...

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$,
while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations ...

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

Numerology for two types of positroid triangulations

Small data: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ good triangulations of $\mathcal{A}_{n,k,2}$.
Thm: max cones of $\text{Trop}^+ Gr_{k+1,n} \leftrightarrow$ regular pos. triangs $\mathcal{D}_P$ of $\Delta_{k+1,n}$.

How could triangulations of $\mathcal{A}_{n,k,2}$ and $\Delta_{k+1,n}$ be related?

Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$,
while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$!

Weird! Nevertheless, compare # of cells comprising the triangulations ...

Karp–W.–Zhang conj: there are $\binom{n-2}{k}$ cells in any triangulation of $\mathcal{A}_{n,k,2}$.
Speyer's 2005 *f-vector conj* implies: there are $\leq \binom{n-2}{k}$ cells in any triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$

Theorem (Speyer–W. 2020)

Every positroidal triangulation $\mathcal{D}_P$ of $\Delta_{k+1,n}$ uses precisely $\binom{n-2}{k}$ cells.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.

Cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ dec perms π on $[n]$ with k antiexcedances,

where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.

One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
Cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.

Cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop. One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
Cells S_π of $(Gr_{k,n})_{\geq 0}$ $\leftrightarrow$ dec perms π on $[n]$ with k antiexcedances,
where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
Cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
Cells S_π of $(Gr_{k,n})_{\geq 0}$ $\leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
Cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Combinatorics of cells of $(Gr_{k,n})_{\geq 0}$ (Postnikov)

A **decorated permutation** is a permutation in which each fixed point is designated either **loop** or **coloop**.
Cells S_π of $(Gr_{k,n})_{\geq 0} \leftrightarrow$ dec perms π on $[n]$ with k antiexcedances, where **antiexcedance** is position i where $\pi(i) < i$ or $\pi(i) = i$ is coloop.
One can read off description of cell S_π from π.

- Triangulations of $\mathcal{A}_{n,k,2}$ consist of $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$, while triangs of $\Delta_{k+1,n}$ consist of $(n-1)$-dim'l cells of $(Gr_{k+1,n})_{\geq 0}$.
- So we need to map $(n-1)$-dimensional cells of $(Gr_{k+1,n})_{\geq 0}$ to $2k$-dimensional cells of $(Gr_{k,n})_{\geq 0}$.

T-duality map on positroid cells

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define

$$\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1}),$$

where any fixed points declared to be loops. Call it **T-duality map.**[a]

[a] The generalization of this map to $m = 4$ is what physicists have already observed as a duality between the formulations of scattering amplitudes for $\mathcal{N} = 4$ SYM in momentum space and in momentum twistor space.

Lemma (Lukowski–Parisi–W.)

The T-duality map gives a bijection

$$\text{loopless cells of } (Gr_{k+1,n})_{\geq 0} \leftrightarrow \text{ coloopless cells of } (Gr_{k,n})_{\geq 0}.$$

Moreover, $\dim(S_{\hat{\pi}}) = \dim(S_\pi) + 2k - (n-1)$.
So it maps cells of dim $n-1$ to cells of dimension $2k$.

T-duality map on positroid cells

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define

$$\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1}),$$

where any fixed points declared to be loops. Call it **T-duality map.**[a]

[a] The generalization of this map to $m = 4$ is what physicists have already observed as a duality between the formulations of scattering amplitudes for $\mathcal{N} = 4$ SYM in momentum space and in momentum twistor space.

Lemma (Lukowski–Parisi–W.)

The T-duality map gives a bijection

$$\text{loopless cells of } (Gr_{k+1,n})_{\geq 0} \leftrightarrow \text{ coloopless cells of } (Gr_{k,n})_{\geq 0}.$$

Moreover, $\dim(S_{\hat{\pi}}) = \dim(S_\pi) + 2k - (n-1)$.
So it maps cells of dim $n-1$ to cells of dimension $2k$.

T-duality map on positroid cells

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define

$$\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1}),$$

where any fixed points declared to be loops. Call it **T-duality map.**[a]

[a] The generalization of this map to $m = 4$ is what physicists have already observed as a duality between the formulations of scattering amplitudes for $\mathcal{N} = 4$ SYM in momentum space and in momentum twistor space.

Lemma (Lukowski–Parisi–W.)

The T-duality map gives a bijection

$$\text{loopless cells of } (Gr_{k+1,n})_{\geq 0} \leftrightarrow \text{ coloopless cells of } (Gr_{k,n})_{\geq 0}.$$

Moreover, $\dim(S_{\hat{\pi}}) = \dim(S_\pi) + 2k - (n-1)$.
So it maps cells of dim $n-1$ to cells of dimension $2k$.

T-duality map on positroid cells

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define

$$\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1}),$$

where any fixed points declared to be loops. Call it **T-duality map.**[a]

[a] The generalization of this map to $m = 4$ is what physicists have already observed as a duality between the formulations of scattering amplitudes for $\mathcal{N} = 4$ SYM in momentum space and in momentum twistor space.

Lemma (Lukowski–Parisi–W.)

The T-duality map gives a bijection

$$\text{loopless cells of } (Gr_{k+1,n})_{\geq 0} \leftrightarrow \text{ coloopless cells of } (Gr_{k,n})_{\geq 0}.$$

Moreover, $\dim(S_{\hat{\pi}}) = \dim(S_\pi) + 2k - (n-1)$.
So it maps cells of dim $n-1$ to cells of dimension $2k$.

T-duality map on positroid cells

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define

$$\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1}),$$

where any fixed points declared to be loops. Call it **T-duality map.**[a]

[a] The generalization of this map to $m = 4$ is what physicists have already observed as a duality between the formulations of scattering amplitudes for $\mathcal{N} = 4$ SYM in momentum space and in momentum twistor space.

Lemma (Lukowski–Parisi–W.)

The T-duality map gives a bijection

$$\text{loopless cells of } (Gr_{k+1,n})_{\geq 0} \leftrightarrow \text{ coloopless cells of } (Gr_{k,n})_{\geq 0}.$$

Moreover, $\dim(S_{\hat{\pi}}) = \dim(S_\pi) + 2k - (n-1)$.
So it maps cells of dim $n-1$ to cells of dimension $2k$.

T-duality map on positroid cells

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define

$$\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1}),$$

where any fixed points declared to be loops. Call it **T-duality map.**[a]

[a] The generalization of this map to $m = 4$ is what physicists have already observed as a duality between the formulations of scattering amplitudes for $\mathcal{N} = 4$ SYM in momentum space and in momentum twistor space.

Lemma (Lukowski–Parisi–W.)

The T-duality map gives a bijection

$$\text{loopless cells of } (Gr_{k+1,n})_{\geq 0} \leftrightarrow \text{ coloopless cells of } (Gr_{k,n})_{\geq 0}.$$

Moreover, $\dim(S_{\hat{\pi}}) = \dim(S_\pi) + 2k - (n-1)$.
So it maps cells of dim $n - 1$ to cells of dimension $2k$.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\text{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \dots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \dots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\mathrm{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\text{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\text{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\text{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\text{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\text{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from Trop$^+$ $Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Conjecture and results

Given loopless decorated permutation $\pi = (a_1, \ldots, a_n)$ on $[n]$, define $\hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$, where any fixed points declared to be loops.

Conjecture (Lukowski–Parisi–W.)

A collection $\{S_\pi\}$ of cells of $Gr^+_{k+1,n}$ gives a triangulation of $\Delta_{k+1,n}$ if and only if the collection $\{S_{\hat{\pi}}\}$ of cells of $Gr^+_{k,n}$ gives a triangulation of $\mathcal{A}_{n,k,2}$.

Regular triangulations of $\mathcal{A}_{n,k,2}$ (Lukowski–Parisi–W.)

- We define a triangulation of $\mathcal{A}_{n,k,2}$ to be **regular** if it comes from $\text{Trop}^+ Gr_{k+1,n}$ – i.e. it is the T-duality image of a regular positroid triangulation of $\Delta_{k+1,n}$.
- The regular triangulations of $\mathcal{A}_{n,k,2}$ behave well at the boundary, i.e. they are good.

Results (Lukowski–Parisi–W.)

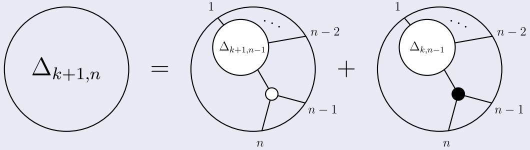
- There is recursion giving many triangulations of $\Delta_{k+1,n}$, and a recursion giving many triangulations of $\mathcal{A}_{n,k,2}$ (Bao-He), and they are in bijection via T-duality.

- We introduce a **momentum amplituhedron** $\mathcal{M}_{n,k,m}$[a] which should give analogous story to what I've explained today, but for any even m.

[a] a generalization of $\mathcal{M}_{n,k,4}$ defined by Damgaard-Ferro-Lukowski-Parisi

Results (Lukowski–Parisi–W.)

- There is recursion giving many triangulations of $\Delta_{k+1,n}$, and a recursion giving many triangulations of $\mathcal{A}_{n,k,2}$ (Bao-He), and they are in bijection via T-duality.

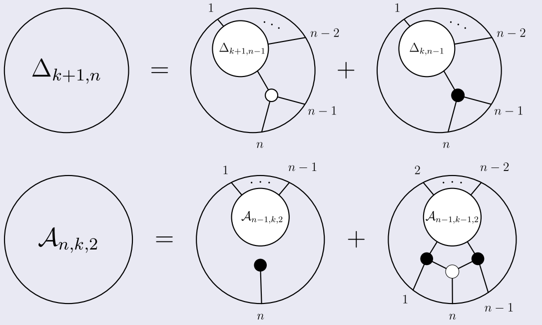


- We introduce a **momentum amplituhedron** $\mathcal{M}_{n,k,m}$[a] which should give analogous story to what I've explained today, but for any even m.

[a] a generalization of $\mathcal{M}_{n,k,4}$ defined by Damgaard-Ferro-Lukowski-Parisi

Results (Lukowski–Parisi–W.)

- There is recursion giving many triangulations of $\Delta_{k+1,n}$, and a recursion giving many triangulations of $\mathcal{A}_{n,k,2}$ (Bao-He), and they are in bijection via T-duality.

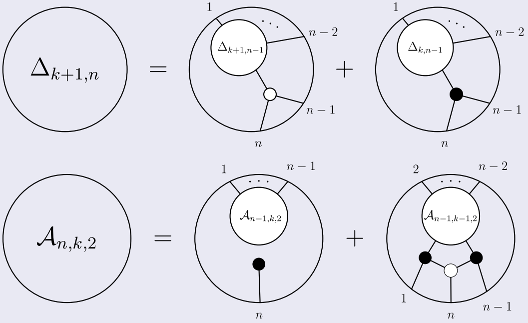


- We introduce a **momentum amplituhedron** $\mathcal{M}_{n,k,m}$[a] which should give analogous story to what I've explained today, but for any even m.

[a] a generalization of $\mathcal{M}_{n,k,4}$ defined by Damgaard-Ferro-Lukowski-Parisi

Results (Lukowski–Parisi–W.)

- There is recursion giving many triangulations of $\Delta_{k+1,n}$, and a recursion giving many triangulations of $\mathcal{A}_{n,k,2}$ (Bao-He), and they are in bijection via T-duality.

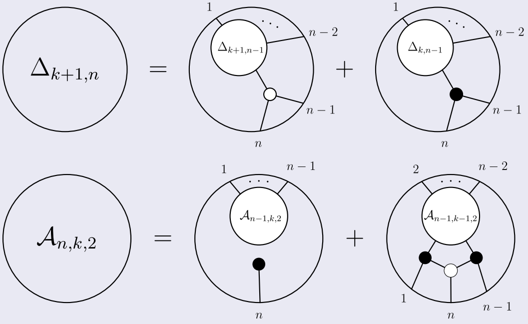


- We introduce a **momentum amplituhedron** $\mathcal{M}_{n,k,m}$[a] which should give analogous story to what I've explained today, but for any even m.

[a] a generalization of $\mathcal{M}_{n,k,4}$ defined by Damgaard-Ferro-Lukowski-Parisi

Results (Lukowski–Parisi–W.)

- There is recursion giving many triangulations of $\Delta_{k+1,n}$, and a recursion giving many triangulations of $\mathcal{A}_{n,k,2}$ (Bao-He), and they are in bijection via T-duality.



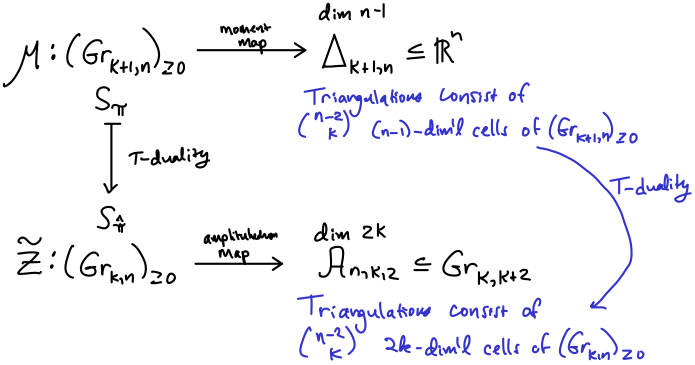
- We introduce a **momentum amplituhedron** $\mathcal{M}_{n,k,m}$[a] which should give analogous story to what I've explained today, but for any even m.

[a] a generalization of $\mathcal{M}_{n,k,4}$ defined by Damgaard-Ferro-Lukowski-Parisi

Summary and questions

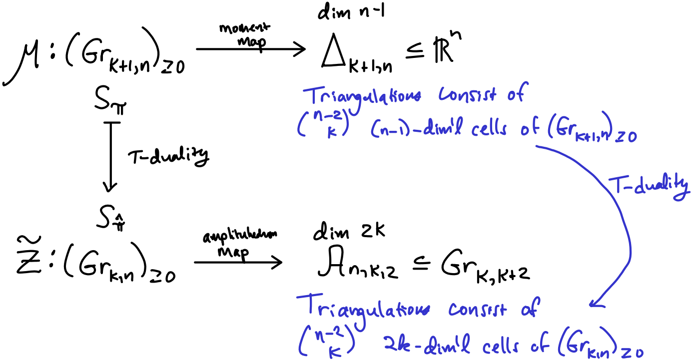
- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?
- In Speyer–W 2005 we found connection of $\text{Trop}^+ Gr_{k,n}$ and cluster algebras. Direct connection of cluster algebras to $\mathcal{A}_{n,k,2}$??

Summary and questions



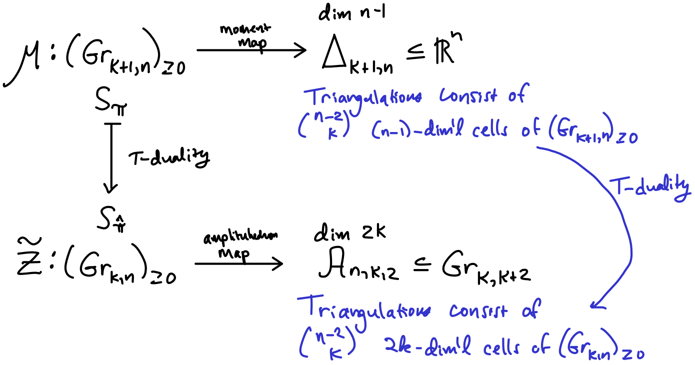
- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?
- In Speyer–W 2005 we found connection of $\text{Trop}^+ Gr_{k,n}$ and cluster algebras. Direct connection of cluster algebras to $\mathcal{A}_{n,k,2}$??

Summary and questions



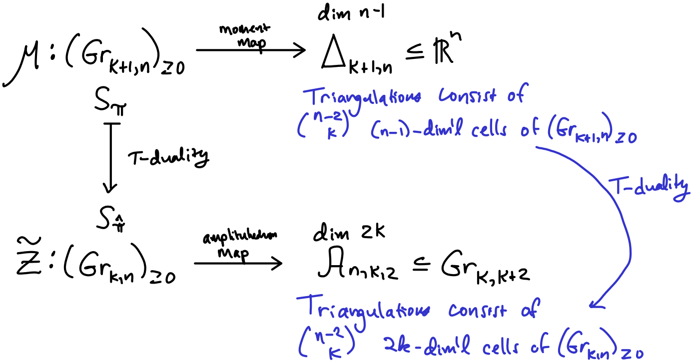
- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?
- In Speyer–W 2005 we found connection of $\mathrm{Trop}^+ Gr_{k,n}$ and cluster algebras. Direct connection of cluster algebras to $\mathcal{A}_{n,k,2}$??

Summary and questions



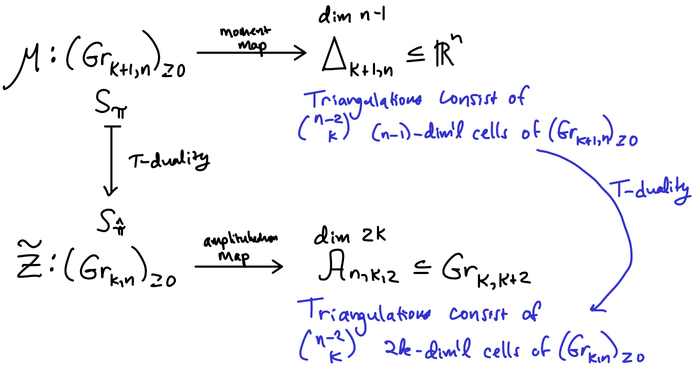
- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?
- In Speyer–W 2005 we found connection of $\text{Trop}^+ \, Gr_{k,n}$ and cluster algebras. Direct connection of cluster algebras to $\mathcal{A}_{n,k,2}$??

Summary and questions



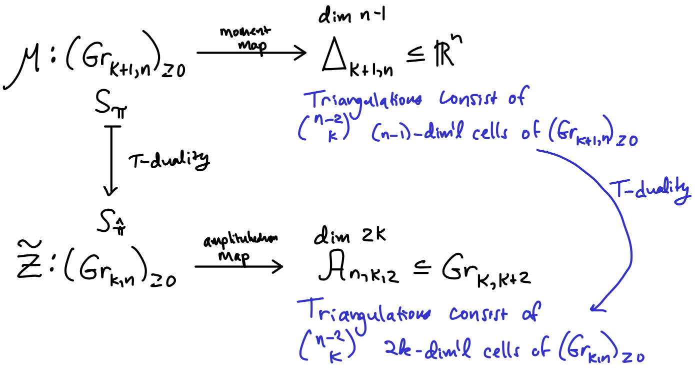
- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?
- In Speyer–W 2005 we found connection of $\mathrm{Trop}^+ Gr_{k,n}$ and cluster algebras. Direct connection of cluster algebras to $\mathcal{A}_{n,k,2}$??

Summary and questions



- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?
- In Speyer–W 2005 we found connection of $\mathrm{Trop}^+ Gr_{k,n}$ and cluster algebras. Direct connection of cluster algebras to $\mathcal{A}_{n,k,2}$??

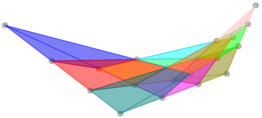
Summary and questions



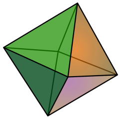
- Why is the moment map related to the amplituhedron map?
- How can we use the relation to better understand the amplituhedron?
- In Speyer–W 2005 we found connection of Trop$^+$ $Gr_{k,n}$ and cluster algebras. Direct connection of cluster algebras to $\mathcal{A}_{n,k,2}$??

Thank you for listening!

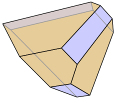
I. Amplituhedron '13



II. Hypersimplex and moment map '87



III. Positive tropical Grassmannian '05



- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Lukowski and Parisi, arXiv:2002.06164
- "The positive Dressian equals the positive tropical Grassmannian," with Speyer, arXiv:2003.10231.
- "The tropical totally positive Grassmannian," with Speyer, arXiv:math/0312297, J. Algebraic Combinatorics, Sept 2005.

How positroid cells are encoded by decorated permutations

Given a $k \times n$ matrix $C = (c_1, \dots, c_n)$ (representing a point of $(Gr_{k,n})_{\geq 0}$) written as a list of its columns, we associate a decorated permutation π as follows.

- Given $i, j \in [n]$, let $r[i,j]$ denote the rank of $\langle c_i, c_{i+1}, \dots, c_j \rangle$, where we list the columns in cyclic order, going from c_n to c_1 if $i > j$.
- We set $\pi(i) := j$ to be the label of the first column j such that $c_i \in \text{span}\{c_{i+1}, c_{i+2}, \dots, c_j\}$.
- If c_i is the all-zero vector, we call i a loop or black fixed point, and if c_i is not in the span of the other column vectors, we call i a coloop or white fixed point.

We define S_π^{tnn} to be the set of all elements $C \in (Gr_{k,n})_{\geq 0}$ which give rise to this π.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \dots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\mathrm{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\mathrm{Trop}^+(f)$ for $f \in I$.

- So we can define $\mathrm{Trop}^+ Gr_{k,n} = \cap \mathrm{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \ldots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\mathrm{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\mathrm{Trop}^+(f)$ for $f \in I$.

- So we can define $\mathrm{Trop}^+ Gr_{k,n} = \cap \mathrm{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \ldots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\mathrm{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\mathrm{Trop}^+(f)$ for $f \in I$.

- So we can define $\mathrm{Trop}^+ Gr_{k,n} = \cap \mathrm{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \ldots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\mathrm{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\mathrm{Trop}^+(f)$ for $f \in I$.

- So we can define $\mathrm{Trop}^+ Gr_{k,n} = \cap \mathrm{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \ldots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\mathrm{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\mathrm{Trop}^+(f)$ for $f \in I$.

- So we can define $\mathrm{Trop}^+ Gr_{k,n} = \cap \mathrm{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \ldots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\text{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\text{Trop}^+(f)$ for $f \in I$.

- So we can define $\text{Trop}^+ Gr_{k,n} = \cap \text{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.

What is the positive tropical Grassmannian?

Theorem (Speyer –W. 2005)

Given ideal $I \subset \mathcal{C}[x_1, \ldots, x_n]$, there are two equivalent ways of defining the positive tropical variety $\text{Trop}^+ V(I)$:

- compute the vanishing set over *positive Puisseux series* $V(I) \subset (\mathcal{C}^+)^n$ and apply a *valuation map* (and take closure);
- take the intersection of all *positive tropical hypersurfaces* $\text{Trop}^+(f)$ for $f \in I$.

- So we can define $\text{Trop}^+ Gr_{k,n} = \cap \, \text{Trop}^+(f)$, where f ranges over all elements in the Plücker ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\text{val}(f) := u$.
Valuation map $\text{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\text{val}(x_1), \dots, \text{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid \text{ coeff. of the lowest term of } x(t) \text{ is positive real}\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then Trop $V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\text{Trop}^+ V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\text{Trop}^+ Gr_{k,n} := \text{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\text{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\text{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\mathrm{val}(f) := u$.
Valuation map $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then Trop $V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\mathrm{Trop}^+ Gr_{k,n} := \mathrm{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\mathrm{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\mathrm{val}(f) := u$.
Valuation map $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then Trop $V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\mathrm{Trop}^+ Gr_{k,n} := \mathrm{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\mathrm{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\text{val}(f) := u$.
Valuation map $\text{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\text{val}(x_1), \dots, \text{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then Trop $V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\text{Trop}^+ V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\text{Trop}^+ Gr_{k,n} := \text{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\text{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\text{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\text{val}(f) := u$.
Valuation map $\text{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\text{val}(x_1), \dots, \text{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then Trop $V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\text{Trop}^+ V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\text{Trop}^+ Gr_{k,n} := \text{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\text{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\text{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\mathrm{val}(f) := u$.
Valuation map $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \ldots, x_n) \mapsto (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \ldots, x_n]$ an ideal, then $\mathrm{Trop}\, V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of $\mathrm{Trop}\, V(I)$ is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define $\mathrm{Trop}\, Gr_{k,n} := \mathrm{Trop}\, V(I)$ (Speyer–Sturmfels '04), and
$\mathrm{Trop}^+ Gr_{k,n} := \mathrm{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\mathrm{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\text{val}(f) := u$.
Valuation map $\text{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \ldots, x_n) \mapsto (\text{val}(x_1), \ldots, \text{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid \text{ coeff. of the lowest term of } x(t) \text{ is positive real}\}$.
If $I \subset \mathcal{C}[x_1, \ldots, x_n]$ an ideal, then $\text{Trop } V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\text{Trop}^+ V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define $\text{Trop } Gr_{k,n} := \text{Trop } V(I)$ (Speyer–Sturmfels '04), and
$\text{Trop}^+ Gr_{k,n} := \text{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\text{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\text{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\mathrm{val}(f) := u$.
Valuation map $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then $\mathrm{Trop}\, V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of $\mathrm{Trop}\, V(I)$ is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define $\mathrm{Trop}\, Gr_{k,n} := \mathrm{Trop}\, V(I)$ (Speyer–Sturmfels '04), and
$\mathrm{Trop}^+ Gr_{k,n} := \mathrm{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\mathrm{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\mathrm{val}(f) := u$.
Valuation map $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then Trop $V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\mathrm{Trop}^+ Gr_{k,n} := \mathrm{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\mathrm{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\text{val}(f) := u$.
Valuation map $\text{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\text{val}(x_1), \dots, \text{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then $\text{Trop}\, V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of $\text{Trop}\, V(I)$ is $\text{Trop}^+ V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define $\text{Trop}\, Gr_{k,n} := \text{Trop}\, V(I)$ (Speyer–Sturmfels '04), and
$\text{Trop}^+ Gr_{k,n} := \text{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\text{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\text{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty}\mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\text{val}(f) := u$.
Valuation map $\text{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \dots, x_n) \mapsto (\text{val}(x_1), \dots, \text{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, then $\text{Trop}\, V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of $\text{Trop}\, V(I)$ is $\text{Trop}^+ V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define $\text{Trop}\, Gr_{k,n} := \text{Trop}\, V(I)$ (Speyer–Sturmfels '04), and
$\text{Trop}^+ Gr_{k,n} := \text{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\text{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\text{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\mathrm{val}(f) := u$.
Valuation map $\mathrm{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \ldots, x_n) \mapsto (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid$ coeff. of the lowest term of $x(t)$ is positive real$\}$.
If $I \subset \mathcal{C}[x_1, \ldots, x_n]$ an ideal, then Trop $V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\mathrm{Trop}^+ Gr_{k,n} := \mathrm{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\mathrm{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ where f ranges over all elements in the ideal I.

More precise defn of the positive tropical Grassmannian

Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$.
If $f \in \mathcal{C}^*$, with lowest term at^u, define $\text{val}(f) := u$.
Valuation map $\text{val} : (\mathcal{C}^*)^n \to \mathbb{Q}^n$, $(x_1, \ldots, x_n) \mapsto (\text{val}(x_1), \ldots, \text{val}(x_n))$.
Let $\mathcal{C}^+ := \{x(t) \in \mathcal{C} \mid \text{ coeff. of the lowest term of } x(t) \text{ is positive real}\}$.
If $I \subset \mathcal{C}[x_1, \ldots, x_n]$ an ideal, then Trop $V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^*)^n)}$.
Positive part of Trop $V(I)$ is $\text{Trop}^+ V(I) := \overline{\text{val}(V(I) \cap (\mathcal{C}^+)^n)}$.

Let I be the Plücker ideal.
Define Trop $Gr_{k,n}$:= Trop $V(I)$ (Speyer–Sturmfels '04), and
$\text{Trop}^+ Gr_{k,n} := \text{Trop}^+ V(I)$ (Speyer–W. '05).

Theorem (Speyer–W. 2005)

Any positive tropical variety $\text{Trop}^+ V(I)$ equals the intersection of the positive tropical hypersurfaces $\text{Trop}^+(f)$ where f ranges over all elements in the ideal I.