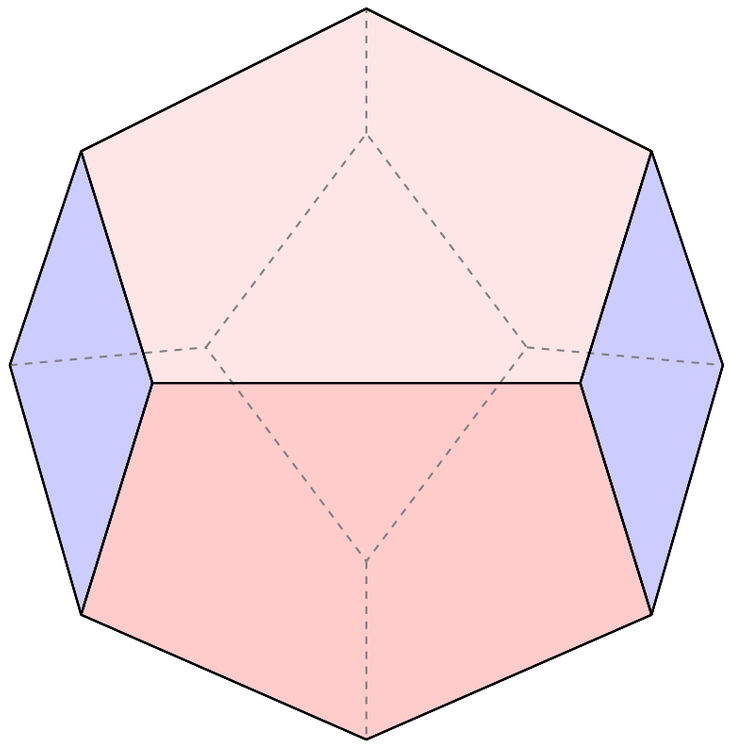
Cluster Algebras, Tropical Geometry and Scattering Amplitudes

James Drummond
University of Southampton

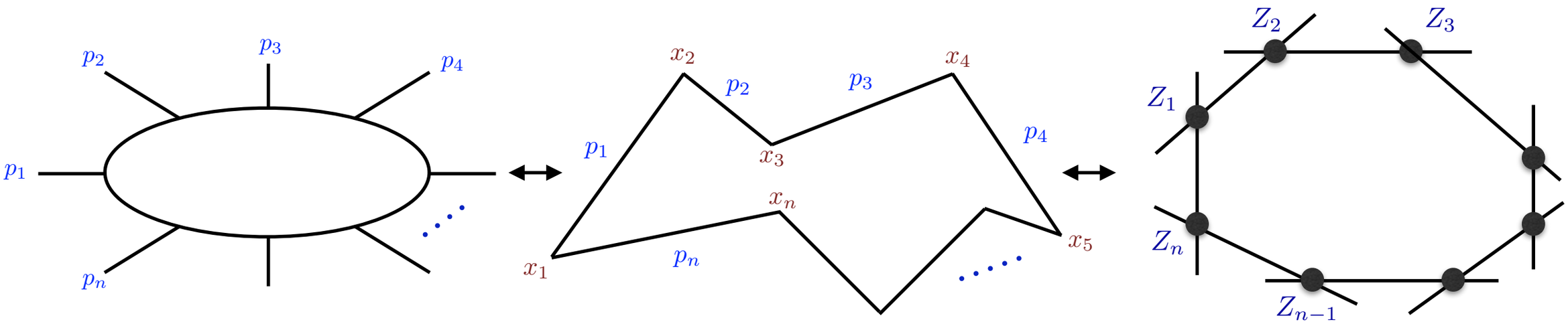


2002.04624 [JMD, Foster, Gürdoğan,Kalousios]
1912.08217 [JMD, Foster, Gürdoğan,Kalousios]
1907.01053 [JMD, Foster, Gürdoğan,Kalousios]

Also related: [Arkani-Hamed, Lam, Spradlin], [Papathanasiou, Henke]

Zoomplitudes 2020

Massless planar scattering kinematics



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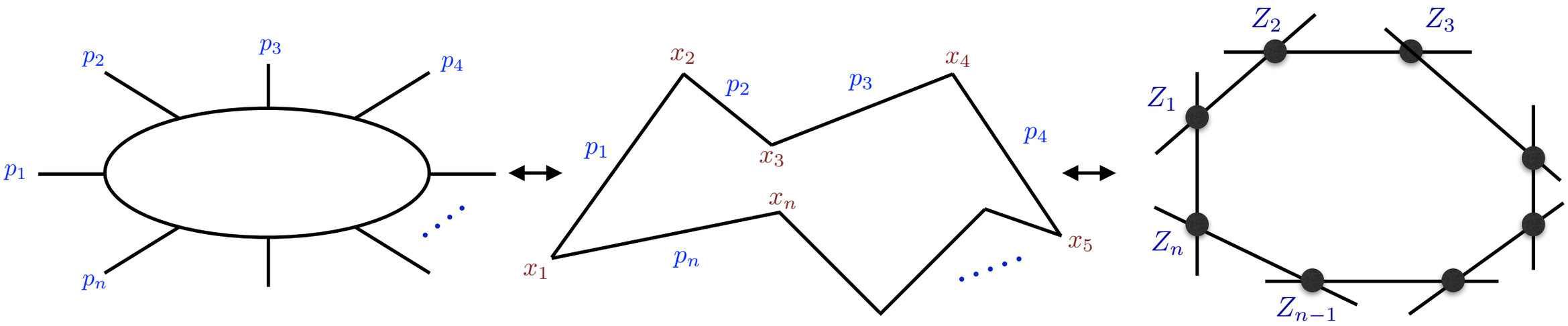

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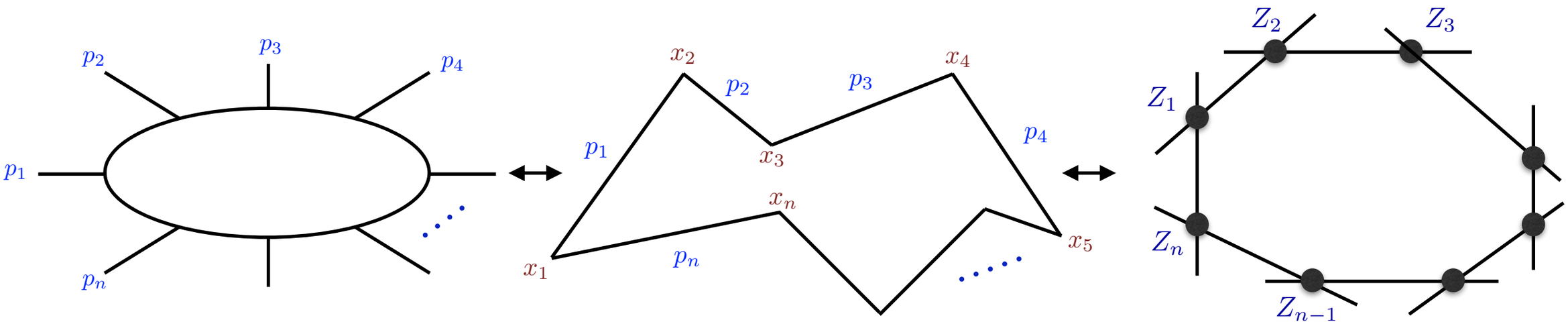

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$$p_i = x_{i+1} - x_i$$

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$$p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}$$

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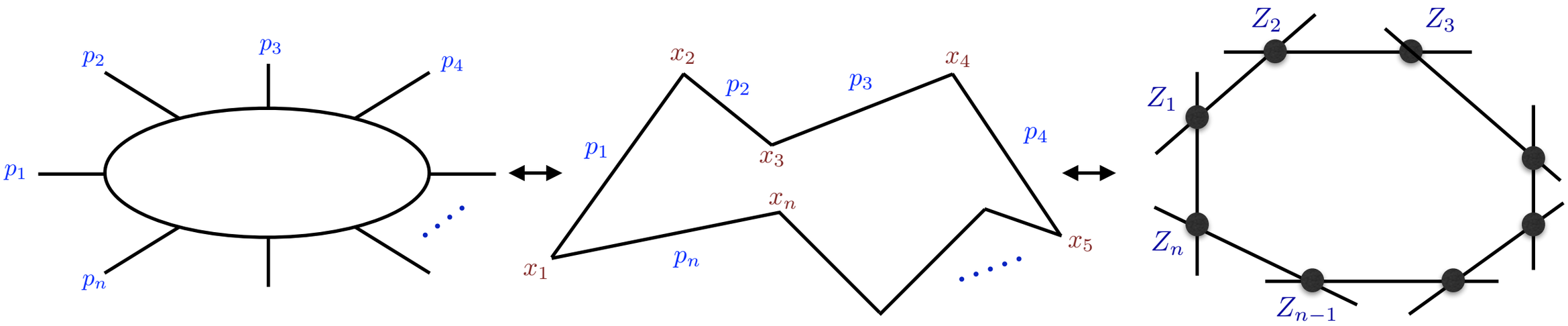

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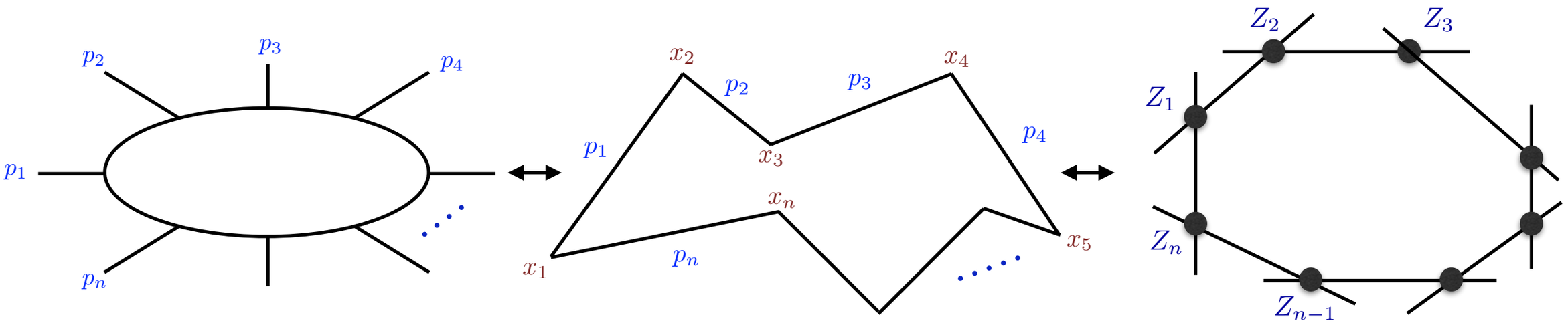

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Dual conformal symmetry: amplitude depends on $\langle Z_i Z_j Z_k Z_l \rangle = \langle ijkl \rangle$

Grassmannian Gr(4,n)

Arrange the twistors into a (4 x n) matrix: (Z_i^A) $Z_i \in \mathbb{P}^3$

Dual conformal symmetry: mod out by sl_4

Grassmannian: dimension = $(3n - 15)$

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Plücker coordinates $\langle Z_i Z_j Z_k Z_l \rangle = \langle ijkl \rangle$

obey relations: $\langle ijk[l\rangle\langle mnpq] \rangle = 0$

$$\langle ij[kl\rangle\langle mnp]q \rangle = 0$$

We can *define* the Grassmannian through these relations

$$\langle ijkl \rangle \longrightarrow p_{ijkl}$$

Tropical Grassmannian Tr(4,n) [Speyer, Sturmfels]

Plücker relations:

$$p_{ijk[l}p_{mnpq]} = 0$$

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(Plücker ideal)

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Tropicalise: multiplication $\longrightarrow$ addition

addition $\longrightarrow$ minimum

example: $xy + wz^2 + 2uv \longrightarrow \min(x + y, w + 2z, u + v)$

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Regions separated by *tropical hypersurfaces*

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Tropical space captures many features of the original

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e.g. in Gr(4,6): $p_{1234}p_{1256} - p_{1235}p_{1246} + p_{1236}p_{1245} = 0$

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[Speyer, Williams]

Positive Tropical Grassmannian $\mathrm{Tr}^+(4,n)$

Choose coordinates (cluster X-coords)

Plückers now become polynomials:

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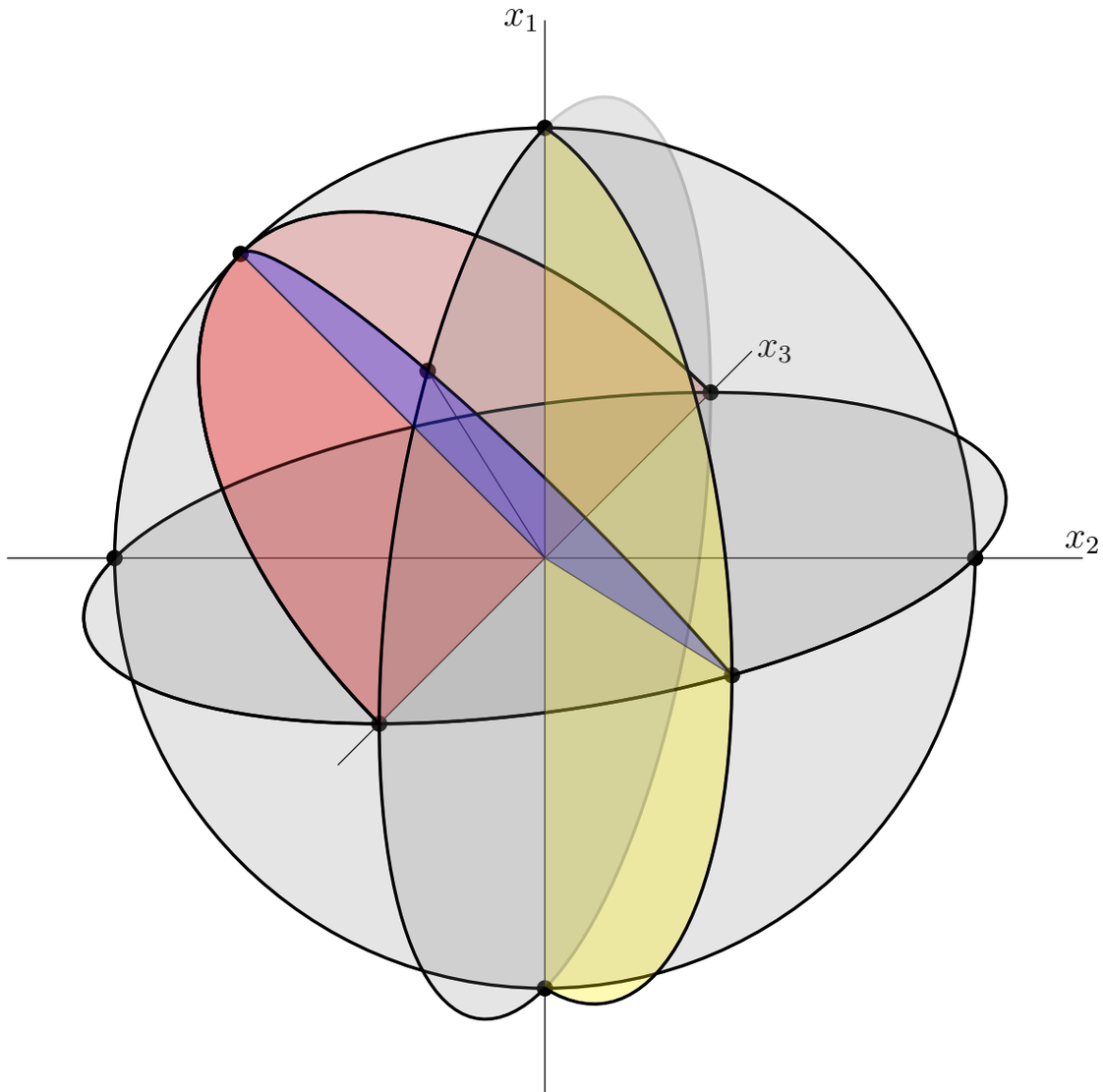

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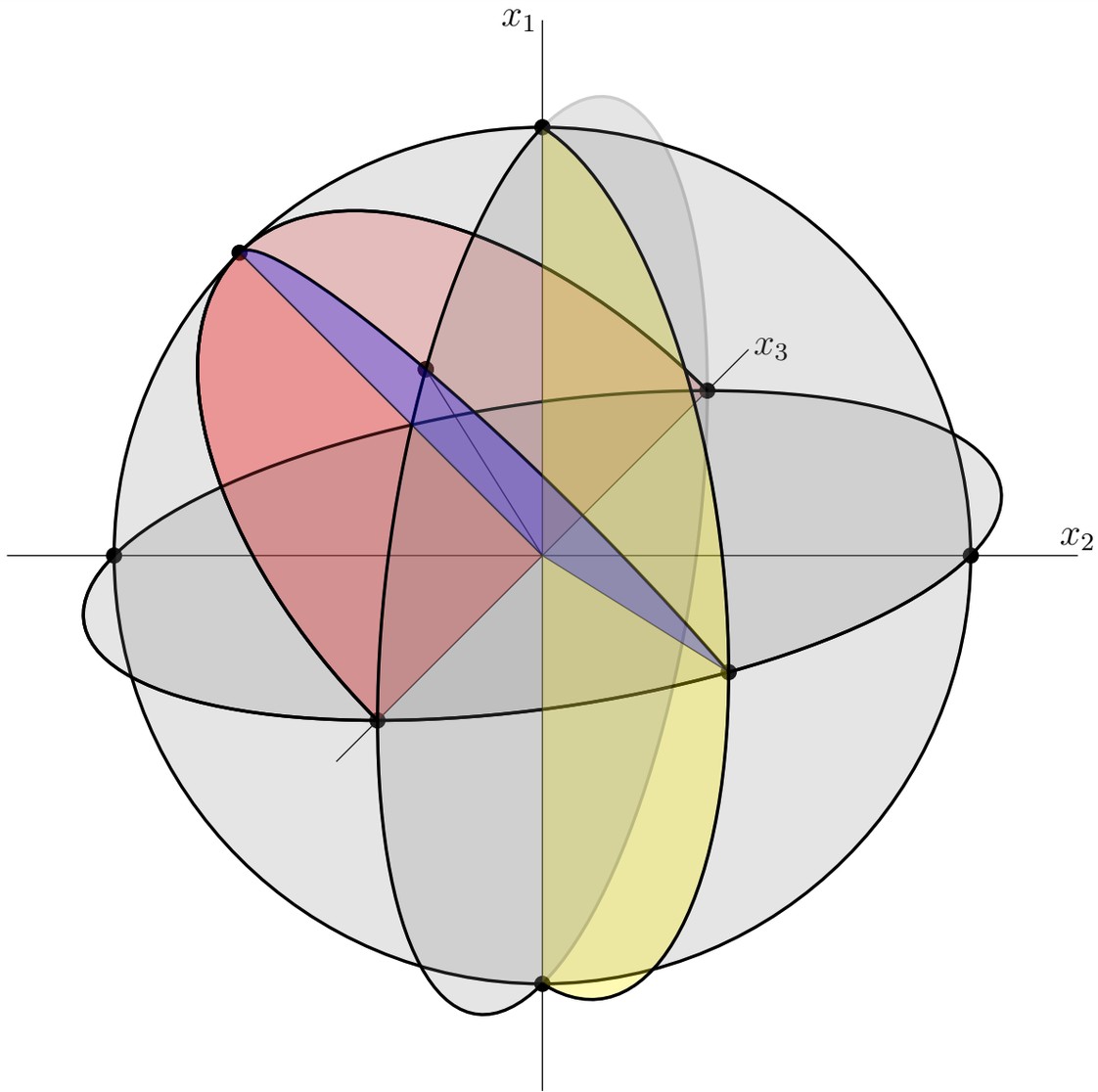

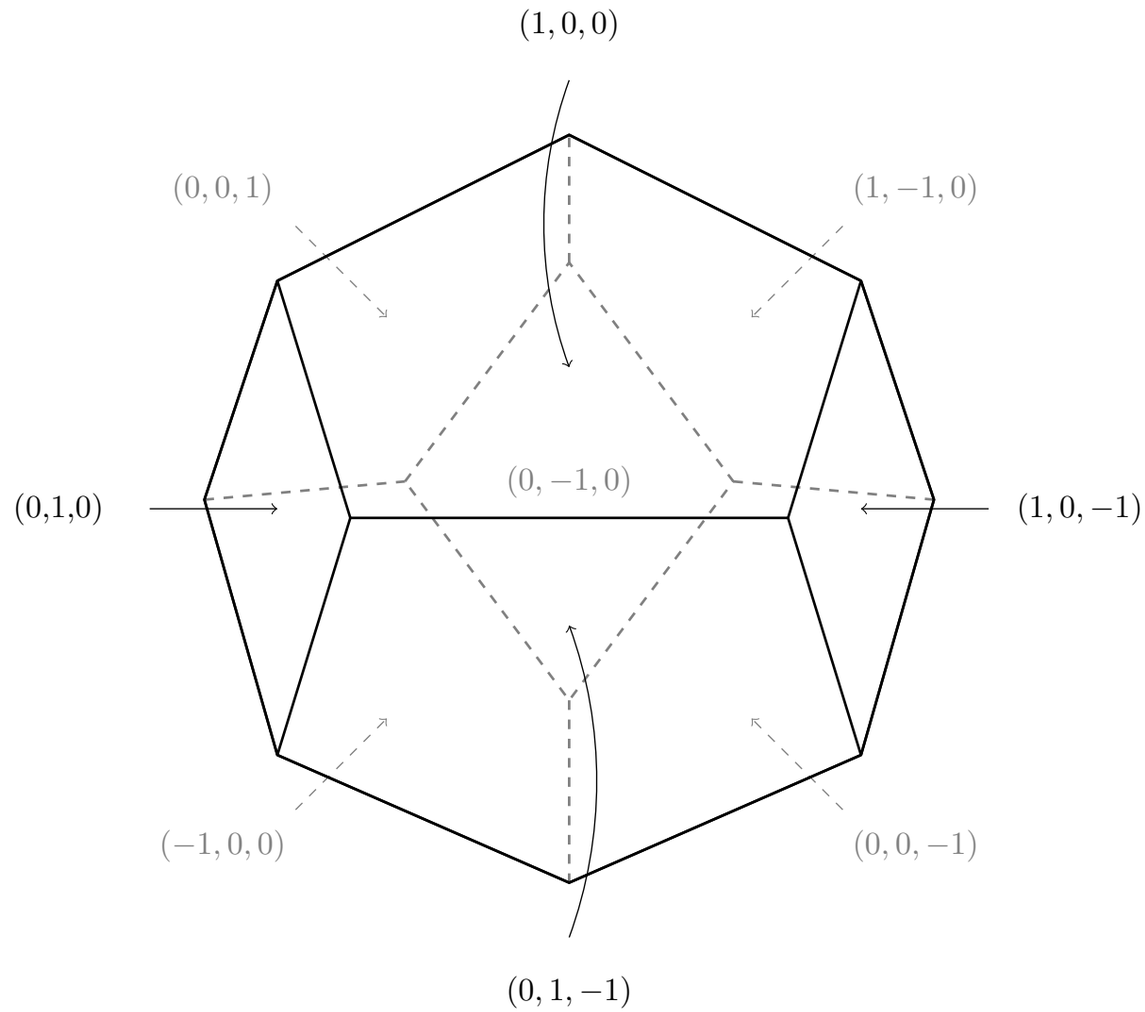

[Fomin, Zelevinsky], [Scott]

Grassmannian Cluster Algebras

Can associate a cluster algebra to the space Gr(4,n) (positive part)

Initial cluster given by specified set of 4-brackets $\langle ijkl \rangle$

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Grassmannian Cluster Algebras

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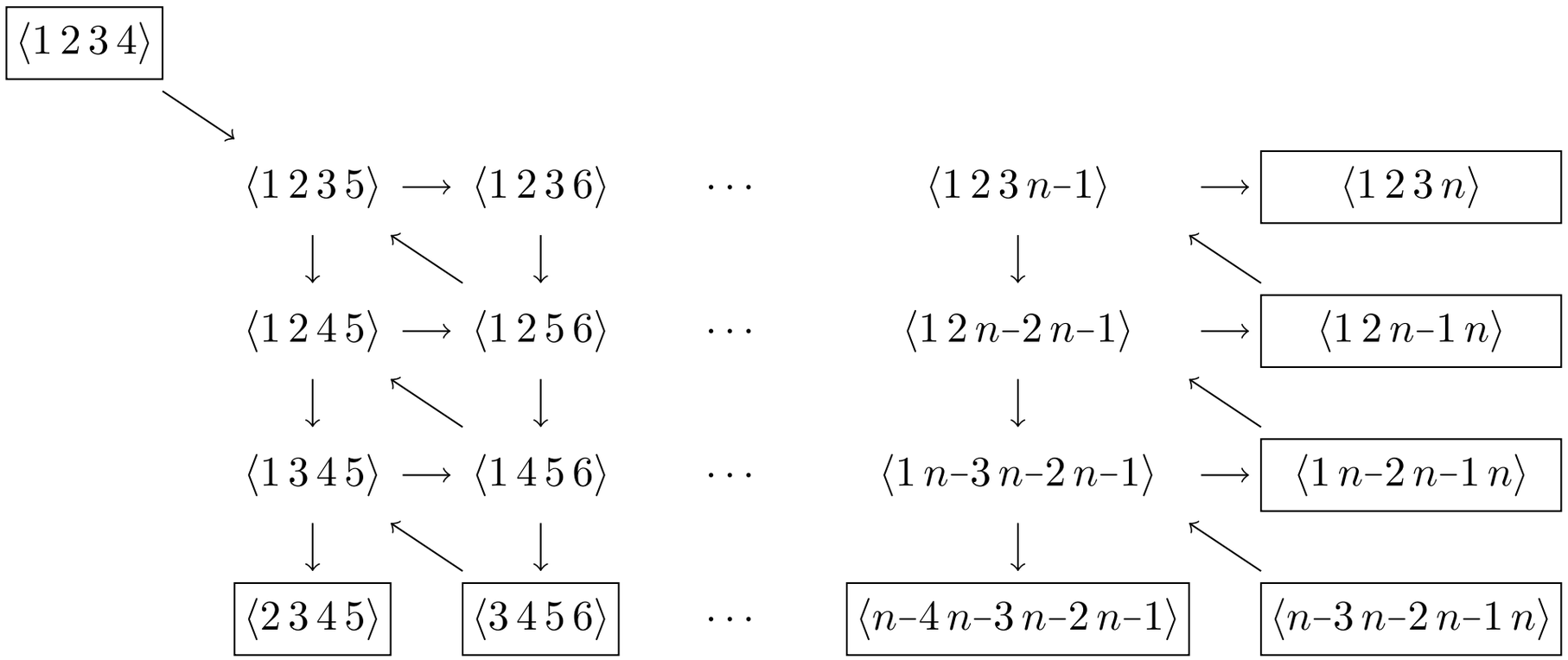

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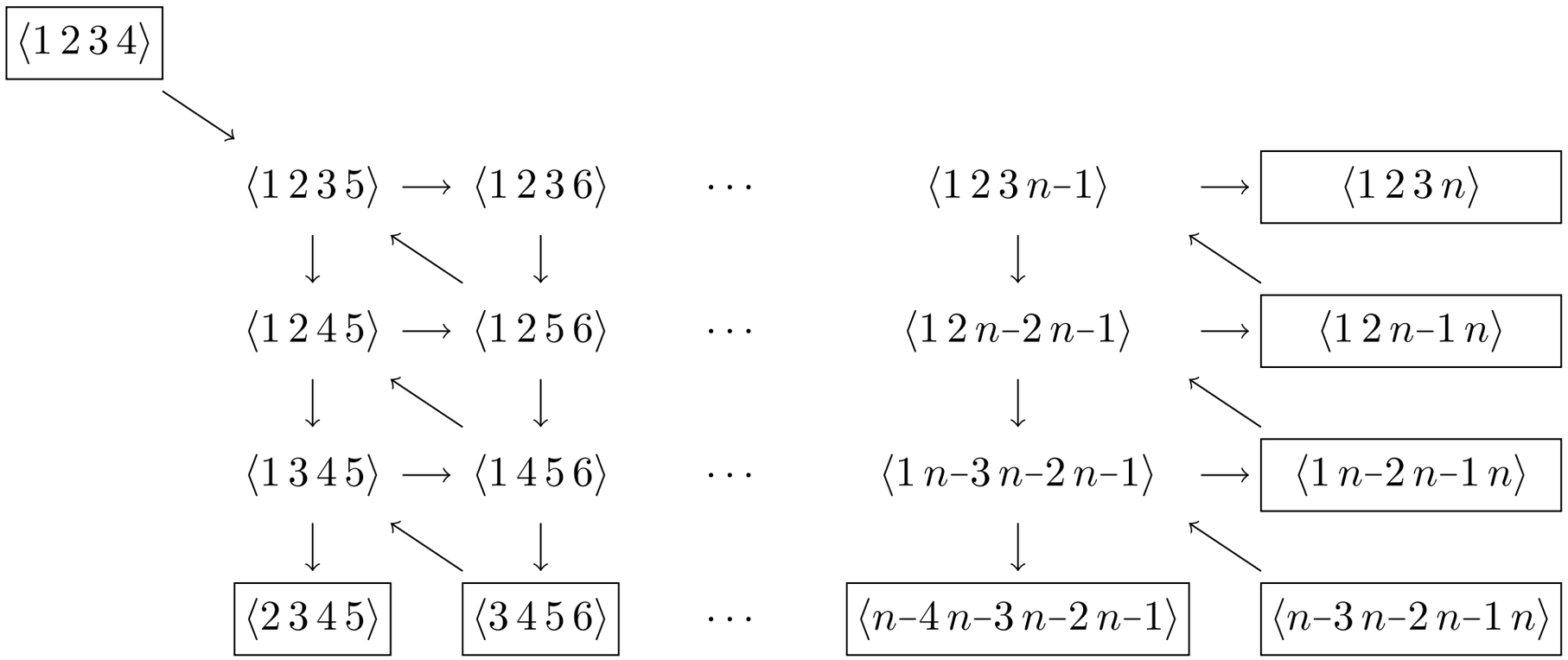

Mutation $a'_k a_k = \prod_{i \to k} a_i + \prod_{k \to i} a_i$ generates more A-coords (polynomials)

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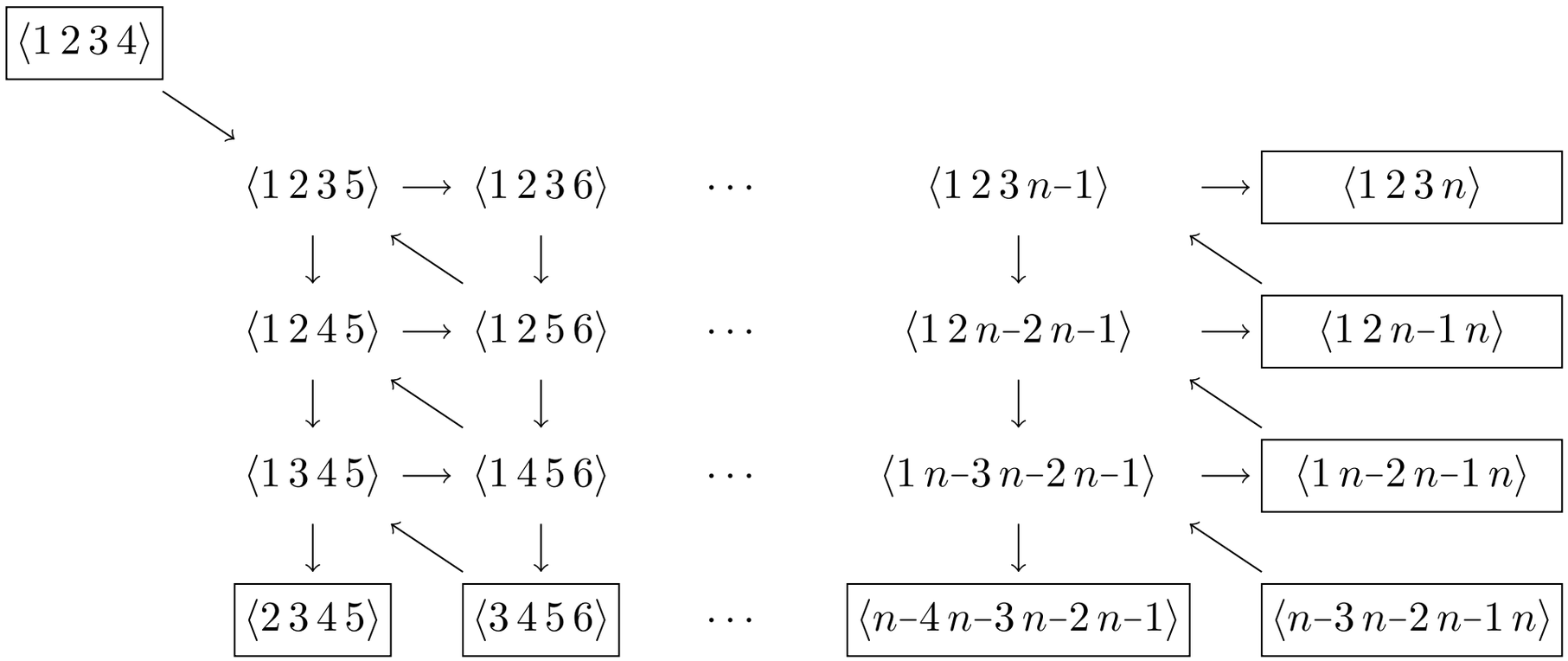

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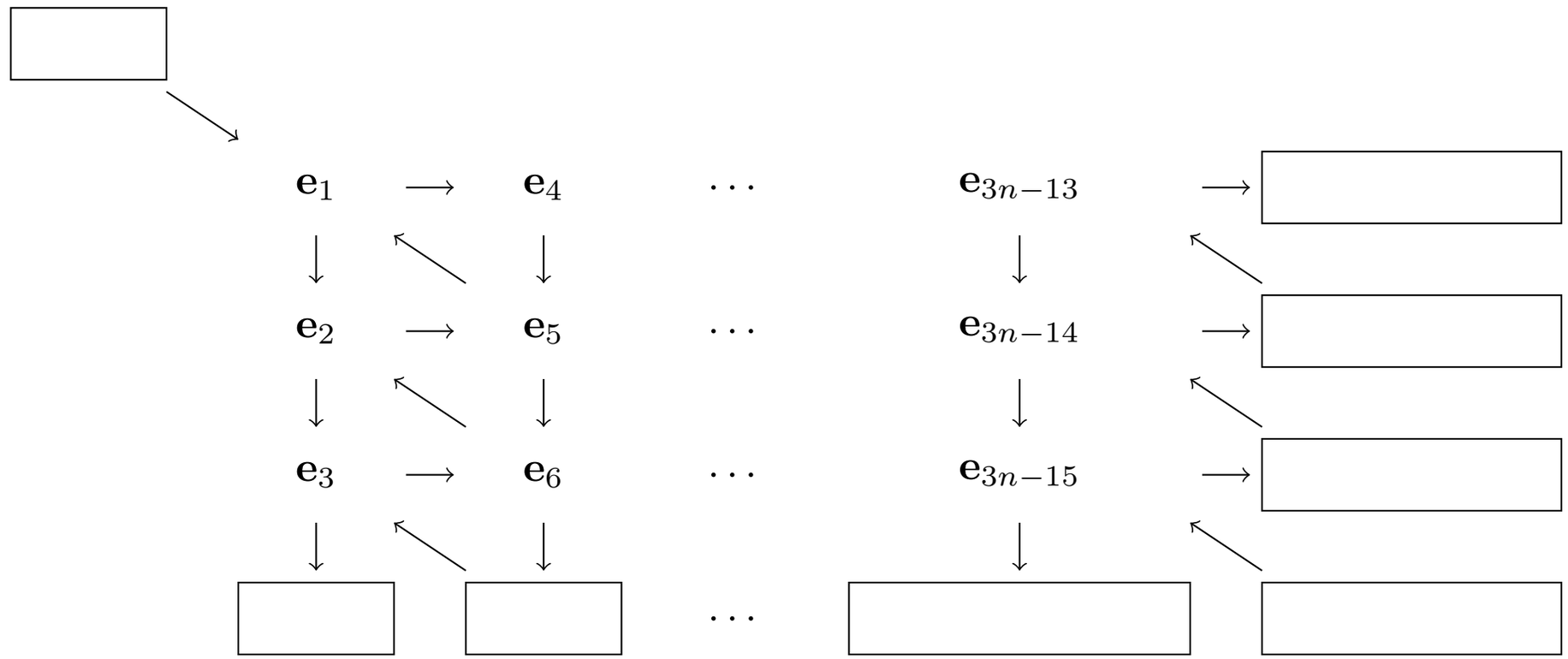
Symbol letters are A-coords [Golden, Goncharov, Spradlin, Vergu, Volovich] using results of [Caron-Huot]

Cluster Adjacency: consecutive letters share a cluster [JMD, Foster, Gürdogan]

Grassmannian Cluster Algebras

Initial cluster assigns unit coordinate g-vectors to each A-coordinate



Mutation generates a g-vector for each A-coordinate $g(a)$

For $\mathrm{Gr}(4,6) \cong \mathrm{Gr}(2,6)$ or $\mathrm{Gr}(2,n)$ Speyer-Williams fan = cluster g-vector fan

Positive Tropical Grassmannians $\mathrm{Tr}^+(4,7)$

For n=7 we have a choice of what to tropicalise - choice of fans.

There are 42 unfrozen A-coordinates:

$$a_{11} = \langle 2367 \rangle \qquad a_{21} = \langle 2567 \rangle \qquad a_{31} = \langle 2347 \rangle \qquad a_{41} = \langle 2457 \rangle$$

$$a_{51} = \langle 1245 \rangle \langle 1367 \rangle - \langle 1345 \rangle \langle 1267 \rangle \qquad a_{61} = \langle 1356 \rangle \langle 1472 \rangle - \langle 1456 \rangle \langle 1372 \rangle$$

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Can also tropicalise quadratic A-coordinates

For n=7 we identified four natural fans generated by:

$\mathcal{S}$	f-vector	triangles	squares
$\{a_{1i}, a_{2i}, a_{3i}\}$	(42, 385, 1393, 2373, 1918, 595)	1379	14
$\{a_{1i}, a_{2i}, a_{3i}, a_{4i}\}$	(42, 392, 1463, 2583, 2163, 693)	1456	7
$\{a_{1i}, a_{2i}, a_{3i}, a_{4i}, a_{5i}\}$	(42, 399, 1540, 2821, 2443, 805)	1540	0
$\{a_{1i}, a_{2i}, a_{3i}, a_{4i}, a_{5i}, a_{6i}\}$	(42, 399, 1547, 2856, 2499, 833)	1547	0

Generalises cluster adjacency: each fan defines a class of polylogarithms

Positive Tropical Grassmannians $Tr^+(4,8)$

For n=8 we have infinite (affine) cluster algebra

1) Need a way to select a finite number of A-coords

2) We know: some letters are not rational functions so cannot be A-coords

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1st problem solved by picking a tropical fan

$\mathcal{S}$	g-vector rays	extra rays
$\{\langle i\, i+1\, j\, j+1\rangle, \langle i-1\, i\, i+1\, j\rangle\}$	272	2
$\{\langle ijkl\rangle\}$	356	4
$\{\langle ijkl\rangle, \langle\overline{ijkl}\rangle\}$	544	4

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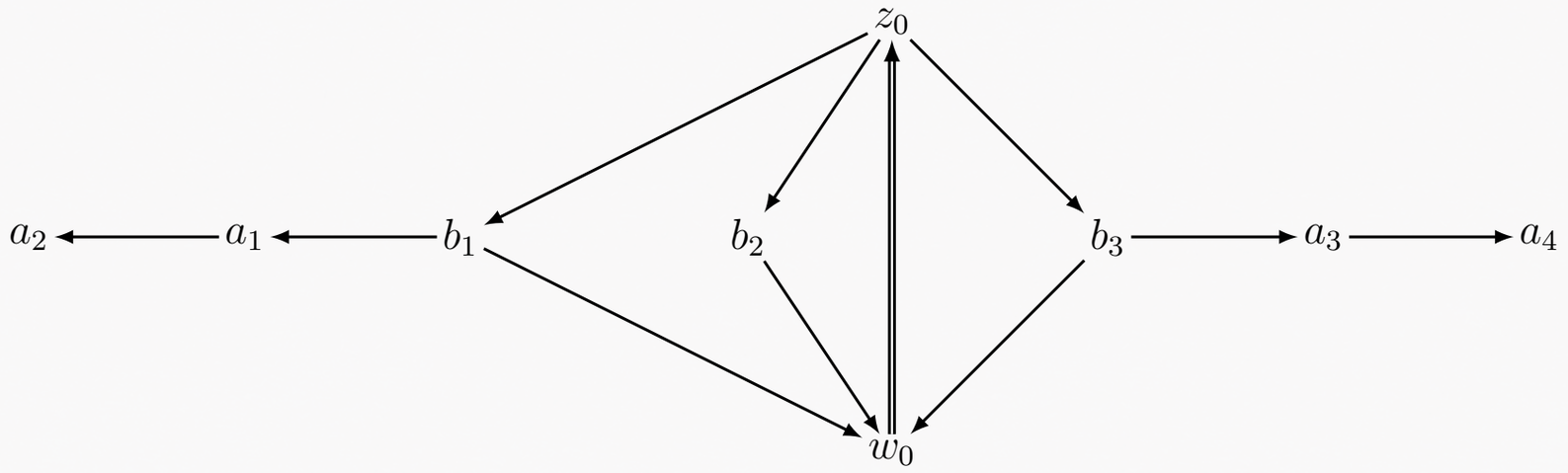
square root letters

2nd problem also solved!

Infinite Affine Sequences

Extra rays are related to infinite sequences of clusters with doubled arrows



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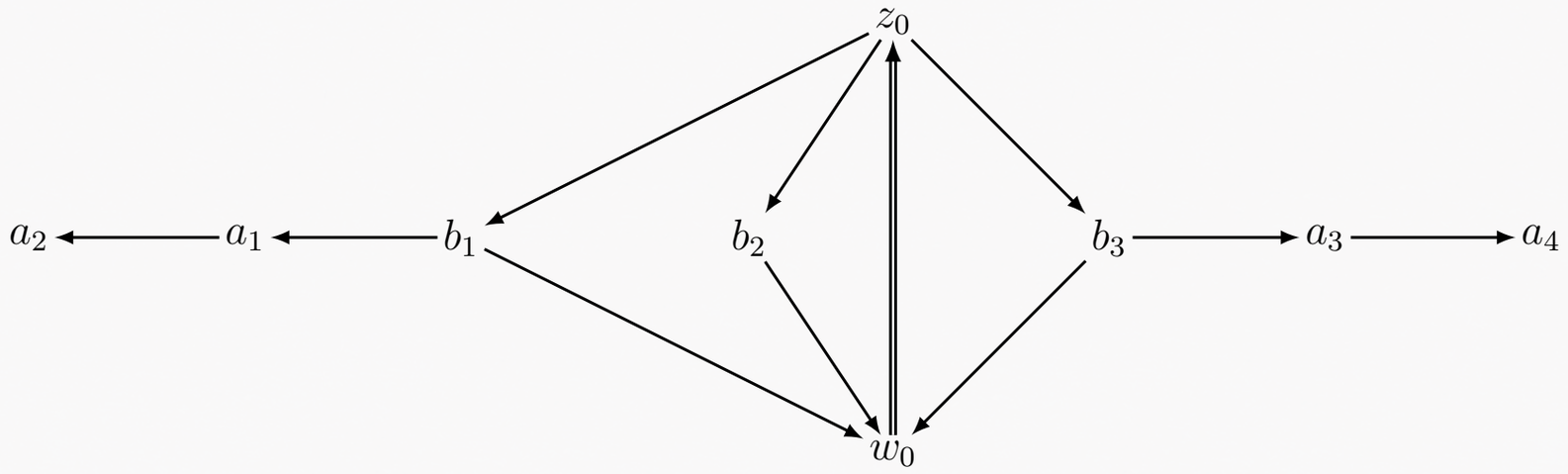

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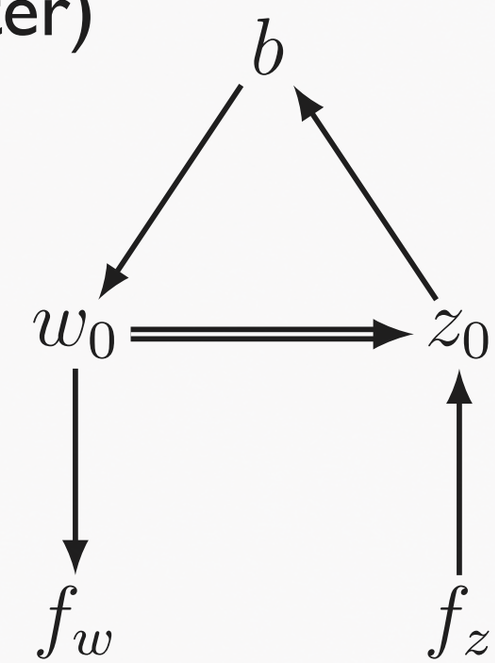
Reduced diagram (origin cluster)



$b = b_1 b_2 b_3$

$f_z,\ f_w$ frozen nodes

Infinite Affine Sequences

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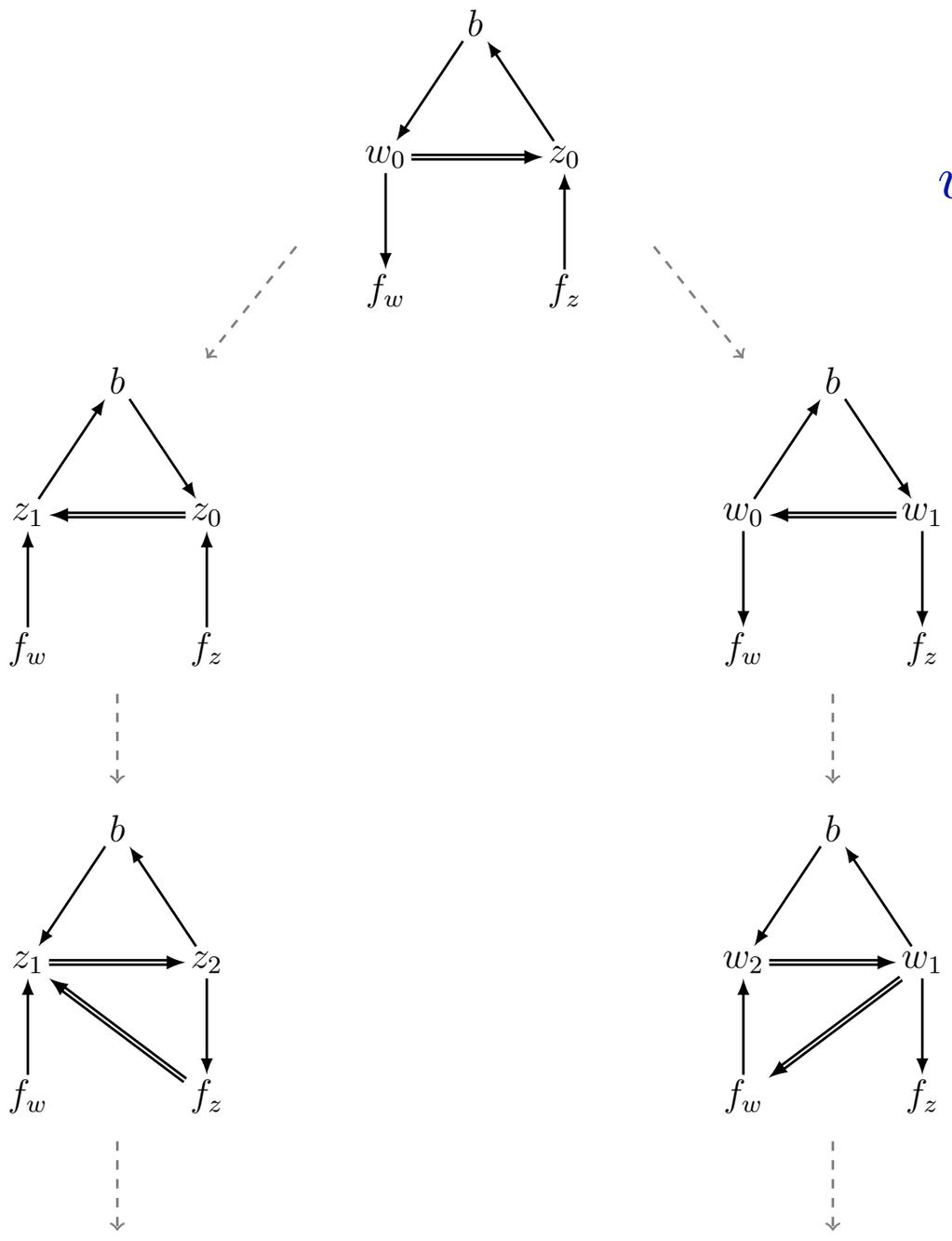
$$w_1 z_0 = b + f_z w_0^2$$



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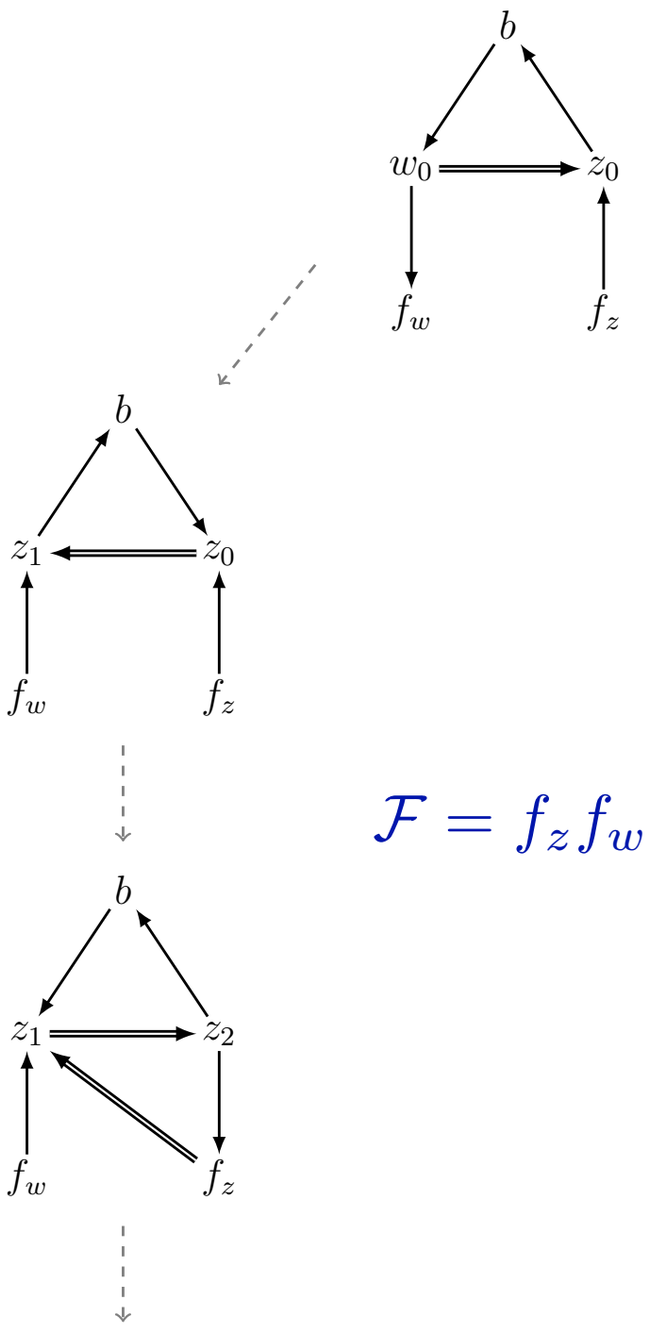

$$w_1 z_0 = b + f_z w_0^2$$

$$z_{n+2} z_n = b f_z \mathcal{F}^n + z_{n+1}^2$$

$$\mathcal{F} = f_z f_w$$

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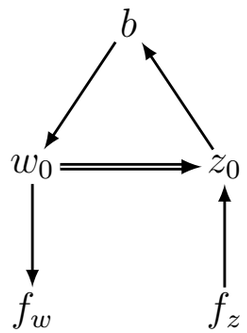

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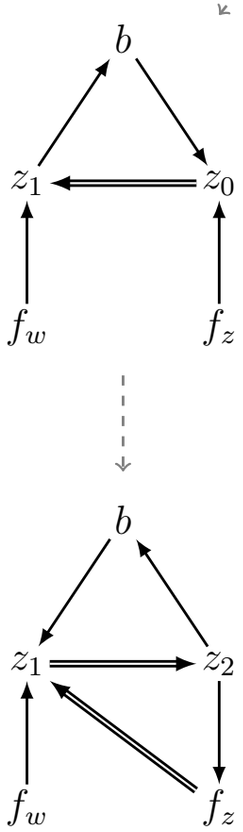

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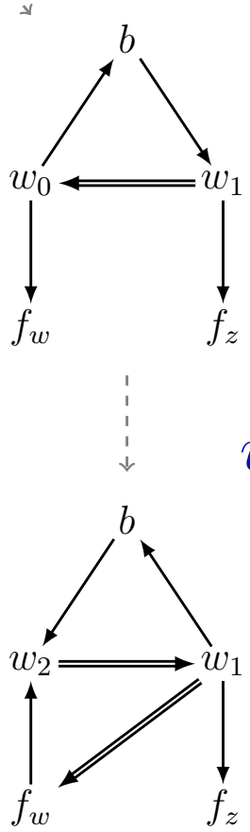

$$w_{n+2} w_n = b f_w \mathcal{F}^n + w_{n+1}^2$$

$$\mathbf{g}(z_{n+1}) - \mathbf{g}(z_n) = \mathbf{g}(w_0) - \mathbf{g}(z_0) = \mathbf{g}(w_{n+1}) - \mathbf{g}(w_n)$$

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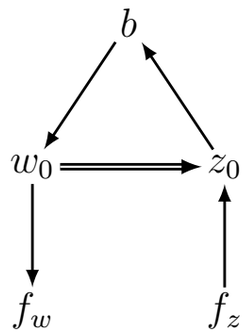

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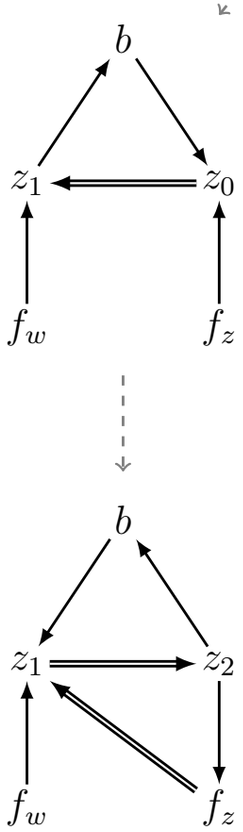

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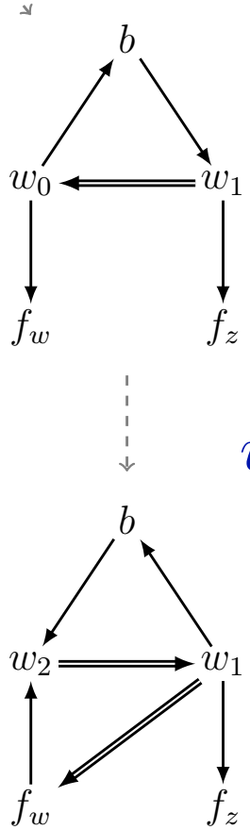

$$w_{n+2} w_n = b f_w \mathcal{F}^n + w_{n+1}^2$$

extra rays!

$$\mathbf{g}(z_{n+1}) - \mathbf{g}(z_n) = \mathbf{g}(w_0) - \mathbf{g}(z_0) = \mathbf{g}(w_{n+1}) - \mathbf{g}(w_n)$$

Infinite Affine Sequences

Quadratic recurrence (because of doubled arrows)

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Which implies a linear recurrence $z_{n+2} = z_{n+1}\mathcal{P} - z_n\mathcal{F}$

Equivalent to original provided $bf_z + z_1^2 + z_0^2\mathcal{F} = z_0 z_1 \mathcal{P}$

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Using the initial mutation rules find $\mathcal{P} = \dfrac{f_z w_0 + z_1}{z_0}$ polynomial!

Infinite Affine Sequences

Linear recurrence $z_{n+2} = z_{n+1}\mathcal{P} - z_n\mathcal{F}$

Would be Fibonacci if we had $\mathcal{P} = -\mathcal{F} = 1$

Infinite Affine Sequences

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Solution $$G_z(x) = \frac{z_1 - z_0\mathcal{F}x}{1 - \mathcal{P}x + \mathcal{F}x^2} = \sum_{n=0}^{\infty} z_{n+1}x^n$$

Implies $$\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = \mathcal{P} + \sqrt{\Delta}\,, \qquad \Delta = \mathcal{P}^2 - 4\mathcal{F}$$

and $$z_n = \frac{1}{2^{n+1}}\left[(z_0 + B_z\sqrt{\Delta})(\mathcal{P} + \sqrt{\Delta})^n + (z_0 - B_z\sqrt{\Delta})(\mathcal{P} - \sqrt{\Delta})^n\right]$$

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$\mathcal{P}, \mathcal{F}$ and hence $\Delta = \mathcal{P}^2 - 4\mathcal{F}$ only depend on limit ray

ϕ_z, ϕ_w depend on path towards limit

For the smallest fan the two limit rays each have 32 x 2 paths

$\sqrt{\Delta}$ is the four-mass box square root

The ϕ_z, ϕ_w are not all multiplicatively independent

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Match 18 letters of 2-loop NMHV octagon!

using results of [He, Li, Zhang]

Summary

Tropical geometry used to study the kinematics of massless scattering

Positive Tropical Grassmannian directly connected to cluster algebras

Beyond Gr(2,n), we have choices of tropical fans to associate to positive tropical Grassmannian

For each fan we have a natural class of tropically adjacenct polylogarithms

Tropical fan and cluster algebra combine to provide algebraic letters associated to limit rays