Generalized Locality in CEGM Amplitudes

Alfredo Guevara (Perimeter)

Zoomplitudes 2020

1903.08904 with Cachazo, Early, Mizera
1909.05291 with Garcia-Sepulveda
1912.09422 with Cachazo, Gimenez-Umbert, Zhang
2005.XXXX with Zhang

CEGM Amplitudes *(See Freddy's talk)*

*Generalize the fascinating connection between moduli and kinematic spaces!

*They were introduced in two representations **[Cachazo, Early, A.G., Mizera '19]**

- Extended CHY integral over $\mathbb{CP}^{k-1}$

$$m_n^{(k)}[\alpha|\beta] = \int \left(\frac{1}{\mathrm{Vol}[\mathrm{SL}(k,\mathbb{C})]} \prod_{a=1}^{n} \prod_{i=1}^{k-1} dx_a^i \right) \prod_{a=1}^{n} {\prod_{i=1}^{k-1}}' \delta\left(\frac{\partial \mathcal{S}_k}{\partial x_a^i} \right) \times \ \mathrm{PT}_n^{(k)}[\alpha]\, \mathrm{PT}_n^{(k)}[\beta]$$

- Sum over (volumes of) facets of the Trop. Grassmannian of Speyer & Williams

$$m_n^{(k)}[\alpha|\beta] = \sum\nolimits_{\mathcal{M} \in J(\alpha) \cap J(\beta)} \mathrm{vol}(\mathcal{M})$$

$J(\alpha)$: Facets of $\mathrm{Tr}_\alpha^+(k,n)$, with canonical ordering α

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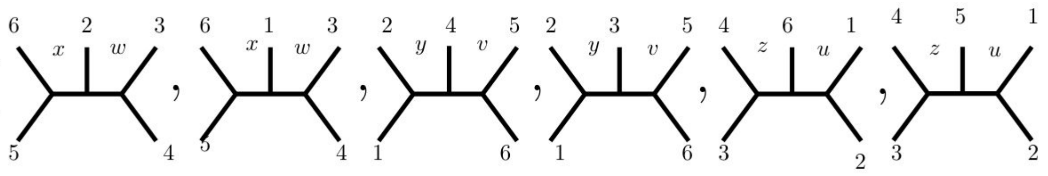
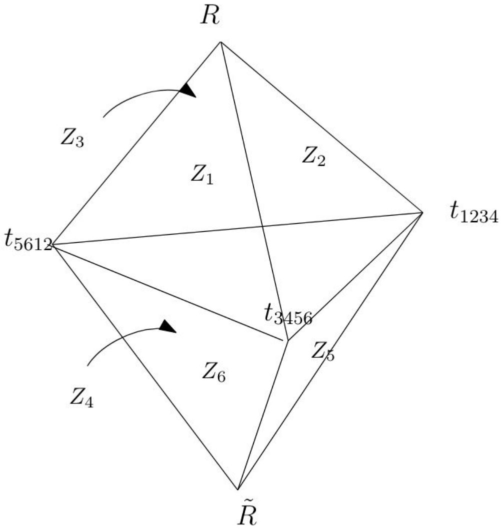
Disclaimer:
In this talk we will focus on the positive part.

Generalized Feynman Diagrams

For **k=3**, the facets of the Tropical Grassmannian are naturally parametrized by **collections** of Feynman Diagrams [Herrmann, Jensen, Joswig & Sturmfels, Borges & Cachazo].

Recall the example of the bipyramid @n=6

$\mathcal{C}^{\text{bip}} = \{$  $\}$



Internal distances degenerate at the following planes:

$$Z_1 : x = 0 \quad , Z_2 : y = 0 \quad , Z_3 : z = 0$$

$$Z_4 : w = 0 \quad , Z_5 : u \equiv y - z + w = 0 \quad , Z_6 : v \equiv x - z + w = 0$$

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The contribution of the CEGM amplitude is obtained via a Laplace transform,

$$\int_{x,y,z,w,u,v>0} dx dy dz dw \, \exp(-\sum_{abc} \mathbf{s}_{abc} d_{bc}^{(a)} = S \cdot F) = \text{Volume of bipyramid projected into plane } S \cdot F = 1$$
$$= \left(\frac{1}{R} + \frac{1}{\tilde{R}}\right) \frac{1}{t_{1234} t_{3456} t_{5612}}$$

where $F = (x, y, z, w)$ and $\sum_{bc} \mathbf{s}_{abc} = 0$ is generalized momentum conservation.

Short Outline

1) Combinatorial Bootstrap

2) Poles

3) Soft/Hard Theorems

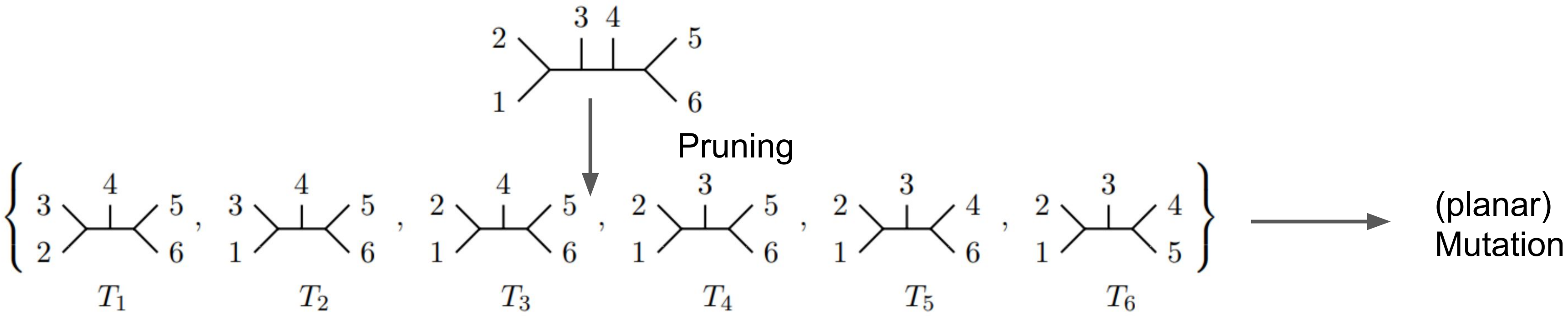
(Bootstraping) Higher-k Feynman Diagrams

[F. Cachazo, A.G., B. Gimenez-Umbert & Y. Zhang]

See also: Fascinating connection with N=4 SYMbology (for k=4) and cluster algebras.

Arkani-Hamed, He, Salvatori, Thomas
Henke, Papathanasiou
Arkani-Hamed, He, Lam
Arkani-Hamed, Lam, Spradlin
Drummond, Foster, Gürdoğan, Kalousios

First Combinatorial Bootstrap *(see also Freddy's talk)*



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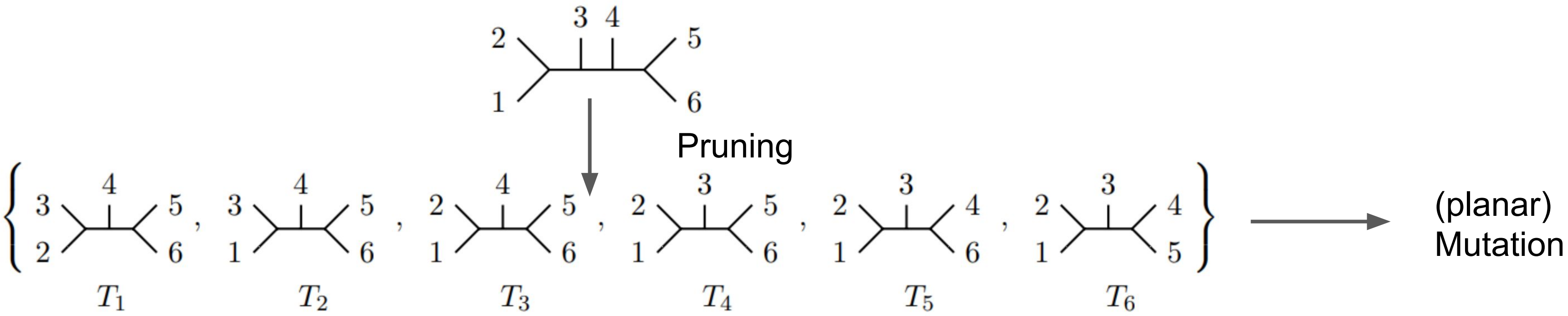

(k,n)	Number of collections	Numbers of collections for each kind					Number of layers
(3,5)	5	2-mut. 5					0
(3,6)	48	4-mut. 46	6-mut. 2				3
(3,7)	693	6-mut. 595	7-mut. 28	8-mut. 70			4
(3,8)	13 612	8-mut. 9 672	9-mut. 1 488	10-mut. 2 280	11-mut. 96	12-mut. 76	8
(3,9)	346 710	10-mut. 186 147	11-mut. 61 398	12-mut. 78 402	13-mut. 12 300	14-mut. 7 668	11
		15-mut. 522	16-mut. 270	17-mut. 3			

Starting from Catalan(n-2) pruned collections we can mutate until recovering all the possible collections (facets) in $\mathrm{Tr}^+(3,n)$ (also checked in higher k).

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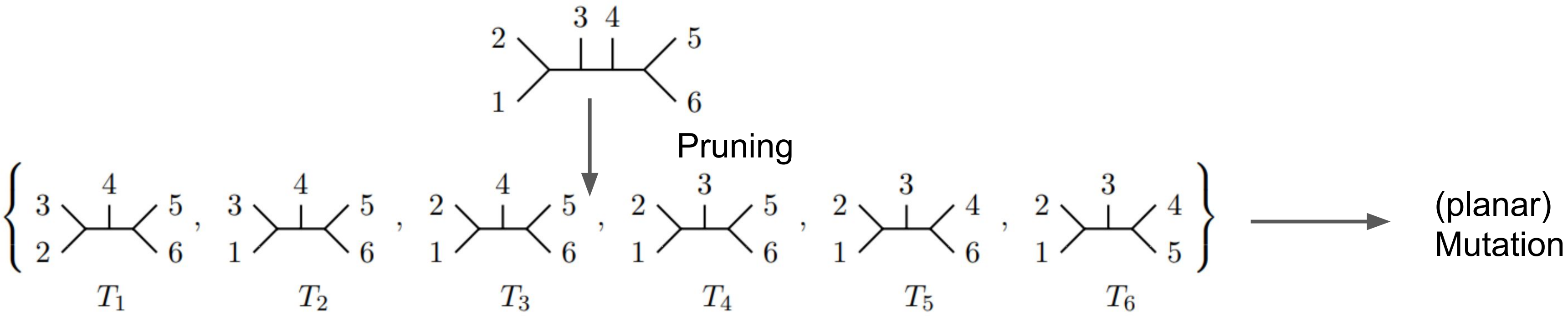

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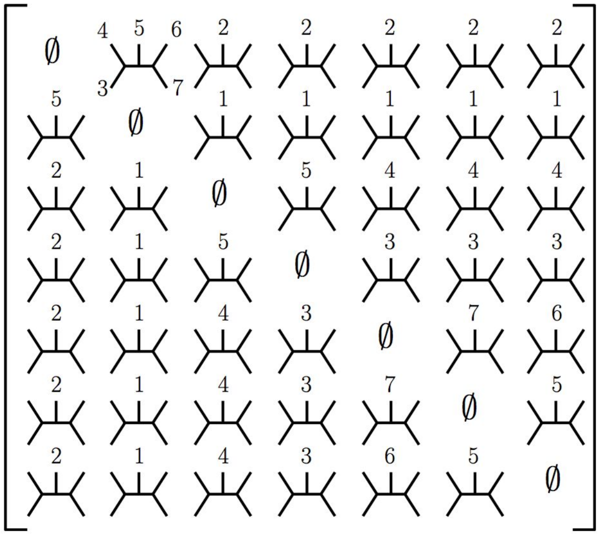
Using 4 kernels, computation of e.g. n=8 takes about 10-15 min.

Generalized Feynman Diagrams

For k=4 and higher, we propose a parametrization of the facets in terms of arrays of Feynman diagrams.

$\mathcal{M} =$



Let $d_{cd}^{(ab)}$ be the distance between labels c and d in the entry $\mathcal{M}^{(ab)}$ of the array. Imposing $d_{cd}^{(ab)} = d_{cd}^{(ba)}$ implies that $\mathcal{M}^{(ab)}$ is a symmetric matrix.

We further impose the distances tensor is completely symmetric via

$$d_{cd}^{(ab)} = d_{bd}^{(ac)} = d_{bc}^{(ad)} = d_{ad}^{(bc)} = d_{ac}^{(bd)} = d_{ab}^{(cd)}$$

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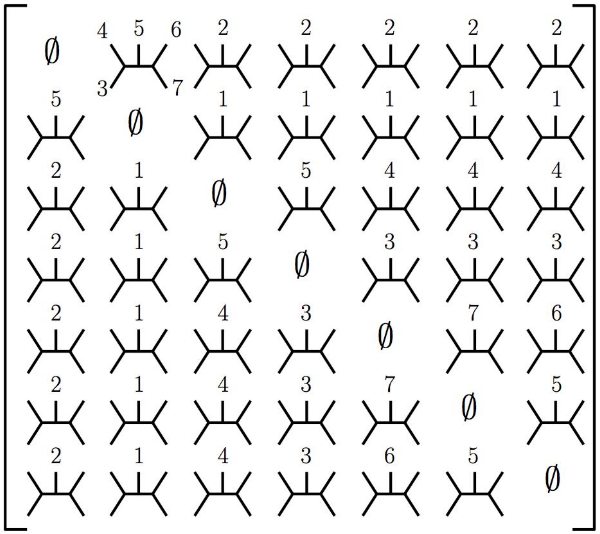

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$$d_{cd}^{(ab)} = d_{bd}^{(ac)} = d_{bc}^{(ad)} = d_{ad}^{(bc)} = d_{ac}^{(bd)} = d_{ab}^{(cd)}$$

The first 2 conditions imply that each column $\mathcal{C}^{(a)} := (\mathcal{M}^{(ai)})_{i \neq a}$ indeed corresponds to a facet of $\mathrm{Tr}^+(3, n-1)$!

Second Combinatorial Bootstrap

We construct all possible symmetric $n \times n$ matrices of the form

$$\mathcal{M} = [\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots, \mathcal{C}^{(7)}]$$

Where $\mathcal{C}^{(a)}$ is a facet of $\mathrm{Tr}^{+}(3, n-1)$. For instance, for n=7, there are 91496 such matrices.

We then impose the compatibility conditions

$$d^{(ab)}_{cd} = d^{(ac)}_{bd} = d^{(ad)}_{bc} = d^{(bc)}_{ad} = d^{(bd)}_{ac} = d^{(cd)}_{/ab}$$

and find that 888 of them have degenerate distances in the graphs. This means they are not full dimensional facets but rather higher codimension objects. Interestingly, this number corresponds to the non regular triangulations of the Amplituhedron $\mathcal{A}_{8,3,2}$ found by **[Lukowski, Parisi, Williams]** (*see Lauren's talk*)

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	$(4,6)$	$(4,7)$	$(4,8)$	$(4,9)$
Planar Matrices	14	693	90 608	30 659 424
Degenerate Matrices	0	0	888	2 523 339

For instance, the 888 degenerate matrices have "volume zero". The volume associated to the other 90608 matrices can be computed via PolyMake in about 2 min, leading to the full amplitude $m_8^{(4)}(\mathbb{I}, \mathbb{I})$. This matches a recent computation using an extended ABHY framework (*see Song's talk*).

In the Second Combinatorial Bootstrap we will write an array of (k,n) as

$$\mathcal{M}^{(k,n)} = [\mathcal{M}_1^{(k-1,n-1)}, \ldots, \mathcal{M}_n^{(k-1,n-1)}]$$

$\mathcal{M}^{(k,n)}$: Rank-(k-2) comp. symmetric tensor.

and impose the compatibility of the induced metric.

This admits the interpretation of a hard kinematical limit, or in mathematical terms, a **boundary map** for matroid subdivisions [see e.g. Early '19, Early '20].

The notion of hard limits is transparent for vertices of the Tropical Grassmannian (= poles of CEGM amplitude).

From full arrays to vertices and back

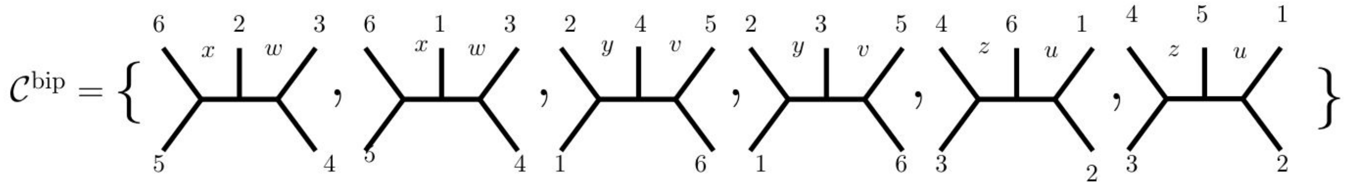
[A.G. & Yong Zhang, to appear]

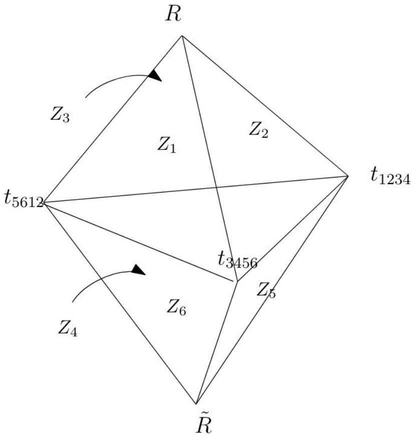
See also:

Early, Dec '19
Lukowski, Parisi & Williams, '20
Arkani-Hamed, Lam & Spradlin, '20

on positroid subdivisions.

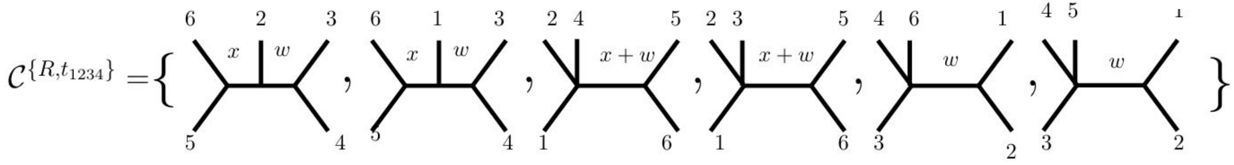
Degenerating full collections





Edges emerge in the intersection of facets:

$y, z \to 0$



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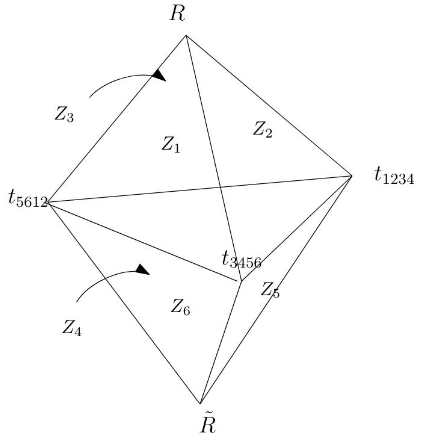

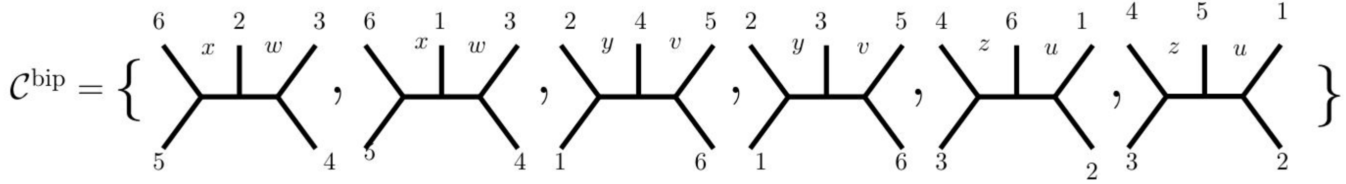

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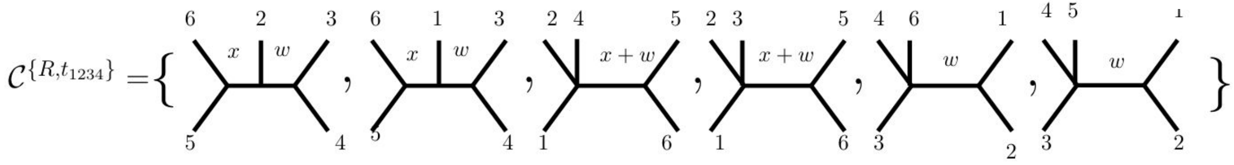
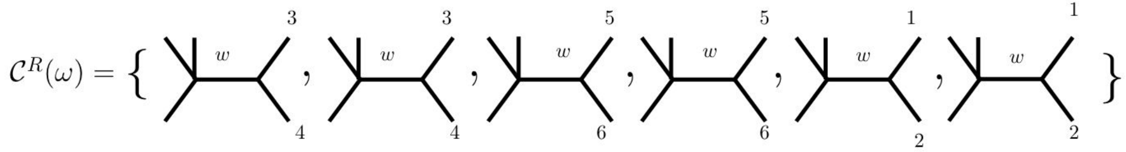


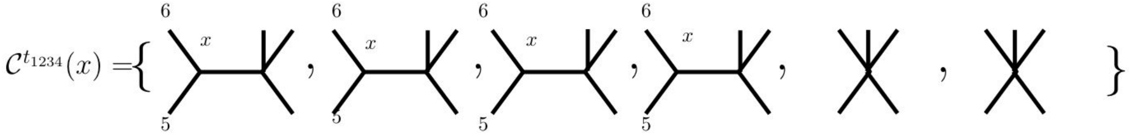
Further degenerations lead to vertices (rays). These correspond to independent solutions of tropical hyperplane equations *(see James' talk)*

$x \to 0$



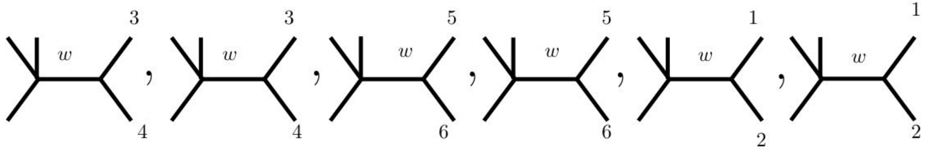
$d_{bc}^{(a)} \sim w\, V_{abc}$

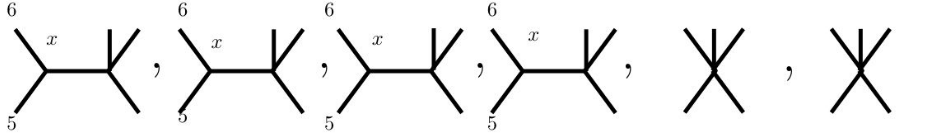
$w \to 0$

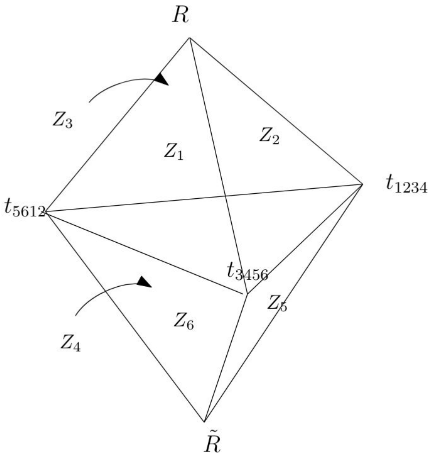


$d_{bc}^{(a)} \sim x\, V'_{abc}$

Degenerating full collections

$\mathcal{C}^R(\omega) = \{$  $\}$

$\mathcal{C}^{t_{1234}}(x) = \{$  $\}$

$\mathcal{M} =$ 

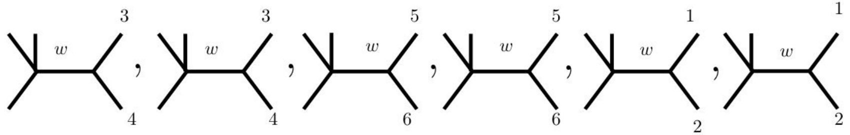
From the volume formula $\mathrm{vol}(\mathcal{M}) \sim \frac{\langle V_1 \cdots V_k \rangle}{\prod_i V_i \cdot S}$, we know that the vertices are in bijection with poles of $m_n^{(k)}[\mathbb{I}_n|\mathbb{I}_n]$!

$$d_{bc}^{(a)} \sim w\, V_{abc} \quad \Rightarrow S \cdot V = \sum \mathbf{s}_{abc} d_{bc}^{(a)} = w(t_{1234} + s_{345} + s_{346}) = wR$$

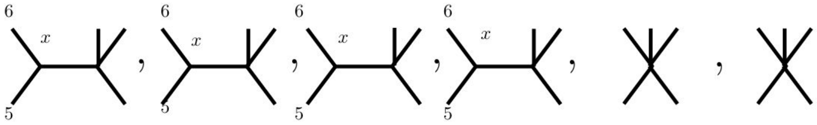
$$d_{bc}^{(a)} \sim x\, V'_{abc} \quad \Rightarrow S \cdot V = \sum \mathbf{s}_{abc} d_{bc}^{(a)} = x(s_{123} + s_{124} + s_{234} + s_{134}) = x t_{1234}$$

From vertices to full collections

We see that vertices arise as collections of degenerate k=2 Feynman diagrams (i.e. not cubic). Edges, facets, etc are less degenerate, and can be recovered via Minkowski sums as follows;

$\mathcal{C}^R(\omega) = \{$ 

$\} \cong w\,\{s_{34}\,, s_{34}\,, s_{56}\,, s_{56}\,, s_{12}\,, s_{12}\}$

$\mathcal{C}^{t_{1234}}(x) = \{$ 

$\} \cong x\,\{s_{56}\,, s_{56}\,, s_{56}\,, s_{56}\,, 0, 0\}$

Observe that each of the components of the collections are compatible as **k=2 poles**. This means that the k=2 Feynman **diagrams can be added** in the space of metrics:

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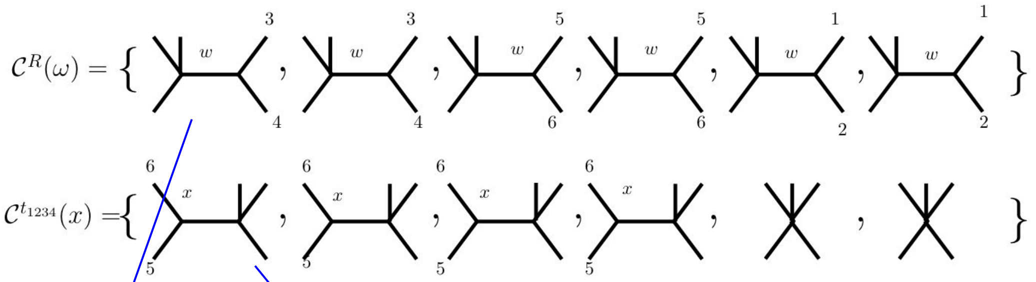

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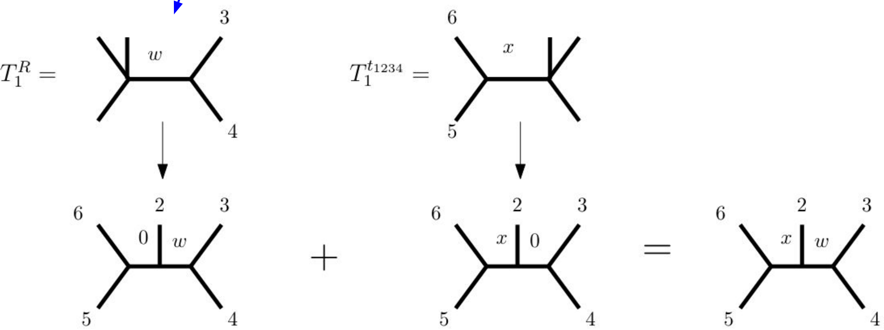

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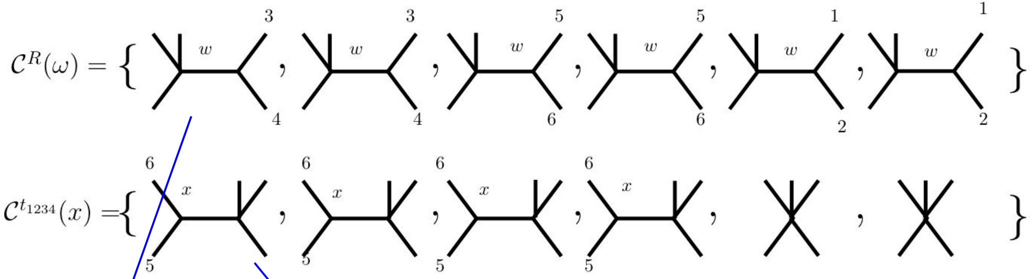

$$\mathcal{C}^R(\omega) \cong w\,\{s_{34}, s_{34}, s_{56}, s_{56}, s_{12}, s_{12}\}$$

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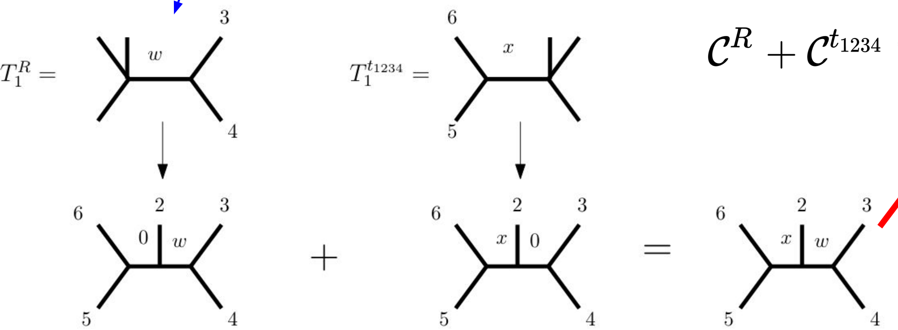
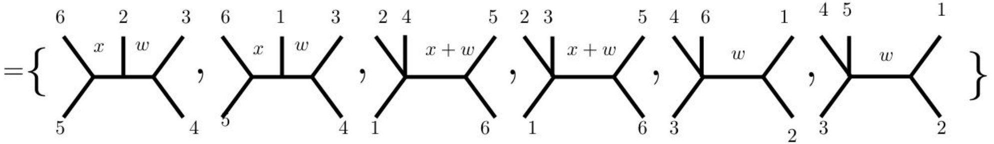


$$\mathcal{C}^R + \mathcal{C}^{t_{1234}} = \{ \ldots \}$$



$$\cong \{xs_{56} + ws_{34}, xs_{56} + ws_{34}, (w+x)s_{56}, (w+x)s_{56}, xs_{12}, xs_{12}\}$$

$$\boxed{d_{bc}^{(a)} \sim w\,V_{abc} + x\,V'_{abc}}$$

$$\sum \mathbf{s}_{abc} d_{bc}^{(a)} = w\,R + x\,t_{1234}$$

The following two facts become evident:

1) *Two one-parameter arrays* $\mathcal{V}$ *and* $\mathcal{V}'$ *of the form*

$$\mathcal{V}^{(k,n)} = [\mathcal{V}_1^{(k-1,n-1)}, \ldots, \mathcal{V}_n^{(k-1,n-1)}]$$ (: Rank-(k-2) symmetric tensor)

are compatible (i.e. add to an edge in tropGr) if their components $\mathcal{V}_i, \mathcal{V}'_i$ *are compatible as rank-(k-3) arrays.*

2) *If a set of such arrays* $\{\mathcal{V}^{A_i}\}$ *is pairwise compatible then they are simultaneously compatible, i.e. their Minkowski sum defines a generalized Feynman diagram (trivial for k=2, then follows* $\forall k$ *).*

$$\pi_{abc}^{\text{bipyramid}}(\alpha^i) \sim \alpha^1 V_{abc}^{R} + \alpha^2 V_{abc}^{\tilde{R}} + \alpha^3 V_{abc}^{t_{1234}} + \alpha^4 V_{abc}^{t_{3456}} + \alpha^5 V_{abc}^{t_{5612}}\,, \quad \alpha^i > 0\,.$$

$$\sim (\alpha^1 + \alpha^5) V_{abc}^{R} + (\alpha^3 + \alpha^2) V_{abc}^{t_{1234}} + (\alpha^4 + \alpha^2) V_{abc}^{t_{3456}} + (\alpha^5 + \alpha^2)(V_{abc}^{t_{5612}} - V_{abc}^{R})$$

For **k=4, n=8** the set of 360 vertices of the positive tropGr can be obtained from e.g. stringy canonical forms/ABHY (**Song's talk)** or from cluster variables (**James, Marcus talks**). They can be written as

$$\mathcal{V}^{(4,8)} = [\mathcal{V}_1^{(3,7)}, \ldots, \mathcal{V}_n^{(3,7)}]$$

where $\mathcal{V}_i^{(3,7)}$ are rays for **k=3, n=7.** The compatibility relations among the k=3 rays are easily implemented in a computer code, from which the graph for k=4 rays can be constructed in minutes.

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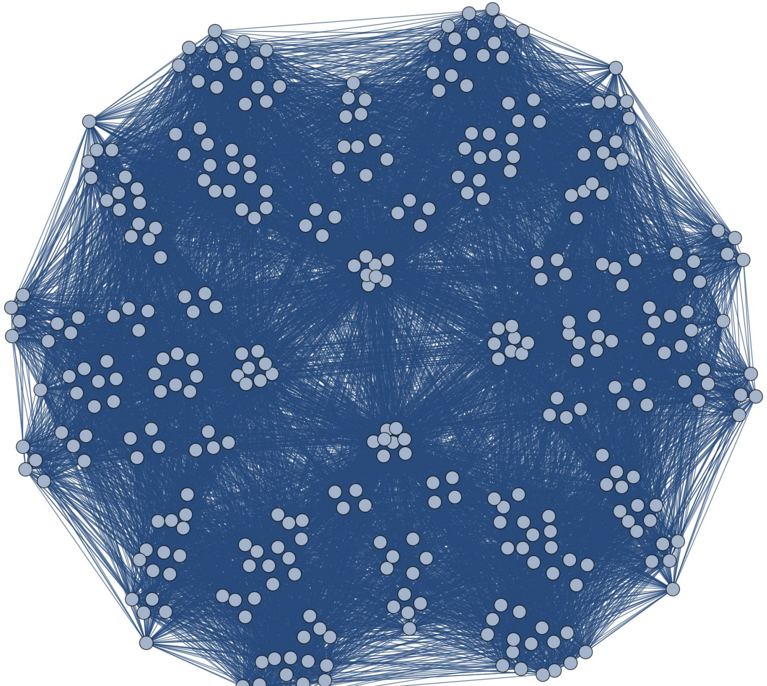

Using the previous facts we inmediately obtain all the maximal compatibles sets of poles (cliques).

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FindClique[Graph[gr48], Infinity, All] // Length
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90 608

Fun with soft limits: New Soft/Hard Factorizations

[Diego Garcia-Sepulveda & A.G. '19]

Soft and Hard Limits

Consider k=3, with $\sum_{bc} \mathbf{s}_{abc} = 0$. We define soft and hard limit on label 1 as

$$\mathbf{s}_{1ab} = \tau \hat{\mathbf{s}}_{1ab}, \quad \tau \to 0, \quad \hat{\mathbf{s}}_{1ab} \text{ fixed.}$$ (Soft)

$$\mathbf{s}_{1ab} = \tau \hat{s}_{ab}\,, \quad \tau \to \infty\,, \quad \hat{s}_{ab} \text{ fixed}$$ (Hard)

These are related via Grassmannian duality $(3,6) \sim (3,6), \mathbf{s}_{123} \sim \mathbf{s}_{456}$ $etc\ldots$

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OBS: Taking the hard limit with respect to all labels defines an n-component vector. This is precisely the collection associated a given kinematic pole!

$$\mathbf{s}_{123} \to \tau\{s_{23}\,, s_{13}\,, s_{12}\,, 0, 0, 0\}\,, \quad \tau \to \infty$$

This operation for vertices $(3,n) \to (2,n-1)$ matches the boundary map for (coarsest) matroid subdivisions of the hypersymplex $\Delta_{k,n} \to \Delta_{k-1,n-1}$ [Early '19]

Soft Theorems

Consider for instance (k,n)=(3,6). The soft limit defines a set of “soft poles”,

$$\{\mathbf{s}_{123}, \mathbf{s}_{156}, \mathbf{s}_{126}, \mathbf{t}_{2345}, \mathbf{t}_{3456}\}$$

and “soft edges” (pairs of compatible soft poles lying inside $\mathrm{Tr}^{+}(3,6)$), that is

$$S_{1,23456} = \left\{\frac{1}{\mathbf{s}_{123}\mathbf{s}_{156}}, \frac{1}{\mathbf{t}_{2345}\mathbf{s}_{156}}, \frac{1}{\mathbf{t}_{2345}\mathbf{s}_{126}}, \frac{1}{\mathbf{t}_{3456}\mathbf{s}_{123}}, \frac{1}{\mathbf{t}_{3456}\mathbf{s}_{126}}\right\}. \sim \mathrm{Tr}^{+}(2,5)$$

Note that the set of soft edges corresponds to the facets of $\mathrm{Tr}^{+}(2,5)$ whose vertices are the soft poles!

We can group the soft edges in the CEGM amplitude

soft **hard**

$$m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6] = \frac{1}{\mathbf{s}_{123}\mathbf{s}_{561}}\left[\frac{1}{R_{45,23,61}\mathbf{s}_{345}} + \frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{t}_{1234}}}{R_{12,34,56}} + \frac{\frac{1}{R_{45,23,61}}+\frac{1}{\mathbf{t}_{1234}}}{\mathbf{t}_{4561}}\right]$$
$$+\frac{1}{\mathbf{t}_{2345}\mathbf{s}_{561}}\left[\frac{1}{R_{45,23,61}\mathbf{s}_{345}} + \frac{\frac{1}{R_{45,23,61}}+\frac{1}{\mathbf{s}_{234}}}{\mathbf{t}_{4561}} + \frac{\frac{1}{\mathbf{s}_{234}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{5612}}\right]$$
$$+\frac{1}{\mathbf{t}_{2345}\mathbf{s}_{612}}\left[\frac{1}{R_{23,45,61}s_{234}} + \frac{\frac{1}{\mathbf{s}_{234}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{5612}} + \frac{\frac{1}{R_{23,45,61}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{6123}}\right]$$
$$+\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{123}}\left[\frac{1}{R_{12,34,56}\mathbf{s}_{345}} + \frac{\frac{1}{R_{12,34,56}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{1234}} + \frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{6123}}\right]$$
$$+\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{612}}\left[\frac{1}{R_{34,12,56}\mathbf{s}_{456}} + \frac{\frac{1}{R_{34,12,56}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{5612}} + \frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{6123}}\right] + \ldots$$

In terms of facets:

$$\mathrm{Tr}^+(3,6) \supset \mathrm{Tr}^+_{soft}(2,5) \times \mathrm{Tr}^+_{hard}(3,5)$$

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$$+\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{123}}\left[\frac{1}{R_{12,34,56}\mathbf{s}_{345}} + \frac{\frac{1}{R_{12,34,56}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{1234}} + \frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{6123}}\right]$$
$$+\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{612}}\left[\frac{1}{R_{34,12,56}\mathbf{s}_{456}} + \frac{\frac{1}{R_{34,12,56}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{5612}} + \frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{6123}}\right] + \dots$$

In terms of facets:

$$\mathrm{Tr}^+(3,6) \supset \mathrm{Tr}^+_{soft}(2,5) \times \mathrm{Tr}^+_{hard}(3,5)$$

Each hard piece has the combinatorics of another $\mathrm{Tr}^+(2,5) \sim \mathrm{Tr}^+(3,5)$. At leading order in the soft deformation $\tau \to 0$ all the hard pieces collapse to a single factor and we get

$$m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6] = \frac{1}{\tau^2}\left[\frac{1}{\hat{\mathbf{s}}_{123}\hat{\mathbf{s}}_{156}} + \frac{1}{\hat{\mathbf{t}}_{2345}}\left(\frac{1}{\hat{\mathbf{s}}_{156}}+\frac{1}{\hat{\mathbf{s}}_{126}}\right) + \frac{1}{\hat{\mathbf{t}}_{3456}}\left(\frac{1}{\hat{\mathbf{s}}_{123}}+\frac{1}{\hat{\mathbf{s}}_{126}}\right)\right]$$
$$\times\left[\frac{1}{\mathbf{s}_{236}\mathbf{s}_{345}} + \frac{1}{\mathbf{s}_{256}}\left(\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{s}_{234}}\right) + \frac{1}{\mathbf{s}_{456}}\left(\frac{1}{\mathbf{s}_{236}}+\frac{1}{\mathbf{s}_{234}}\right)\right] + \mathcal{O}\left(\frac{1}{\tau}\right)$$

The factorization occuring at $1/\tau^2$ for **k=3**

$$m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6] = S^{(3)}_{1,23456} \times m_5^{(3)}[23456|23456] + \mathcal{O}\left(\frac{1}{\tau}\right),$$

$$S^{(3)} = \frac{1}{\tau^2}\left[\frac{1}{\hat{\mathrm{s}}_{123}\hat{\mathrm{s}}_{156}} + \frac{1}{\hat{\mathrm{t}}_{2345}}\left(\frac{1}{\hat{\mathrm{s}}_{156}} + \frac{1}{\hat{\mathrm{s}}_{126}}\right) + \frac{1}{\hat{\mathrm{t}}_{3456}}\left(\frac{1}{\hat{\mathrm{s}}_{123}} + \frac{1}{\hat{\mathrm{s}}_{126}}\right)\right] \sim \hat{m}_5^{(2)}[\mathbb{I}|\mathbb{I}]/\tau^2$$

generalizes the well known **k=2** Soft Theorem at $1/\tau$ (for the biadjoint scalar)

$$m_n^{(2)}[\mathbb{I}_n|\mathbb{I}_n] = S^{(2)} \times m_{n-1}^{(2)}[2\ldots n|2\ldots n] + \mathcal{O}(\tau^0).$$

$$S^{(2)} = \frac{1}{\tau}\left(\frac{1}{\hat{s}_{n1}} + \frac{1}{\hat{s}_{12}}\right) \sim \hat{m}_4^{(2)}[\mathbb{I}|\mathbb{I}]/\tau$$

Soft Theorem [D. Garcia-Sepulveda & A.G.]: The general soft behaviour of the CEGM amplitude for any (k,n) is

$$m_n^{(k)}[\mathbb{I}_n|\mathbb{I}_n] = \frac{1}{\tau^{k-1}}\hat{m}_{k+2}^{(2)}[\mathbb{I}|\mathbb{I}] \times m_{n-1}^{(k)}[2\ldots n|2\ldots n] + \mathcal{O}(1/\tau^{k-2})$$

Proof: Use CHY representation + Global Residue Theorem.

This entails that all "standard" k=2 biadjoint amplitudes are nothing but soft factors of higher-k amplitudes!

Corollary: We can obtain the Soft Factorization for

$$m_n^k[1\alpha|1\beta] = \sum_{\Upsilon \in J_n(1\alpha) \cap J_n(1\beta)} \Upsilon$$

where $J_n(\alpha)$ corresponds to the (volume of) facets associated to $\mathrm{Tr}^+(k,n)$ with ordering α.
From our Soft Theorem, the facets contributing to the soft limit form a **cartesian product**

$$S_{1\alpha} \times J_{n-1}(\alpha) \subset J_n(1\alpha)$$

$$(\sim \mathrm{Tr}^+_{soft}(2, k+2) \times \mathrm{Tr}^+_{hard}(k, n-1) \subset \mathrm{Tr}^+(k,n))$$

Thus we get

$$m_n^k[1\alpha|1\beta] \to \sum_{\Upsilon \in (S_{1\alpha} \cap S_{1\beta}) \times (J_{n-1}(\alpha) \cap J_{n-1}(\beta))} \Upsilon$$

$$= \sum_{\Theta \in S_{1\alpha} \cap S_{1\beta}} \Theta \times \sum_{\Upsilon \in J_{n-1}(\alpha) \cap J_{n-1}(\beta)} \Upsilon$$

$$=: S[1\alpha|1\beta] \times m_{n-1}^{(k)}[\alpha|\beta]$$

Soft/Hard Duality

Due to Grassmannian duality $\mathrm{Tr}^+(k,n) \sim \mathrm{Tr}^+(n-k,n)$, CEGM amplitudes satisfy

$$m_n^{(k)}[\mathbb{I}|\mathbb{I}] \sim m_n^{(n-k)}[\mathbb{I}|\mathbb{I}]$$

after relabeling kinematics $s_{a_1 \ldots a_k} \sim s_{b_1 \ldots b_{n-k} = \overline{a_1 \ldots a_k}}$. This exchanges soft and hard deformations. Thus we can derive the **hard theorem** from the following diagram:

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$$\begin{tikzcd}
m_n^{(k)}[\mathbb{I}|\mathbb{I}] \arrow[d, swap, "\text{Hard Thm}"] & \sim & m_n^{(n-k)}[\mathbb{I}|\mathbb{I}] \arrow[d, "\text{Soft Thm}"] \\
m_{n-1}^{(k-1)}[\mathbb{I}|\mathbb{I}] & \sim & m_{n-1}^{(n-k)}[\mathbb{I}|\mathbb{I}]
\end{tikzcd}$$

$$m_n^{(k)} \longrightarrow \begin{cases} S^{(k)} \times m_{n-1}^{(k)} + \mathcal{O}(1/\tau^{k-2}) & \text{(Soft Theorem)} \\ H^{(k)} \times m_{n-1}^{(k-1)} + \mathcal{O}(1/\tau^{n-k-2}) & \text{(Hard Theorem)} \end{cases}$$

where $S^{(k)}$ is given by $m_{k+2}^{(2)}$ whereas $H^{(k)}$ is given by $m_{n-k+2}^{(2)}$

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Soft/Hard thms are **k preserving/decreasing** respectively, hinting a connection to the hypersimplex $\Delta_{k,n}$ or Amplituhedron $\mathcal{A}_{n,k-1,2}$ pictures.

Two questions for the future...

- Generalized Feynman diagrams provide a notion of locality in terms of Schwinger parameters. Is there a notion of quantum theory that determines the compatible collections/arrays?

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- Can we bootstrap the full amplitude $m_n^{(k)}$ from its soft/hard factorizations?
  - The subleading order is also controlled by a **CHY formula** [Afkhami-Jeddi '14, Volovich & Schwab '14, D. Garcia-Sepulveda & A.G. '19]
  - Alternatively, our second Combinatorial Bootstrap corresponds to an inverse hard limit from $\mathrm{Tr}^+_{k-1,n-1} \to \mathrm{Tr}^+_{k,n}$

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… thanks!