Positive and Cluster Configuration Spaces

Amplitudes 2020 May 13

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arXiv: 2003.03904

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arXiv: 1912.08707

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Configuration Space

$\mathrm{Conf}(k,n)$ = configuration space of n distinct labeled points in $\mathbb{P}^{k-1}$

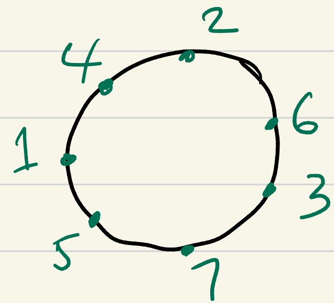
$$= \left\{ z_1, z_2, \dots, z_n \in \mathbb{P}^{k-1} \;\middle|\; z_i \neq z_j \right\} \Big/ PGL(k)$$

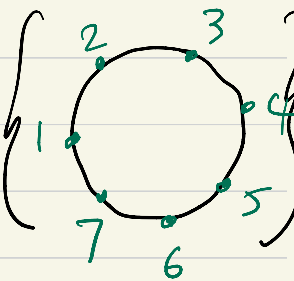
How do we compactify?

$\parallel$

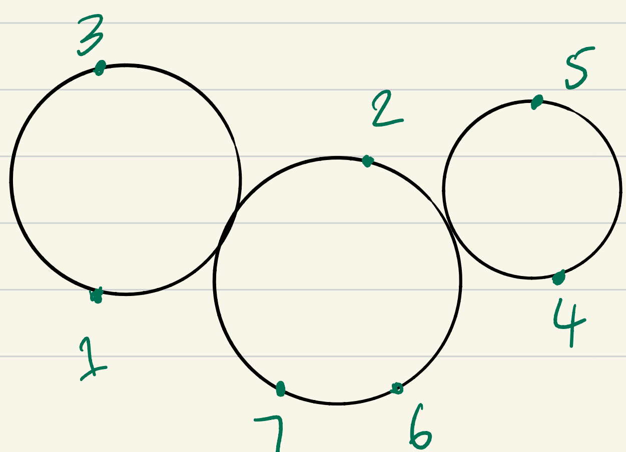
What happens when points collide?

$\mathcal{M}_{0,n} = \mathrm{Conf}(2,n)$ dim = n-3

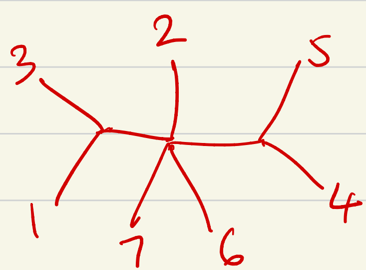
$\mathcal{M}_{0,n}(\mathbb{R}) = \Big\{$ 

$\Big\}$ $(\mathcal{M}_{0,n})_{>0} = \Big\{$ 

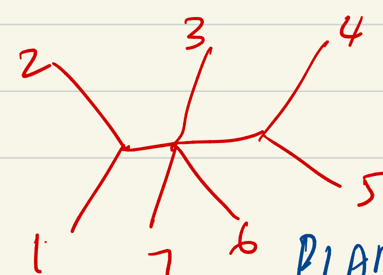
$\Big\}$

$\overline{\mathcal{M}_{0,n}}(\mathbb{R}) \ni$ 

$(\mathcal{M}_{0,n})_{\geq 0}$



Trees with n leaves



PLANAR

Open string amplitude

$$I_n(s) = \int_{(\mathcal{M}_{0,n})_{>0}} \underbrace{\frac{d^{n-3}z}{(z_1-z_2)\cdots(z_n-z_1)}}_{\text{Parke-Taylor form}} \underbrace{\prod_{1\le a<b\le n} (z_a-z_b)^{\alpha' s_{ab}}}_{\text{Koba-Nielsen factor}}$$

Canonical form of positive geometry

$s_{ab} = (p_a + p_b)^2$ Mandelstam variables

"Factorization" of $I_n(s)$ matches combinatorics of $\overline{\mathcal{M}}_{0,n}$

$$\boxed{\text{Why } \overline{\mathcal{M}}_{0,n} \text{ and } (\mathcal{M}_{0,n})_{\ge 0} \text{ and not some other compactification?}}$$

Answer 1: positive parametrization [Song's talk + Nima last week]

$$I_n = \int_{\mathbb{R}^{n-3}_{>0}} \prod_i \frac{dx_i}{x_i} \prod_{a<b} \underbrace{p_{ab}(x)}_{\text{Laurent polynomials}}^{\alpha' s_{ab}}$$

P = Minkowski sum (Newton polytope (p_{ab}))

is an associahedron A_n. Faces $\overset{1:1}{\longleftrightarrow}$ planar trees with n leaves

I_n is "Stringy canonical form" of polytope A_n

<u>Answer 2</u>: Explore $I_n(s)$ in region that it converges.

$$I = \int_{(\mathcal{M}_{0,n})_{>0}} \frac{d^{n-3} z}{(z_1 - z_2) \cdots (z_n - z_1)} M$$

$$M = \prod_{i<j} (z_i - z_j)^{s_{ij}}$$

<u>Def'n</u> M nearly converges if it is limit of integrands that do converge.

M nearly converges $\Longleftrightarrow$ $\mathrm{Trop}(M) \geq 0$

Trop: $(+, \times, \div) \longmapsto (\min, +, -)$

e.g. $$I_s = \int \frac{dx}{x}\frac{dy}{y} \underbrace{x^{\alpha' a}\, y^{\alpha' b}\, (1+x)^{-\alpha' c_1} (1+x+xy)^{-\alpha' c_2} (1+y)^{-\alpha' c_3}}_{M}$$

$$\mathrm{Trop}(M) = \alpha' \Big(aX + bY - c_1 \min(0,X) - c_2 \min(0, X, X+Y) - c_3 \min(0,Y) \Big)$$

$\underset{\text{for all } X,Y}{\mathrm{Trop}(M) \geq 0}$ $\Longrightarrow$ collection of linear ineq on a, b, c_1, c_2, c_3

SEMIGROUP: M, M' nearly converges $\Longrightarrow$ MM' nearly converges

$R := \mathbb{C}[M \mid M$ rational and nearly convergent$] \subset$ rational functions

$\widetilde{\mathcal{M}}_{0,n} = \mathrm{Spec}(R) =$ "the biggest possible space such that every such M does not blow up"

Turns out: $\mathcal{M}_{0,n} \subset \widetilde{\mathcal{M}}_{0,n} \subset \overline{\mathcal{M}_{0,n}}$

e.g. $n=4$

$$\int_0^1 \frac{dy}{y(1-y)} y^s (1-y)^t$$

Converges when $s>0,\ t>0$

- $R = \mathbb{C}[y, 1-y] = \mathbb{C}[y]$ ($\frac{1}{y}$ and $\frac{1}{1-y}$ not allowed) $\xrightarrow{\mathrm{Spec}} \mathbb{C}^1$
- $\mathbb{C}[y, 1-y, \frac{1}{y}, \frac{1}{1-y}] \xrightarrow{\mathrm{Spec}} \mathcal{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$

$$\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\} \subset \widetilde{\mathcal{M}}_{0,4} = \mathbb{C} \subset \mathbb{P}^1 = \overline{\mathcal{M}_{0,4}}$$

$$(\mathcal{M}_{0,n})_{\geq 0} := \text{closure of } (\mathcal{M}_{0,n})_{>0} \text{ in } \widetilde{\mathcal{M}}_{0,n}$$

curved associahedron

no reference to $\overline{\mathcal{M}}_{0,n}$!!!

$$(\mathcal{M}_{0,n})_{\geq 0} \xrightarrow{\text{CHY scattering equations}} A_n$$

(Arkani-Hamed, Bai, He, Yan, Arkani-Hamed, He, L.)

A_n associahedron living in kinematic space (usual polytope)

Cluster String Integral

$$\int_{X_D/T} \prod_i \frac{dx_i}{x_i} \prod_\gamma x_\gamma^{\alpha' s_\gamma}$$

cluster variety

$\mathcal{M}_D \subset \tilde{\mathcal{M}}_D$

cluster configuration space

cluster variables

PROBLEM: Most cluster algebras have infinitely many cluster variables

D = Dynkin diagram

$A_{n-3} \longrightarrow$ n-point string amplitude $\quad \mathrm{Conf}(2,n) = \mathcal{M}_{0,n}$

$D_4, E_6, E_8 \longrightarrow$ related to $\mathrm{Conf}(3,6)$, $\mathrm{Conf}(3,7)$, $\mathrm{Conf}(3,8)$

Grassmannian string integral

$$\int_{Gr(k,n)/T} \Omega \prod_I \Delta_I^{\alpha' s_I}$$

Plucker coord.

$Gr(k,n)/T$ "=" $Conf(k,n)$

$\rightsquigarrow Ch(k,n)_{\geq 0}$.

Positive configuration space

Cachazo - Early - Guevara - Mizera

Drummond - Foster - Gürdoğan - Kalousios

Henke - Papathanasiou

Cachazo - Rojas

Sepulveda - Guevara

$\vdots$

(Mark: applications to $N=4$ SYM?)

[Arkani-Hamed, L., Spradlin]

Theorem. $Ch(k,n)_{\geq 0}$ has a stratification by positive Chow cells and is homeomorphic to a polytope $P(k,n)$ as a stratified space.

Theorem. Positive Chow cells are indexed by:

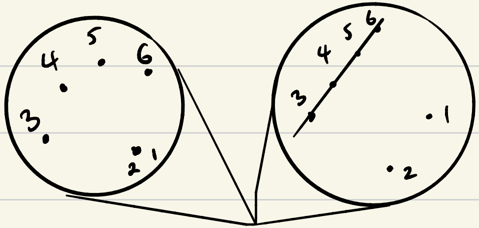
Freddy's
cf. Lauren's
James'
talks

(a) regular subdivisions of the hypersimplex into positroid polytopes
[L.-Postnikov, Early, Lukowski-Parisi-Williams, ...]

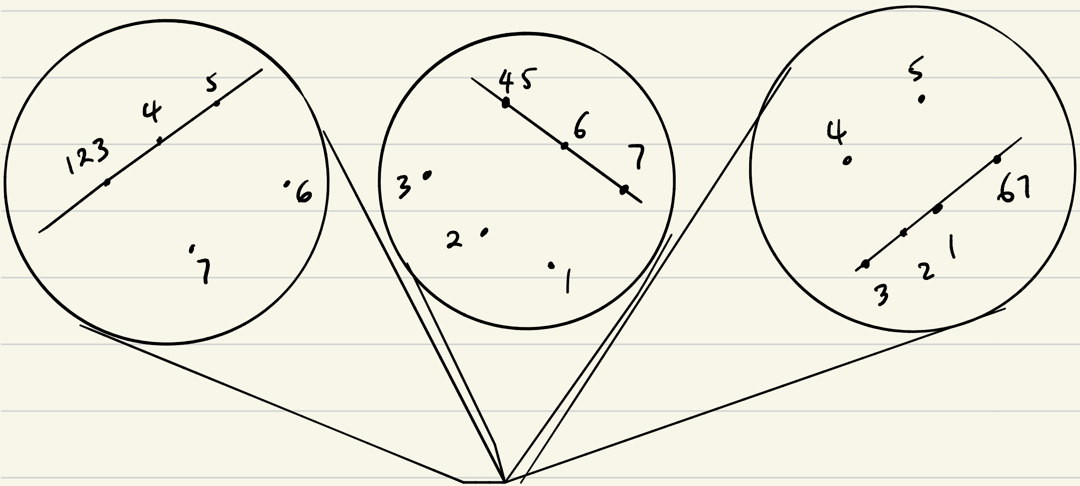
(b) cones of the positive tropical Grassmannian $\mathrm{Trop}\, Gr(k,n)_{\geq 0}$
[Speyer-Sturmfels, Speyer-Williams, ...]

(c) cones of the positive Dressian $Dr(k,n)_{\geq 0}$

[Speyer, Hermann-Jensen-Joswig-Sturmfels, Olarte-Panizzut-Schröter, ...]



$\in Ch(3,6)_{\geq 0}$



$\in Ch(3,7)_{\geq 0}$

Cluster vs. Grassmannian

(1) Polytope is simple | Polytope is not simple

(2) # u-vars = # cluster vars | # u-vars $\geq$ # Plücker variables

generators of cone of convergent $\mathcal{M}$-s

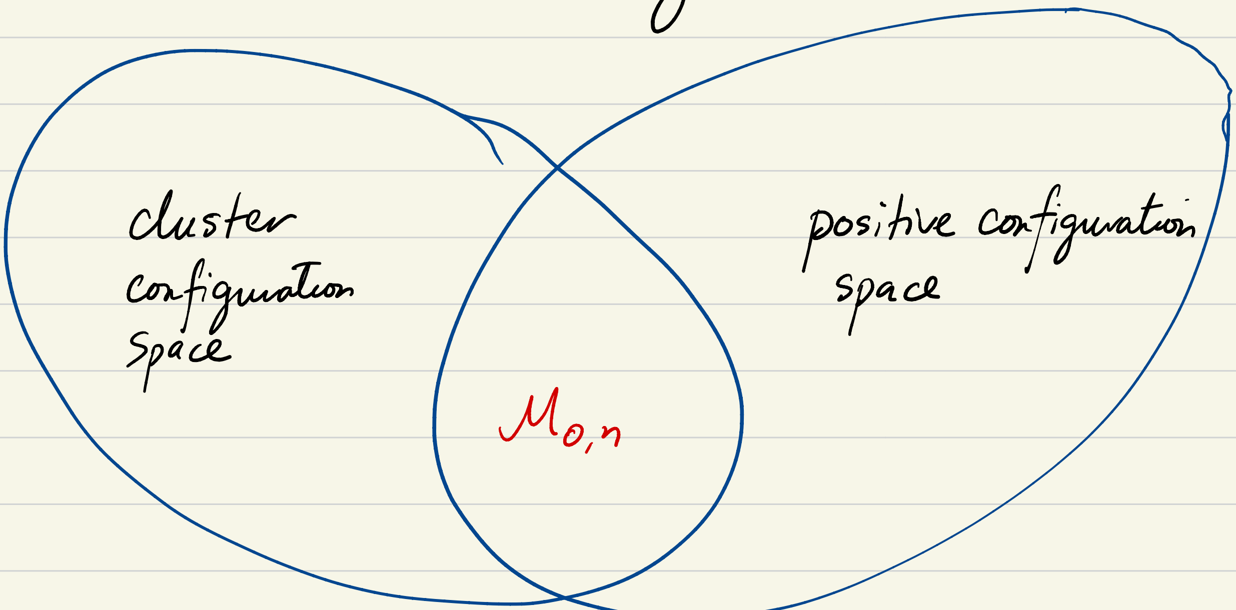
$$\int \Omega \prod u_\gamma^{\alpha' s_\gamma}$$

$s_\gamma > 0$

(3) $u_\gamma + \prod_\tau u_\tau^{(\tau \| \gamma)} = 1$ | ???

(4) $\widetilde{\mathcal{M}}_D$ is smooth and $\widetilde{\mathcal{M}}_0 \setminus \mathcal{M}_D$ has normal crossings | geometry of $\widetilde{Ch(k,n)}$?

Thank you!!


cluster configuration space
positive configuration space
$\mathcal{M}_{0,n}$