Non-Perturbative Geometries for SYM Amplitudes

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This talk: Arkani-Hamed, Lam, MS 1912.08222
and work in progress with L. Ren, A. Volovich.

Introduction and Motivation

Our work touches on several threads in the recent literature involving positive tropical Grassmannians, including applications to the physics of generalized bi-adjoint ϕ^3 theory [Cachazo, Early, Guevara Mizera 1903.08904; Cachazo, Rojas, 1906.05979; Drummond, Foster, Ö. Gürdoğan, Kalousios 1907.01053; Sepúlveda, Guevara 1909.05291; Borges, Cachazo 1910.10674; Cachazo, Guevara, Umbert, Zhang 1912.09422; Early 1912.13513; Cachazo, Early 2003.07958

mathematical work on the positive Dressian and positroidal subdivisions of the hypersimplex [Jorge Alberto Olarte, PhD Thesis; Early, 1910.11522; Łukowski, Parisi, Williams 2002.06164; Speyer, Williams 2003.10231]

and (most closely) the structure of scattering amplitudes in planar SYM theory [Drummond, Foster, Ö. Gürdoğan, Kalousios 2002.04624 and 1912.08217; Henke, Papathanasiou 1912.08254].

Introduction and Motivation

The questions that motivate our work are:

What is the natural kinematic domain for amplitudes in SYM theory?

To what extent does the mathematical structure of this space imprint on, or perhaps even determine, amplitudes?

Introduction and Motivation

Think of planar SYM theory as an "encyclopedia" filled with collections of functions with remarkable properties and interrelationships.

In fact, the properties are so remarkable that these functions "barely exist", given all the constraints they satisfy...

... to the extent that it some aspire to identify a purely mathematical problem to which these functions are the solution.

Introducing the Cast of Characters

In my talk I'll focus almost exclusively on the simplest types of amplitudes A_n called "MHV".

They are indexed by an integer $n \geq 6$ that labels the number of particles ($n = 3, 4, 5$ are special cases) and several continuous variables.

One of these variables, λ, is special and is usually used to study the perturbative series

$$A_n(\lambda) = 1 + \sum_{L=1}^{\infty} \lambda^L A_n^{(L)}$$

(which is believed to have a finite and non-zero radius of convergence) where L is called the loop order.

Introducing the Cast of Characters

The other continuous variables that the amplitude A_n depends on parameterize the space of energies and momenta of n particles, therefore called the kinematic domain.

Each (massless) particle travels at the speed of light in some particular direction in $\mathbb{R}^3$ so we need $3n$ variables

- minus 10 because of the isometry group of Minkowski space $\mathbb{R}^{1,3}$ [Lorentz; Poincaré]
- minus another 5 because of dual conformal symmetry [Drummond, Henn, Korchemsky, Sokatchev],

so the kinematic domain is $3(n-5)$-dimensional.

Kinematic Space

A convenient parameterization of this kinematic space is provided by momentum twistor variables [Hodges]:

One takes n unconstrained points Z_i in projective space $\mathbb{P}^3$ assembled into a $4 \times n$ matrix

$$Z = \begin{pmatrix} | & | & & | \\ Z_1 & Z_2 & \cdots & Z_n \\ | & | & & | \end{pmatrix}$$

modulo $PGL(4)$ acting from the left. So the kinematic domain is

$$\mathrm{Gr}(4,n)/(\mathbb{C}^*)^{n-1} \simeq \mathrm{Conf}(4,n)$$

> The amplitude A_n is a multivalued function on $\mathrm{Conf}(4,n)$.

Cluster Algebras and Amplitudes

The space Conf$(4, n)$ has the structure of a cluster Poisson variety [Gekhtman, Shapiro, Vainshtein].

> To what extent does the mathematical structure of kinematic space imprint on, or perhaps even determine, amplitudes?

The first concrete example, based on the results of [Goncharov, MS, Vergu, Volovich; Caron-Huot], was the observation in [Golden, Goncharov, MS, Vergu, Volovich] that the symbol alphabet of $A_n^{(L)}$ consists of cluster variables [Fomin, Zelevinsky] of the associated Gr$(4, n)$ cluster algebra [Scott].

This is true for all currently-known $A_n^{(L)}$, and expected to be true for all amplitudes with $n < 8$.

Indeed, the hypothesis that this is true for all amplitudes with $n < 8$ is an underlying assumption of the amplitude bootstrap program that has enabled the computation of six- and seven-particle amplitudes to high loop order.

Work of Caron-Huot, Del Duca, Dixon, Druc, Drummond, Duhr, Dulat, Liu, Marzucca, McLeod, Papathanasiou, Pennington, MS, Verbeek, von Hippel.

The S-matrix Bootstrap

Following ideas that go back to Heisenberg, a goal of the “old” S-matrix program is to be able to determine scattering amplitudes based on a few physical principles and a thorough knowledge of their analytic structure.

It is necessary to identify some “initial subset” of the kinematic domain on which one has a mechanism to provide a definition of scattering amplitudes (for example, as convergent power series expansions in the kinematic variables). Then one can (in principle) defined them throughout the kinematic domain by analytic continuation.

In the “new” approach we allow ourself the luxury of eschewing the need to worry about whether there is anything “physical” about this “initial” domain — for example, maybe it covers real energies and momenta in $(2,2)$ rather than $(3,1)$ signature.

The Landau Singularity Locus

In the case of planar SYM theory we have two strong pieces of evidence suggesting that this "new" S-matrix program is ~~guaranteed to succeed~~ not doomed to fail:

- (as already mentioned) the theory is expected to have a finite radius of convergence in λ, and
- for every n, there is a finitely generated codimension-one subvariety $\mathcal{S}_n \subset \mathrm{Conf}(4, n)$ on which all singularities of A_n must lie [Dennen, Prlina, MS, Stanojevic, Volovich; Prlina, MS, Stanojevic].

[The nontrivial content of the second statement that the subvariety does not grow without bound in complexity as L increases; there is a single $\mathcal{S}_n$ for $A_n^{(L)}$ at arbitrarily high L.]

Positive Configuration Space

The positive Grassmannian $\mathrm{Gr}(k,n)_+$ [Lusztig; Postnikov] is the open subspace of the real Grassmannian $\mathrm{Gr}(k,n,\mathbb{R})$ cut out by the condition that all Plücker coordinates are positive

$$\langle i_1 i_2 \cdots i_k \rangle = \det(Z_{i_1} Z_{i_2} \cdots Z_{i_k}) > 0 \quad \forall i_1 < i_2 < \cdots < i_k$$

Positive configuration space $\mathrm{Conf}(k,n)_+$ is the image of $\mathrm{Gr}(k,n)_+$ in $\mathrm{Conf}(k,n)$.

Central Conjecture: $\mathrm{Conf}(4,n)_+$ has empty intersection with $\mathcal{S}_n$, but its boundary lies entirely on $\mathcal{S}_n$.

In other words, amplitudes are singular everywhere on the boundary of positive configuration space, and nowhere on the interior.

Non-Perturbative Geometries

In other words, amplitudes are singular everywhere on the boundary of positive configuration space, and nowhere on the interior.

These words are very reminiscent of amplituhedra [Arkani-Hamed, Trnka]: The integrand of the amplitude $A_n^{(L)}$ is a rational form that is positive everywhere inside, and has logarithmic (dx/x) singularities on (and only on) the boundaries of the amplithedron.

The main difference here is that we have a *single* positive configuration space, not a different space for each loop order.

So, the positive configuration space $\mathrm{Conf}(4, n)_+$ is a candidate non-perturbative geometry for SYM theory.

The Boundary of Positive Configuration Space

However: Because, as mentioned above, we are interested in a thorough knowledge of the analytic structure of amplitudes, and because their singularities can only occur on the boundary (or outside!) of $\mathrm{Conf}(4, n)_+$, we are particularly keen to understand its boundary structure.

That is tantamount to identifying a suitable closure or compactification of $\mathrm{Conf}(4, n)_+$.

Moreover, motivated by similar problems that arise in physics (the open string moduli space), we are motivated to seek polytopal realizations of these compactifications, which manifest all of their boundary structure.

Example: $n = 6$

Let's use coordinates

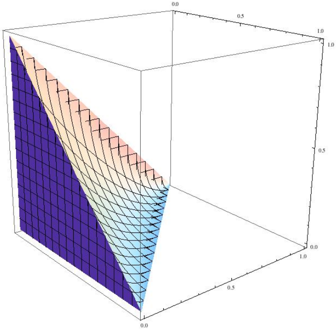
$$u_2 = \frac{\langle 1234\rangle\langle 1456\rangle}{\langle 1245\rangle\langle 1346\rangle},\ u_3 = \frac{\langle 1256\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 2356\rangle},\ u_1 = \frac{\langle 1236\rangle\langle 3456\rangle}{\langle 1346\rangle\langle 2356\rangle}$$

commonly used in the hexagon bootstrap literature.

In terms of these, positive configuration space is the part of the (u_1, u_2, u_3) unit hypercube satisfying

$$1 - u_1 - u_2 - u_3 > 2\sqrt{u_1 u_2 u_3}$$

Positive Domain for $n = 6$



The positive domain inside the (u_1, u_2, u_3) unit hypercube.

The tip at $(0, 0, 0)$ is the "origin" of six-particle kinematics studied in [Basso, Dixon, Papathanasiou].

The point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is discussed in `#heptagons`.

Blowup

This picture is perhaps fine if, for example, the class of functions you are intereseted in consists only of polynomials in the u_i. However, consider the one-parameter family

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & -t & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

which lies in $\mathrm{Conf}(4,6)_+$ for $t \in (0,1)$.
For all t, these points lie together at the collinear point

$$(u_1, u_2, u_3) = (1, 0, 0)$$

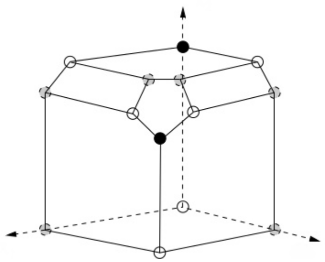
but there are other coordinates which separate them; for instance

$$\frac{\langle 1234\rangle\langle 3456\rangle}{\langle 1346\rangle\langle 2345\rangle} = \frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle} = \frac{\langle 1235\rangle\langle 2456\rangle}{\langle 1246\rangle\langle 2345\rangle} = \frac{t}{1-t}$$

Blowup

In this case you expose an entire line segment "hidden" at the point $(1, 0, 0)$.

The most natural blowup of the full space (the Deligne-Mumford compactification of $\mathcal{M}_{0,6}$), obtained by considering arbitrary cross-ratios, gives the $\mathrm{Gr}(2, 6)$ (or A_3) cluster polytope



[see F. Brown, from which this figure was taken.] The two black vertices are the two "origins", related to each other by the cyclic transformation $Z_i \to Z_{i+3}$.

Seeking Candidate Geometries

Which (if any) compactification of $\text{Conf}(4, n)_+$ underlies the structure of the amplitude A_n?

If we knew A_n, we could, in principle, read off the answer ...

... but instead, we look at some mathematically natural constructions and see if they are promising candidates.

In 1912.08222 we explored candidates that are naturally constructed using "stringy canonical forms" [Arkani-Hamed, He, Lam] and are dual to (coarsenings of) certain fans associated to the positive tropical Grassmannian [Speyer, Williams].

Polytope Construction

Let $Z(x_1, \ldots, x_d)$ be a parameterization of $\mathrm{Conf}(4, n)_+$ in terms of the cluster variables $x_1, \ldots, x_d$ of the initial cluster of $\mathrm{Gr}(4, n)$. (Here $d = 3(n-5)$.) [Postnikov]

For a Plücker coordinate P evaluated on $Z(x_1, \ldots, x_d)$, let $\mathrm{Newt}(P)$ be its Newton polytope in $\mathbb{R}^d$ with respect to $(x_1, \ldots, x_d)$.

Define the polytope $C^\dagger(4, n)$ as the Minkowski sum of all $\mathrm{Newt}(P)$'s for P's of the form

$$\langle i\, i{+}1\, j\, j{+}1 \rangle \quad \text{or} \quad \langle i\, j{-}1, j\, j{+}1 \rangle.$$

(Using *all* Plücker coordinates would give the dual of the Speyer-Williams fan.)

Polytope Data

This gives polytopes with f-vectors

$$
\begin{aligned}
(4,6): &\quad (1, 14, 21, 9, 1) \\
(4,7): &\quad (1, 595, 1918, 2373, 1393, 385, 42, 1) \\
(4,8): &\quad (1, 49000, 249306, 536960, 635176, 447284, \\
&\qquad 189564, 46312, 5782, 274, 1)
\end{aligned}
$$

Choosing the cluster parameterization on the previous slide ties the geometry of the polytope $C^{\dagger}(4, n)$ to the associated $\mathrm{Gr}(4, n)$ cluster algebra.

For the finite cases $n = 6, 7$, the normal rays to the facets of the polytope are generated by the g-vectors [Fomin, Zelevinsky IV] of the cluster algebra.

As previously noted, the corresponding cluster variables constitute (conjecturally) the complete symbol alphabets for six- and seven-particle amplitudes.

Exceptional Rays

In the case $(4, 8)$ where the cluster algebra is infinite, only 272 normal rays are generated by g-vectors, 2 are not. These are (in our conventions)

$$\mathbf{v} = (-1, 1, 0, 1, 0, -1, 0, -1, 1)$$

and its cyclic partner. See also [Drummond, Foster, Ö. Gürdoğan, Kalousios; Henke, Papathanasiou].

We can check that this is not associated to a cluster variable using a recent algorithm [Chang, Duan, Fraser, Li] that computes, for each lattice point $p \in \mathbb{Z}^d$, a (Lusztig) canonical basis element $\mathcal{B}(p)$.

$\Longrightarrow$ $\mathcal{B}(p)$ is a cluster monomial if and only if $\mathcal{B}(2p) = \mathcal{B}(p)^2$.

CDFL noted that the $\mathbf{v}$ given above fails this test.

A Quadratic Generating Function

Using the CDFL character formula one can (in principle) compute a basis element associated to each integer point along the ray generated by $\mathbf{v}$:

$$\mathcal{B}(1\mathbf{v}) = A$$
$$\mathcal{B}(2\mathbf{v}) = A^2 - B$$
$$\mathcal{B}(3\mathbf{v}) = A^3 - 2AB$$

(first two from CDFL, third in 1912.08222) where

$$A = \langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle$$
$$B = \langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

Conjecture (1912.08222): these are generated by

$$\sum_{s\geq 0} t^s \mathcal{B}(s\mathbf{v}) = \frac{1}{1 - At + Bt^2}$$

which, in particular, is a rational function of t. (Hard to test further because the computational complexity is $\mathcal{O}((4s)!)$.)

Four-Mass Box

The roots of this quadratic denominator $1 - At + Bt^2$ involve precise the square roots $A \pm \sqrt{A^2 - 4B}$ of four-mass box type that are known to appear in the symbols of eight-particle amplitudes [Britto, Cachazo, Feng].

This provides some evidence supporting the idea that these polytopes have some direct connection to the structure of amplitudes.

Drummond, Foster, Ö. Gürdoğan, Kalousios presented a construction that precisely reproduces all symbol letters of all currently-known eight-point amplitudes, including those of the two-loop MHV [Caron-Huot] (which has only rational letters) and NMHV [Zhang, Li, He] (which has 18 algebraic letters) amplitudes.

MHV versus non-MHV Regions

Why do we expect the "MHV region" that I have been describing to know about the singularities of non-MHV amplitudes?

Because if you work to all loop order, they are the same: MHV amplitudes are usually missing certain singularities at low loop order due to accidental cancellations, but any singularity of any n-particle L-loop non-MHV ampiltude is guaranteed to show up in the MHV amplitude $A_n^{(L')}$ for some sufficiently high (but still finite) loop order L'.

It is an interesting open problem to study the boundary of non-MHV regions. Only for $n = 6$ do we know the answer: it has the topology of a solid torus that surrounds the MHV region, touching it along the three square facets [Arkani-Hamed talk at Amplitudes 2018].

Wrapping Up

Studying amplitudes in SYM theory to all orders in perturbation theory is a timely problem that is receiving increasing attention.

(See for example [Caron-Huot, Dixon, von Hippel, McLeod, Papathanasiou 1806.01361, Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 1912.00188, Basso, Dixon, Papathanasiou 2001.05460].

A suitable compactification of the positive configuration space $\mathrm{Conf}(4, n)_+$ will play a starring, but still far from completely understood, role in solving this problem.

Conclusion

To do:

- Understand which choice of compactification (or, in the dual language, which tropical fan) “most faithfully” exhibits the properties of amplitudes, and in what precise manner.
- Harness this knowledge to learn more about amplitudes, or compute new ones!
- Is there some canonical way in which these spaces beg to have certain functions naturally associated to them (in a manner analogous to the way that positive geometries have naturally associated canonical forms [Arkani-Hamed, Bai, Lam]), such that those functions turn out to precisely be the amplitudes of SYM theory?

Big mystery: what is the role or origin of λ in this story?