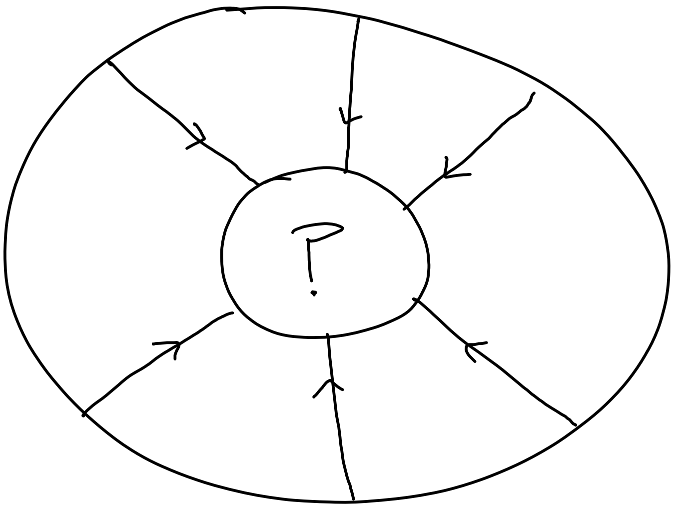
Scattering Amplitudes

and

Clusterhedra in Kinematic Space

w/ S. He, T. Lam, M. Spradlin;
H. Frost, P-G. Plamondon, H. Thomas + G. Salvatori



What is Q in:
Kinematic Space of
on-shell data labelling
scattering process,
to which A(mplitude)
is the Answer?

Planar [$\mathcal{N}=4$ S] YM Kin. Space

$$Z = m{=}4 \begin{pmatrix} Z_1 & \cdots & Z_n \end{pmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \tilde{\lambda}_1 & & \tilde{\lambda}_n \end{bmatrix} 4$$

* Integrand: $+AB_1, \ldots, AB_L$

$G_k[m, n, L]$

"Winding" def. of Amplitudehedron in Kin Space.

$\Omega_{4(k+L)}$: Amp Form.

* Non-pert geometry?

$G_k(m,n)$ / Little Group T

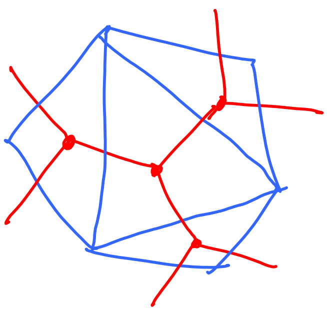
$\downarrow (k=0, \ldots)$

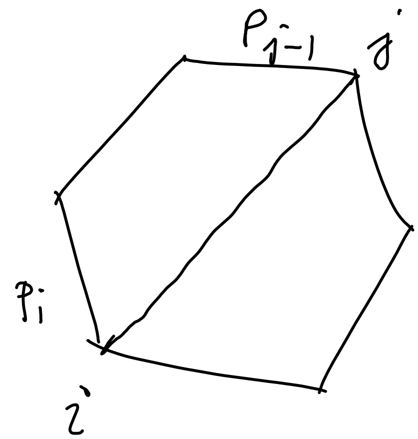
$G_+(4,n)/T$

Taming Cluster Infinity

$\phi^3_{B\text{-}A}$

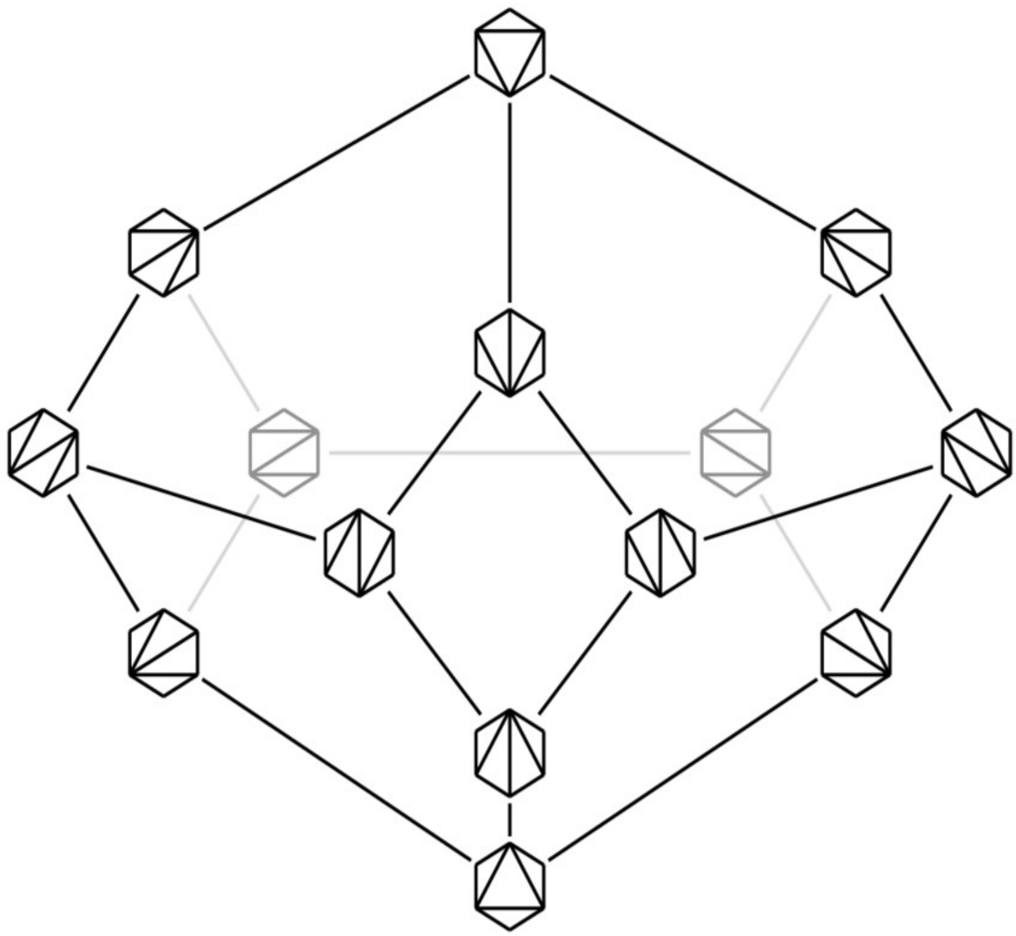
* Tree-level



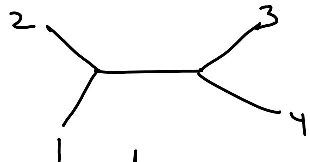
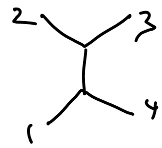


$$X_{ij} = (p_i + \cdots + p_{j-1})^2 = X_{ji}$$

$$[X_{i\,i+1} = 0]$$

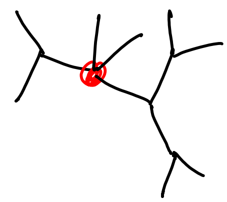


Hidden Symmetry: Projective Invariance

$n=4$  

$$\frac{dX_{13}}{X_{13}} - \frac{dX_{24}}{X_{24}} = d\log X_{13}/X_{24}.$$

General n: invariance of $\Omega_{(n-3)}$ under $X_{ij} \to \alpha(X) X_{ij}$

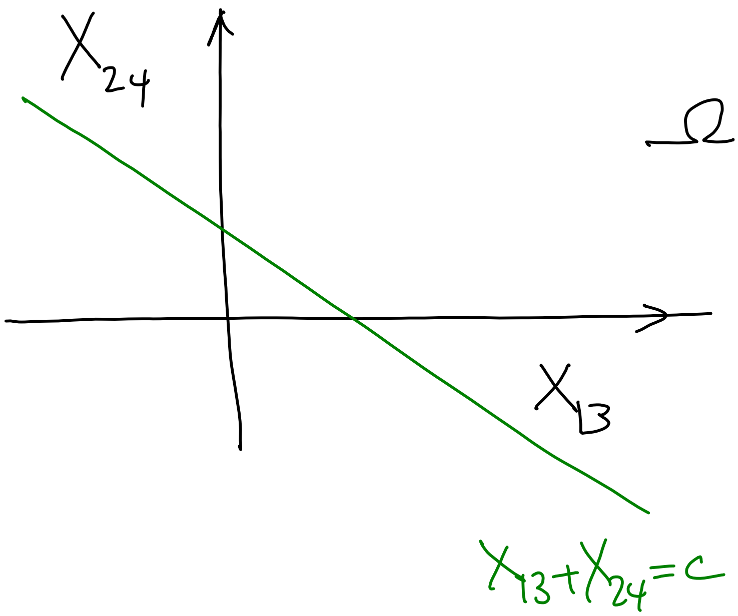
$\Longrightarrow d\log X_{ij} \to d\log X_{ij} + d\log\alpha \longrightarrow$ for each set of $(n-4)$ comp. propagators coeff. of $d\log\alpha$.  vanishes.

Many more eqns than unknown signs.

Remarkably: signs can be chosen so $\Omega_{(n-3)}$ is projective!

[Analog of Dual Conf Inv. in $\mathcal{N}=4$ SYM]

Pullback to Amplitude

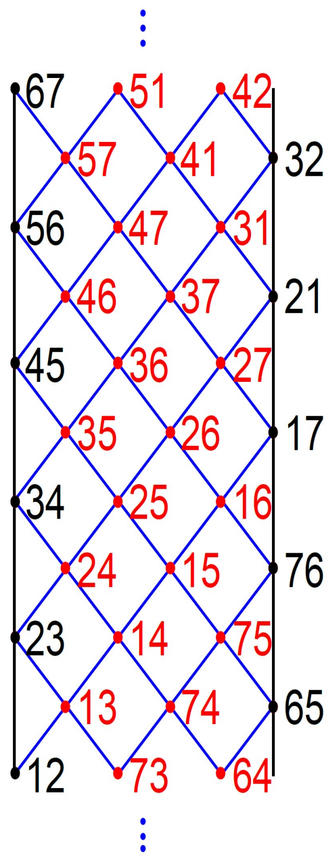


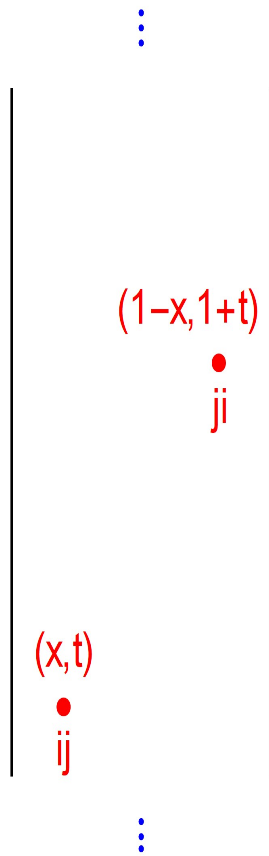
$$\Omega\Big| = dX_{13}\left(\frac{1}{X_{13}}+\frac{1}{X_{24}}\right)$$

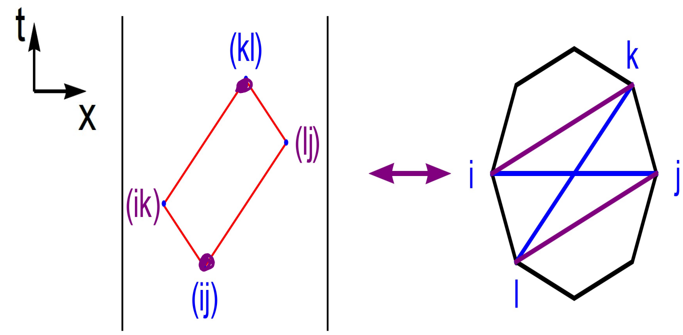
General n: $\Omega\Big|_{ABHY} = d^{(n-3)}X \cdot (\text{Amplitude})$

[All + signs]

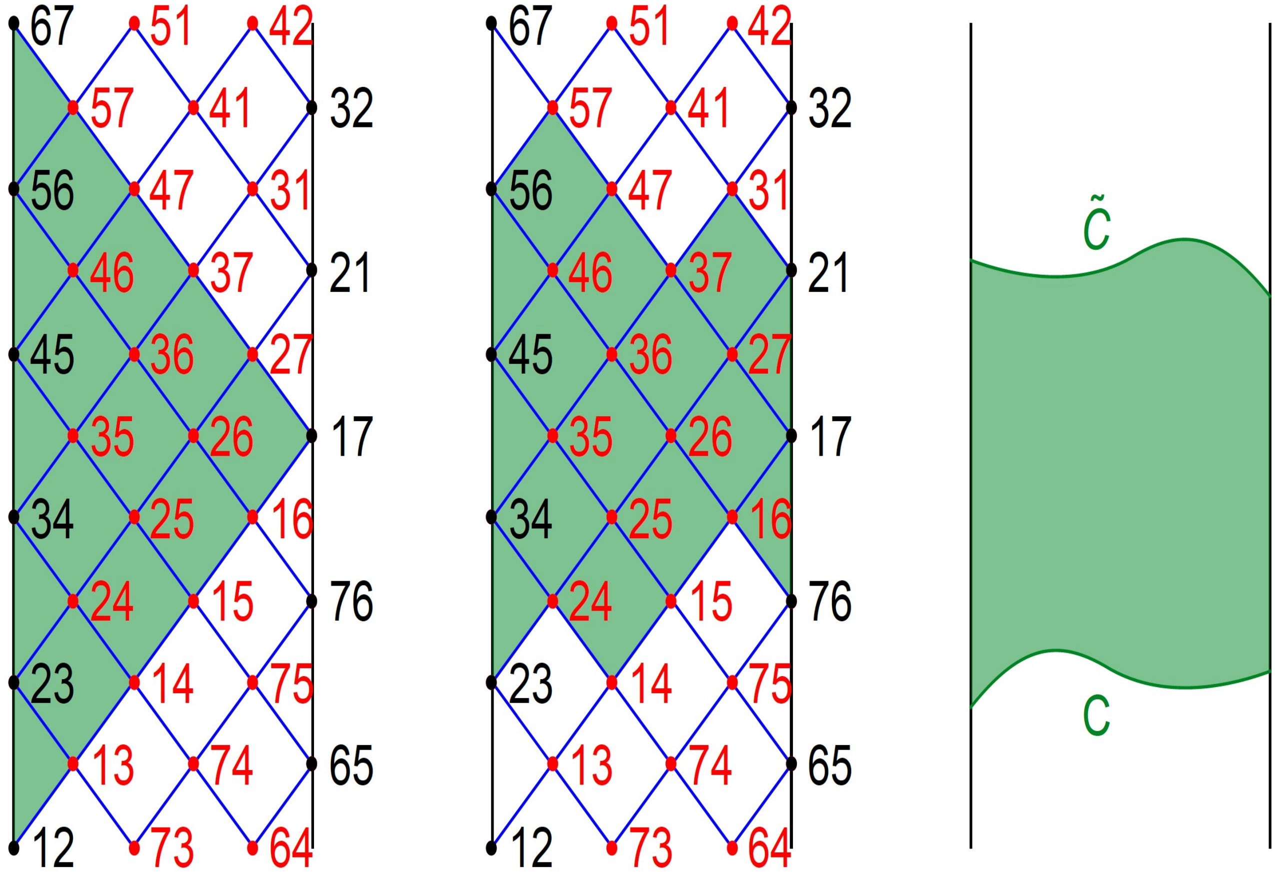
Kinematic "Spacetime"







P, F corners of causal diamond $\longleftrightarrow$ incomp. chords


67 51 42
57 41 32
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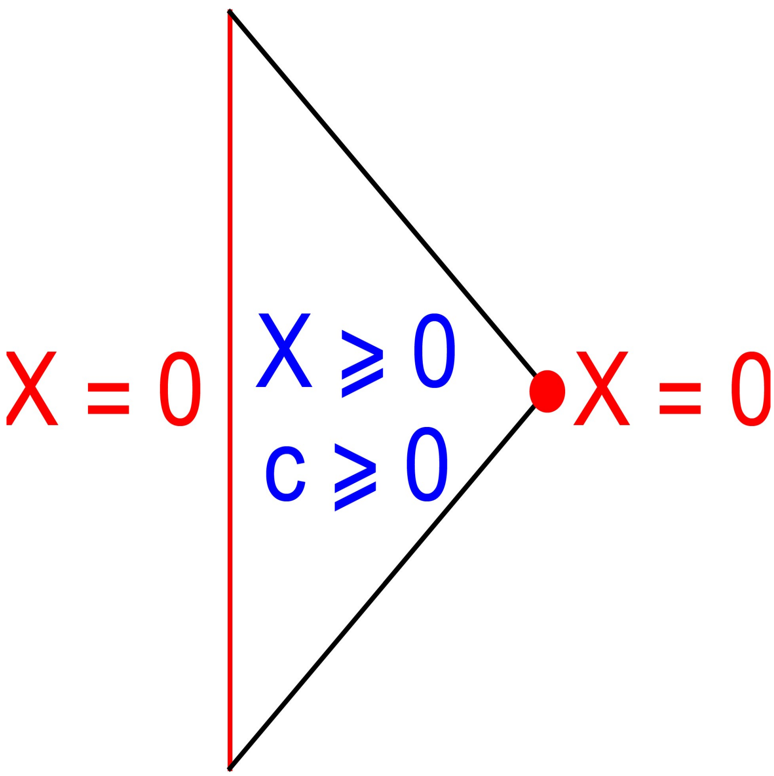
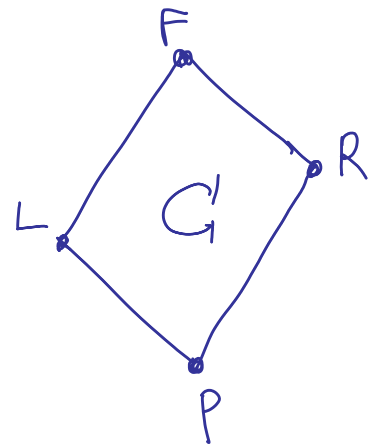
46 37 21
45 36 27
35 26 17
34 25 16
24 15 76
23 14 75
13 74 65
12 73 64
67 51 42
57 41 32
56 47 31
46 37 21
45 36 27
35 26 17
34 25 16
24 15 76
23 14 75
13 74 65
12 73 64
$\tilde{C}$
C

Positive Wave Equation

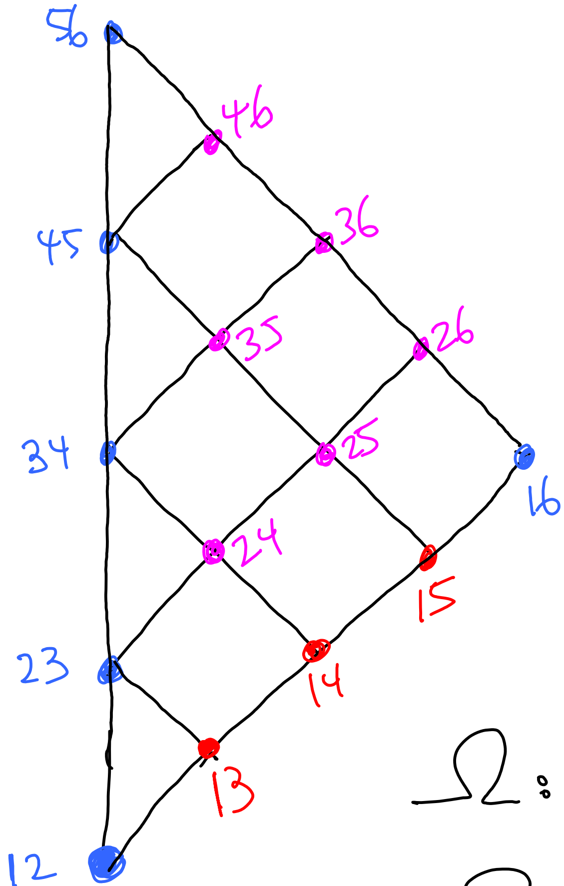
$$\partial_u \partial_v X = c$$

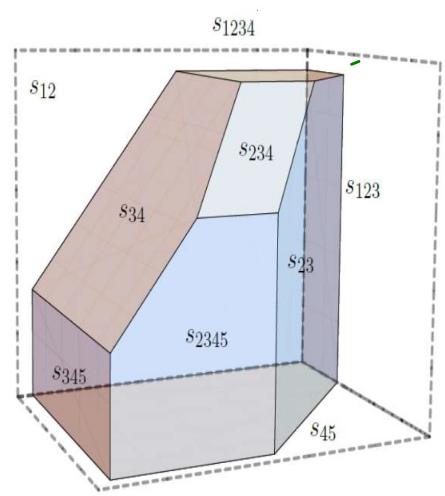




$$X_P + X_F - X_L - X_R = C$$

Positive WE $\longrightarrow$ ABHY Associahedron in Kinematic Space.





Ω: Fully determined by

$$\Omega\big|_{WE} = \omega^{can}[\text{Assoc.}]$$

Some Cluster Algebra Basics

- Quiver

$$\begin{tikzcd}
x_1 \arrow[dr] & & & x_5 \\
& x_3 \arrow[dl] & x_4 \arrow[l, shift left] \arrow[l, shift right] \arrow[ur] & \\
x_2 \arrow[urr] & & &
\end{tikzcd}$$

- "Mutation" @ v

1) Reverse all arrow
2) If $a \to v \to b$ add $a \to b$.
3) Delete $\rightleftarrows$

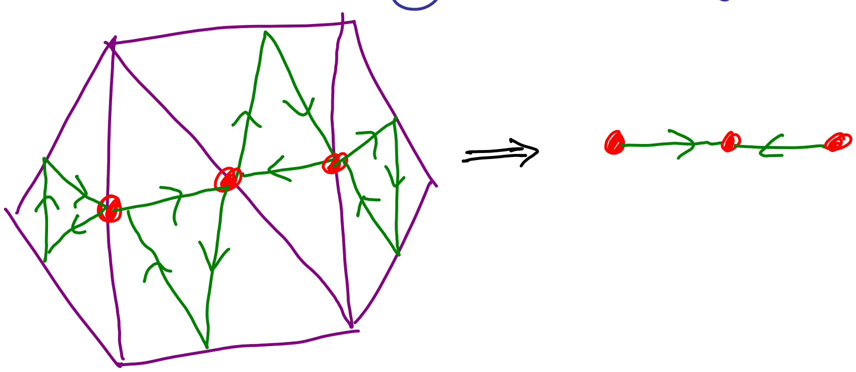
- $x_v x_{v'} = \prod_{w \to v} x_w + \prod_{w \leftarrow v} x_w$

- Also "tropicalized", pair:

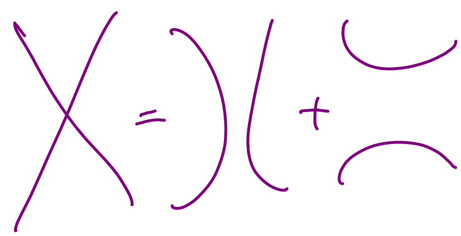
$$\left\{ X_v + X_{v'} - \sum_{w \leftarrow v} X_w \,, \quad X_v + X_{v'} - \sum_{w \to v} X_w \right\}$$

Surface Cluster Algebras

* Quivers from triangulations of surface



* Mutation $\longleftrightarrow$ Skein relations :



g-vectors

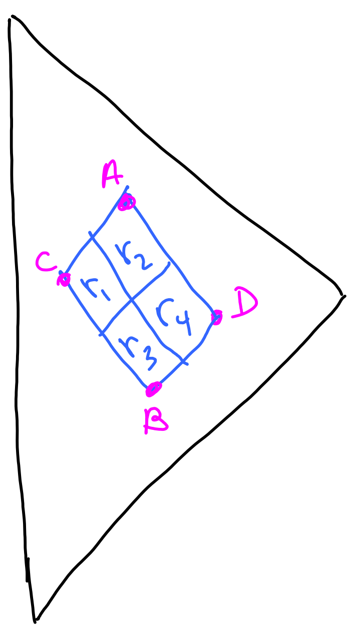
- Canonical way of labelling cluster variables, given choice of initial cluster $(x_1, \ldots, x_n)$.
- For each mutation μ, there is one of
$$\left\{ X_v + X_{v'} - \sum_{w \to v} X_w,\ X_v + X_{v'} - \sum_{w \leftarrow v} X_w \right\}$$
that is satisfied by (+ determines) g-vectors.
- We'll call this set of relations [again, dep. on choice of initial cluster], $\{R_\mu\}$.

Emergence of "Kinematic Spacetime" from Cluster Algebra.

Consider all "$X + X - X - \cdots$" relations $\{R_\mu\}$.

There is a tiny subset of them R_{μ_*} that generate all the rest by positive lin combination:

$R_\mu = \sum c_{\mu,\mu_*} R_{\mu_*}$. R_{μ_*} are "pixels" or "meshes" in "Kin Spacetime".

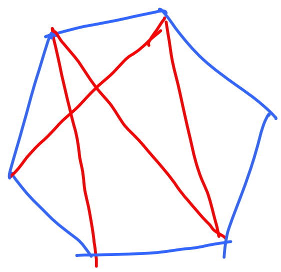


Mut. rel. $A + B - C - D$

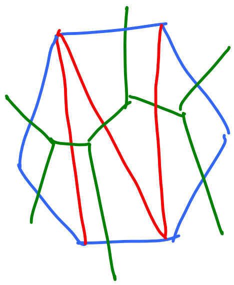
$= r_1 + r_2 + r_3 + r_4$

Understood for all finite-type cluster algebras

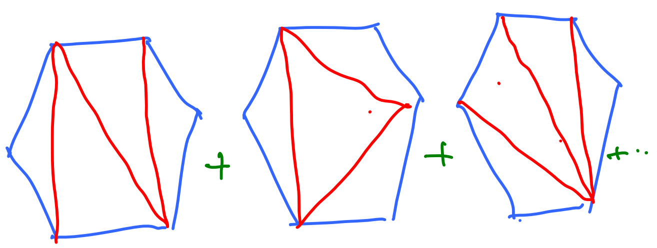
Spacetime + QM $\leftrightarrow$ Cluster g-vector Cones + Fan



Propagators X_{ij}

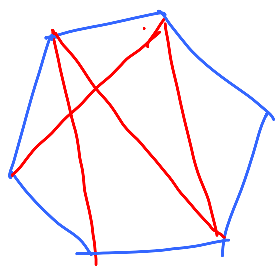


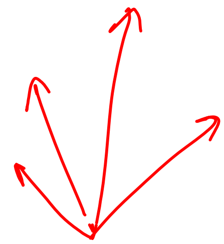
Spacetime process



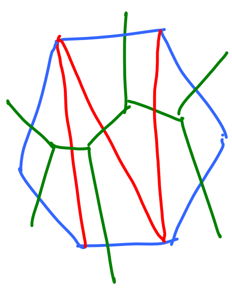
Add an amplitude for all processes with "+1", because of QM

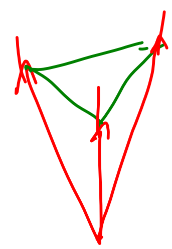
Spacetime + QM $\leftrightarrow$ Cluster g-vector Cones + Fan



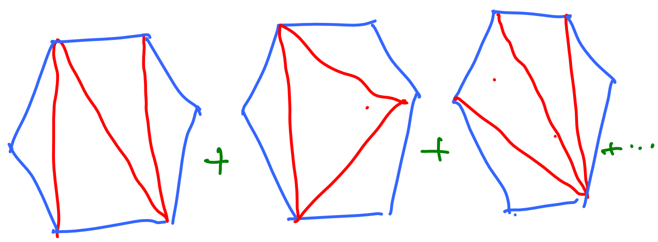


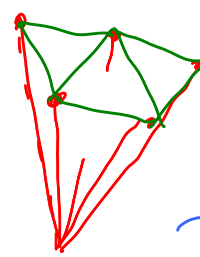
g-vectors.





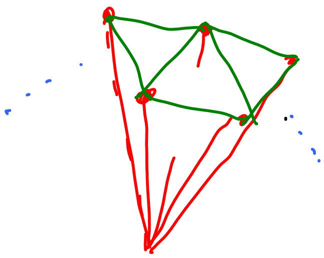
g-vector cone





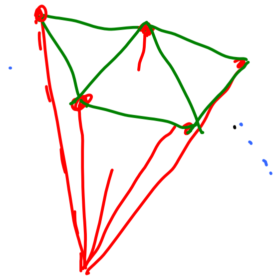
g-vector cones tile a fan

Amazing + Deep Fact!



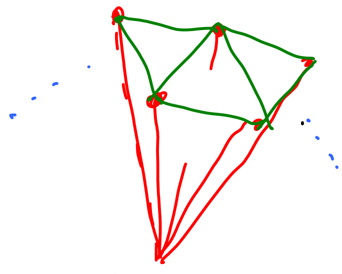
$$\Omega = \sum \operatorname{sgn}\left| g_{a_1} \cdots g_{a_n} \right| \, d\log X_{a_1} \cdots d\log X_{a_n}$$

is projective invariant



$$\Longrightarrow \quad \Omega\Big|_{X_a = g_a^{\,i} X_i + k_a} = \left(\prod_i dX_i\right)$$

$$\times \underbrace{\sum_{\text{diag}} (+1) \prod \frac{1}{X's}}_{\text{Amplitude}}$$

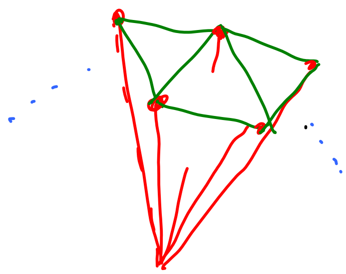


$$\Longrightarrow \quad \Omega\Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i\right)$$

$$\times \sum_{\text{diag}} +1 \cdot \prod \frac{1}{X's}$$

Because

$$\left| g_{a_1} \cdots g_{a_m} \right| = \pm 1 \; !$$

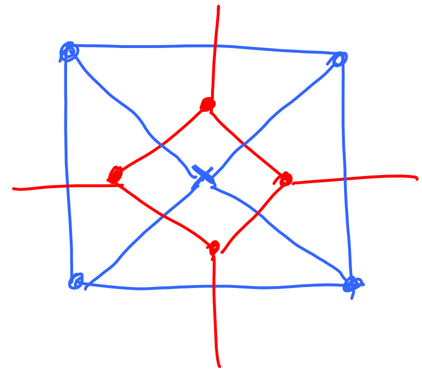


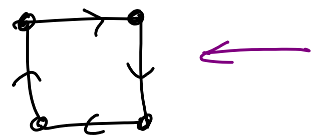
$$\Rightarrow \quad \Omega\Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i\right)$$

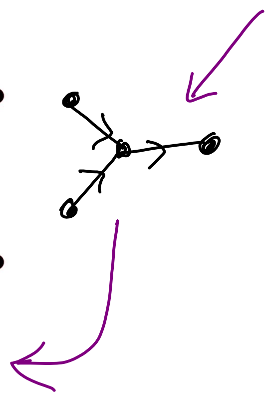
[g-vectors are normal to facets of polytope] $\times \sum_{\text{diag}} \pm 1 \, \prod \frac{1}{X's}$

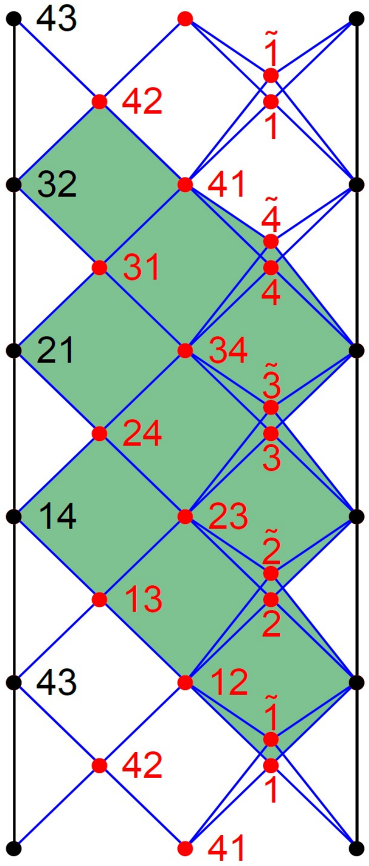
ABHY construction automatically orders k_a such that Polytope given by $X_A \geq 0$ has correct shape!

D_n = 1-loop








43
42
1̃
1
32
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31
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1
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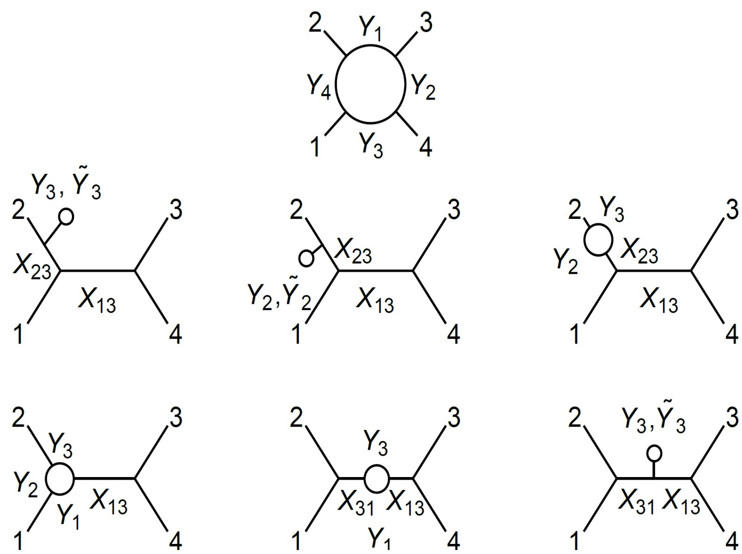
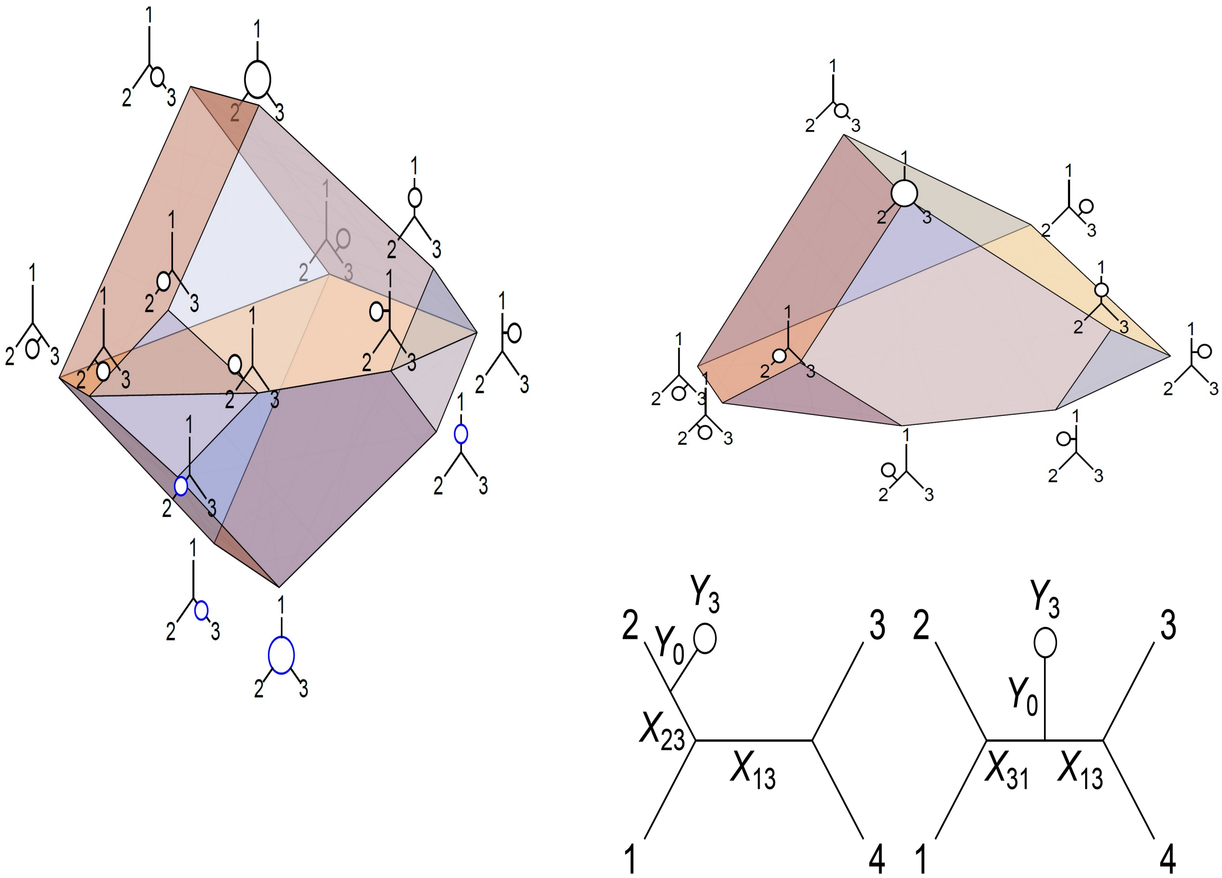
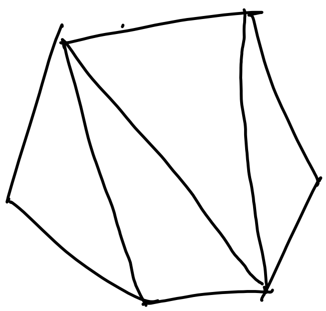

Y_1 Y_2 Y_3 Y_4
$Y_3, \tilde{Y}_3$ X_{23} X_{13}
$Y_2, \tilde{Y}_2$ X_{23} X_{13}
Y_2 Y_3 X_{23} X_{13}
Y_2 Y_3 Y_1 X_{13}
Y_3 X_{31} X_{13} Y_1
$Y_3, \tilde{Y}_3$ X_{31} X_{13}

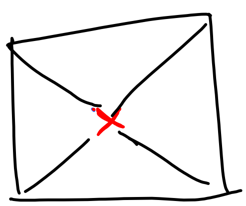


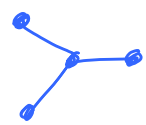
Y_3
Y_0
X_{23}
X_{13}
X_{31}
1
2
3
4

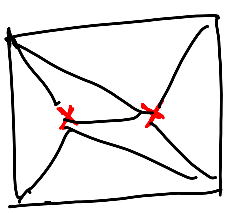
All-loop ϕ^3_{BA}

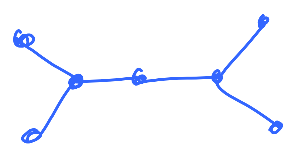


trees $\longrightarrow$  A_{n-3}



1-loop $\longrightarrow$  D_n

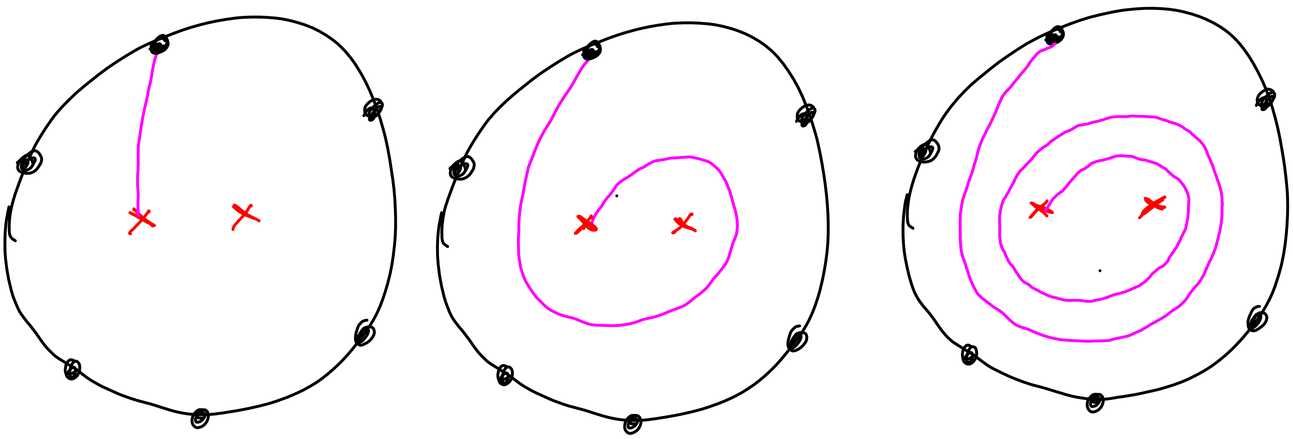


2-loop $\longrightarrow$  $\tilde{D}_{n+2}$

infinite type

Amusing we have to "tame infinity" for $\begin{cases} \mathcal{N}=4 \quad G(4,n)/T \\ \phi^3_{L\geq 2} \end{cases}$

Infinite Winding



* Infinite action of Mapping class groups, Braid L
* Also, g-vector cones still fit snugly together, but leave lower-codim. holes in the space

[$\Longrightarrow$ Projective invariance works as "telescopic cancellation" but we have to "compactify" the infinite space, somehow].

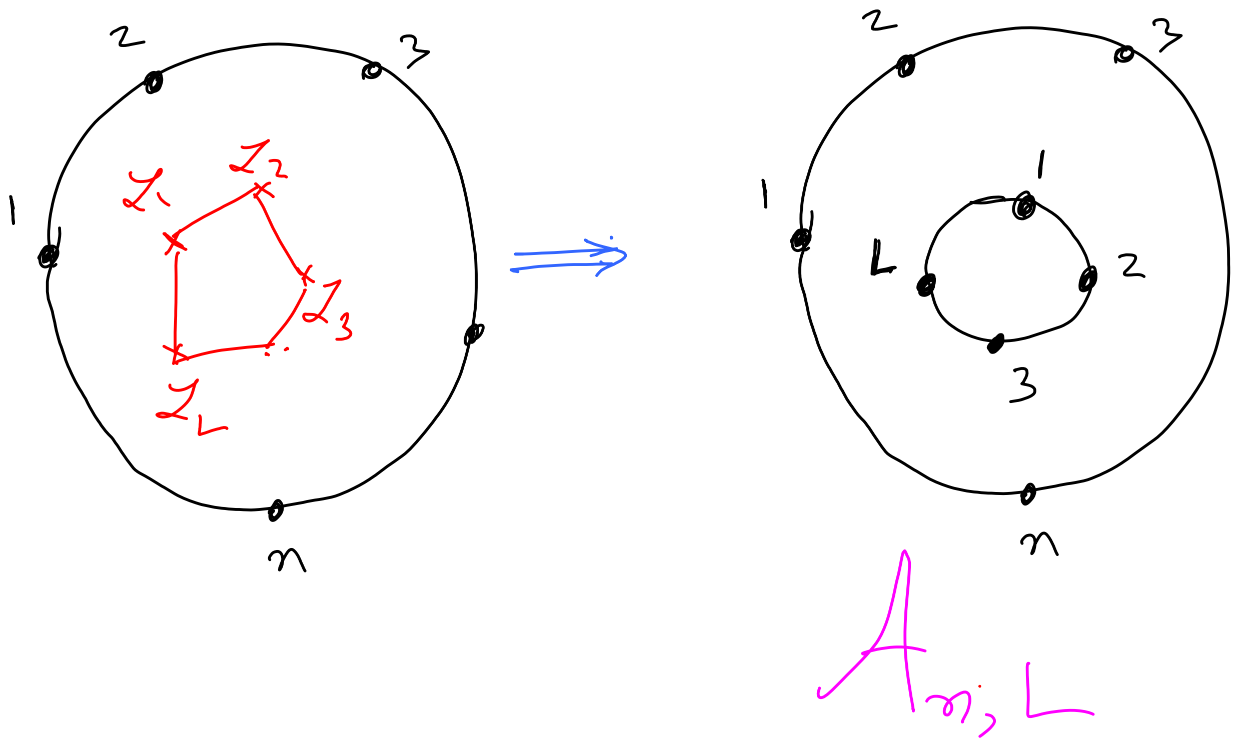
"Clusterhedra"

Are the polytopes where:

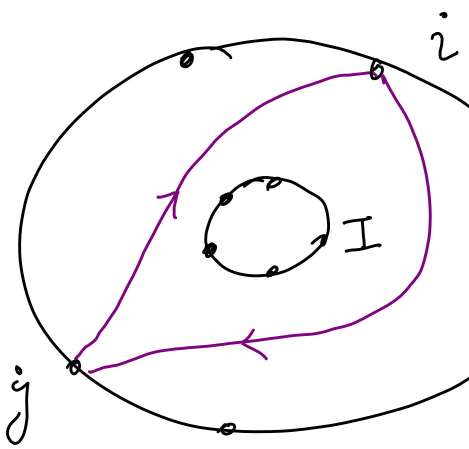
* Vertices unambiguously associated w/ diagrams/clusters.
* g-vectors are (among) facet normals.
* Projective [trivial]
* Pull back to amplitude with all +1's (= all facet normal dets $= \pm 1$, non-trivial!)

* At loop level, "Kinematic Spacetime" is richer — and we will find Clusterhedra canonically living there!

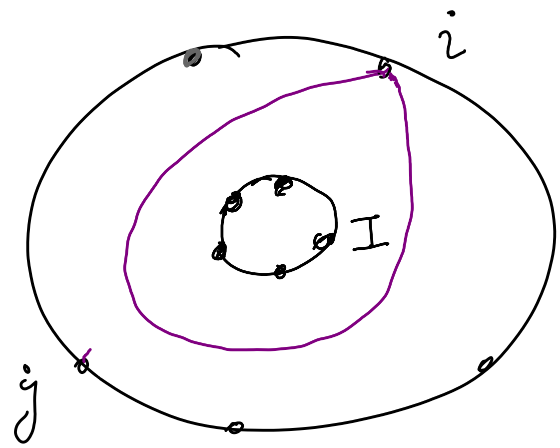
Class of All-loop Cuts



Variables for $A_{n,L}$



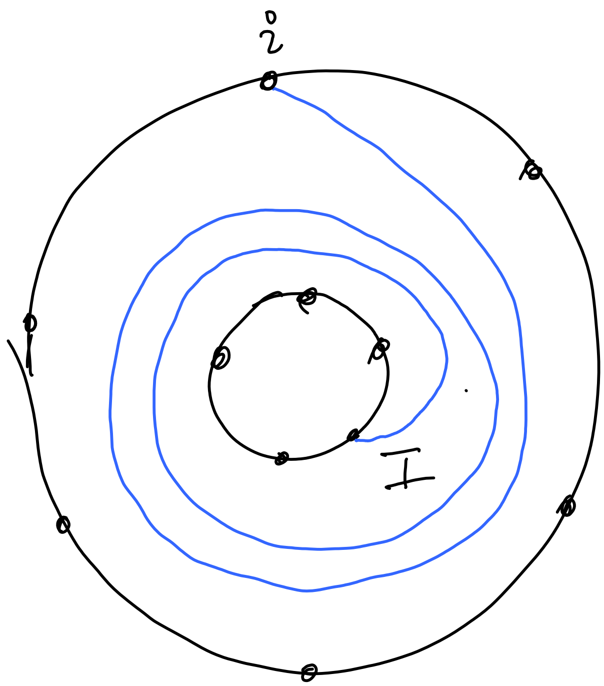
$X_{ij} \neq X_{ji}$



$X_{i\,i+n}$

"Out/Out" or "In/In"

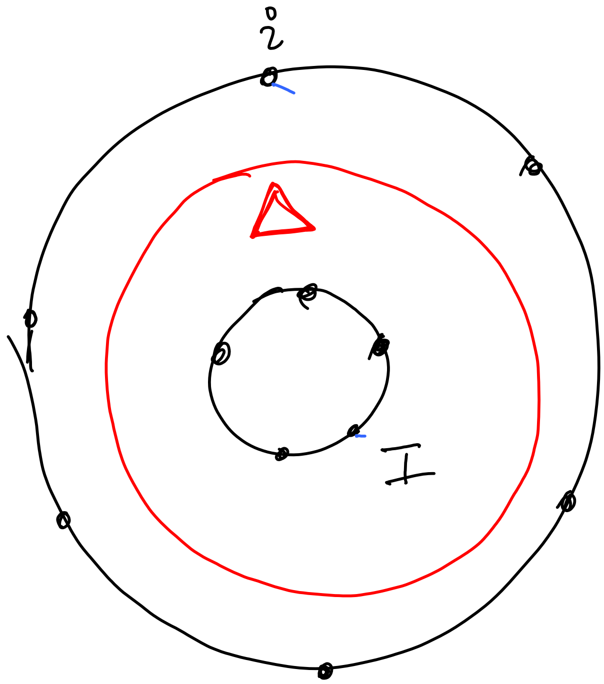
Variables for $\mathcal{A}_{n,L}$



$$X^{\omega}_{iI}$$

"In – Out"

Variables for $A_{n,L}$



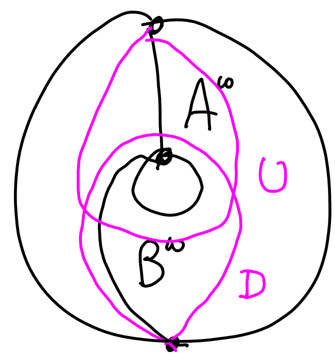
$$\left\{ \begin{array}{l} X^{+\omega} + \Delta = X^{+\omega+1} \\ X^{-\omega} + \Delta = X^{-\omega-1} \end{array} \right\}$$

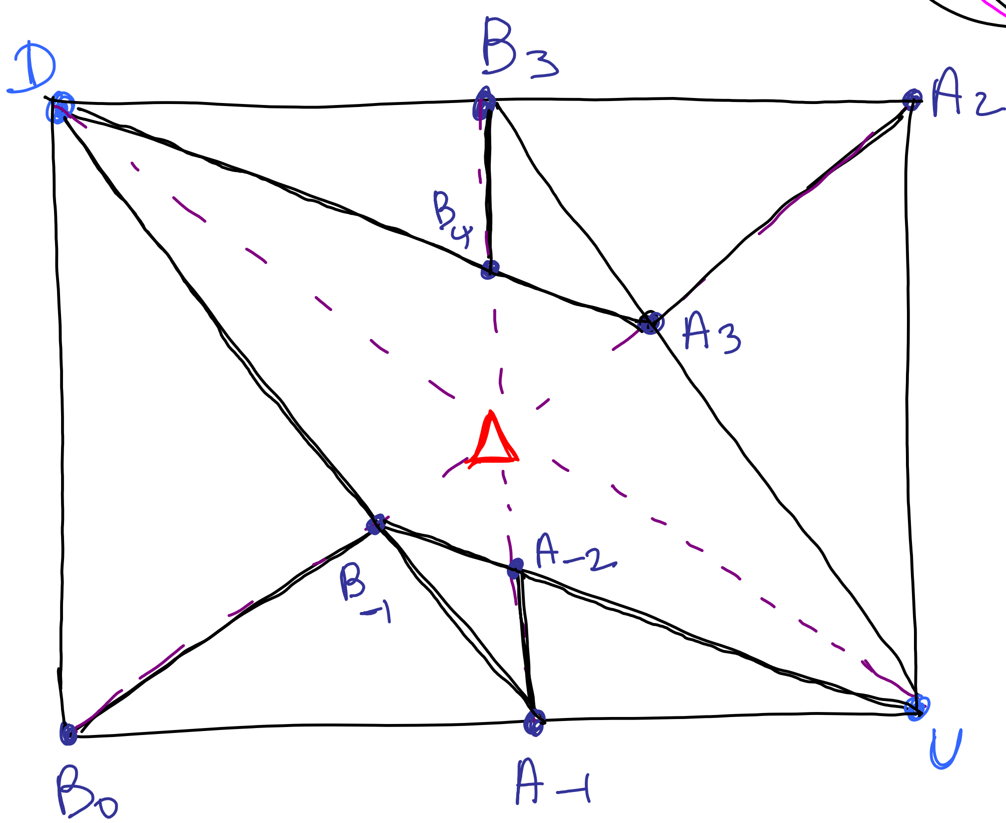
Δ : Not Cluster Variable, but crucial part of story.

[Also starred as the $\sqrt{\ }$ of the 4-mass box in $\mathcal{N}=4$ story!]

The g-vectors asymptote to Δ

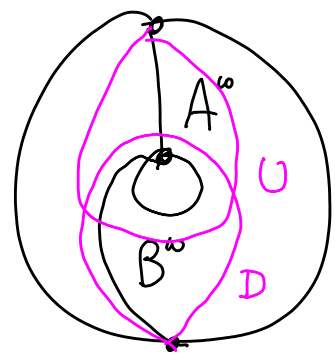
For example in $A_{2,1}$:

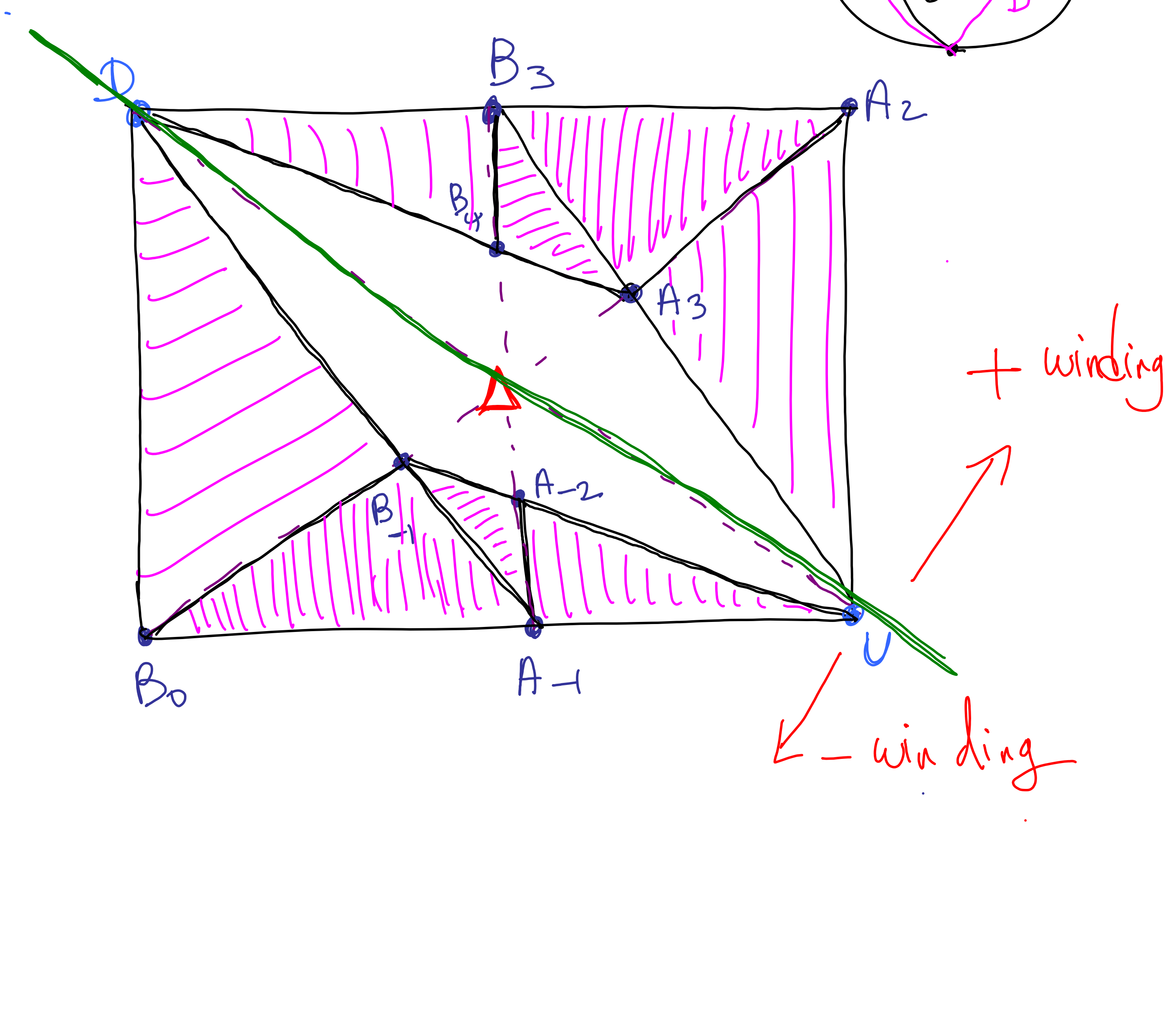




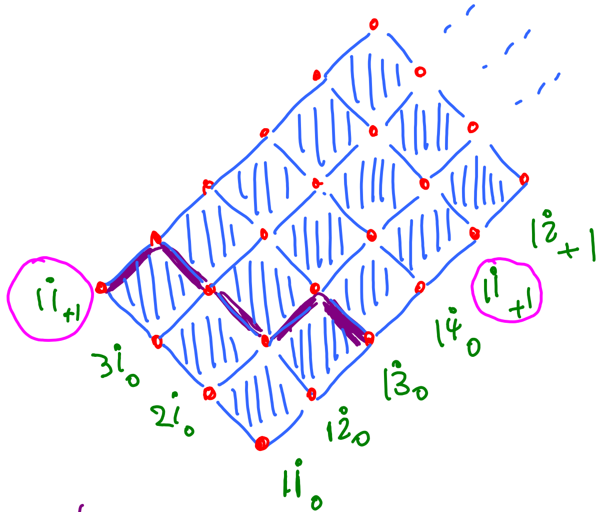
The g-vectors asymptote to Δ

For example in $A_{2,1}$:

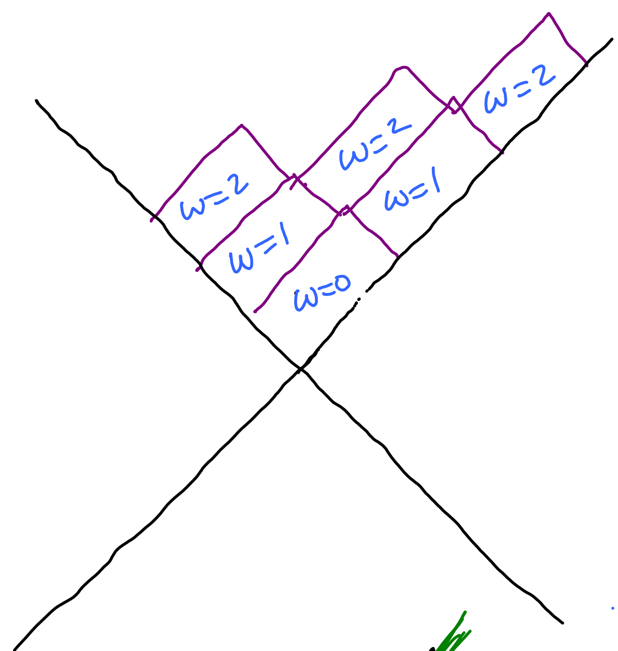




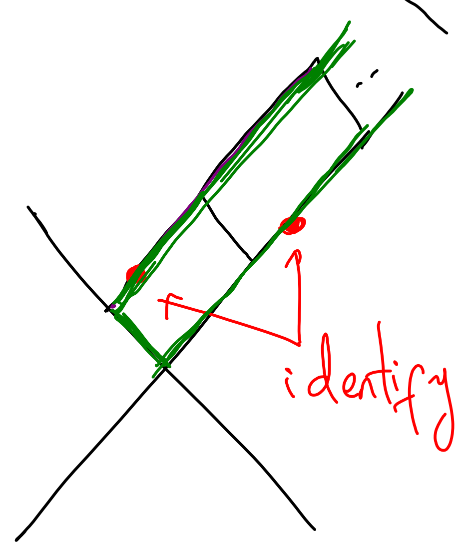
In–Out "Kinematical Spacetime"



Initial "Cauchy Surface" set by initial cluster

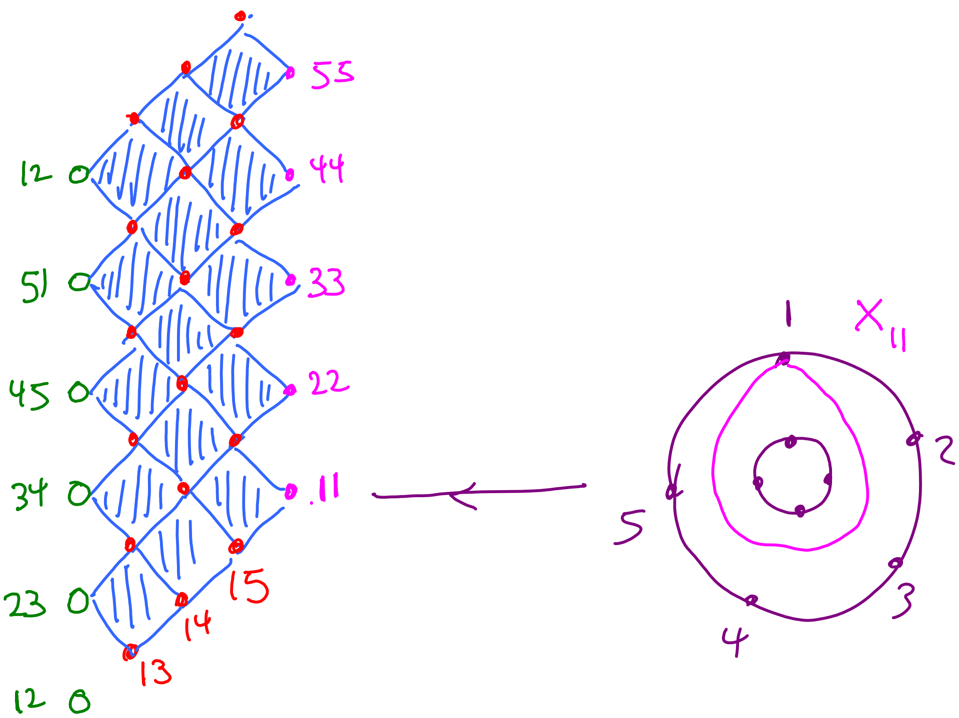


* So, we have an infinite cylinder:

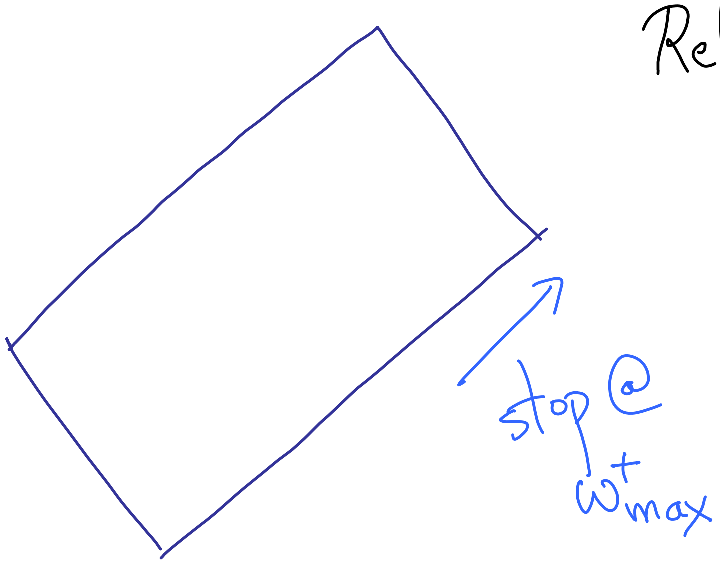


* Two disconnected cylinders for + and − winding.

In-In / Out-Out → "Kinematic Spacetime"



"Compactifying" winding with Δ



Relation $X_\omega + X_{\omega'} - X_{\cdot\cdot}$

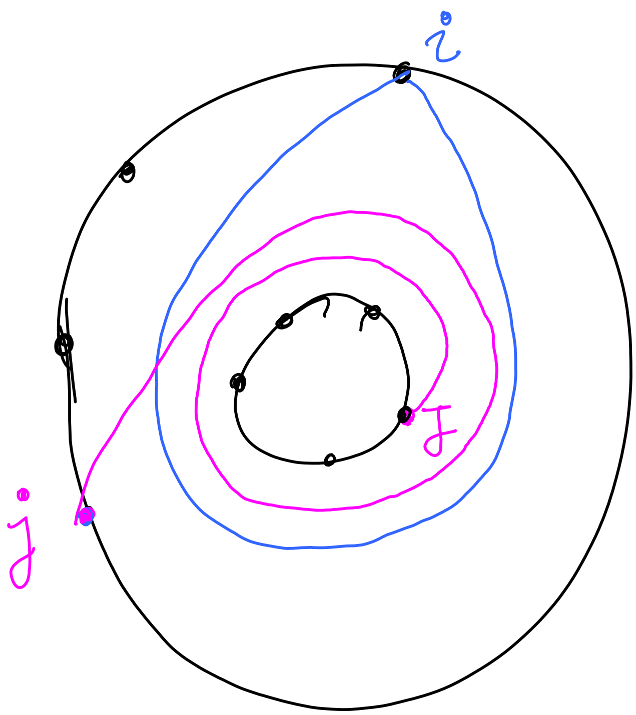
missing!

$\Downarrow$ Replace with.

$$X_\omega + \Delta - X_{\omega+1}$$

Gluing "In-In/Out-Out" to "In-Out"

The "mesh" relations are
$$\left\{ \begin{array}{l} X_{ii} + X_{jI}^{\omega^+_{max},\,\omega^-_{min}} - \cdots \\ X_{II} \end{array} \right\}$$

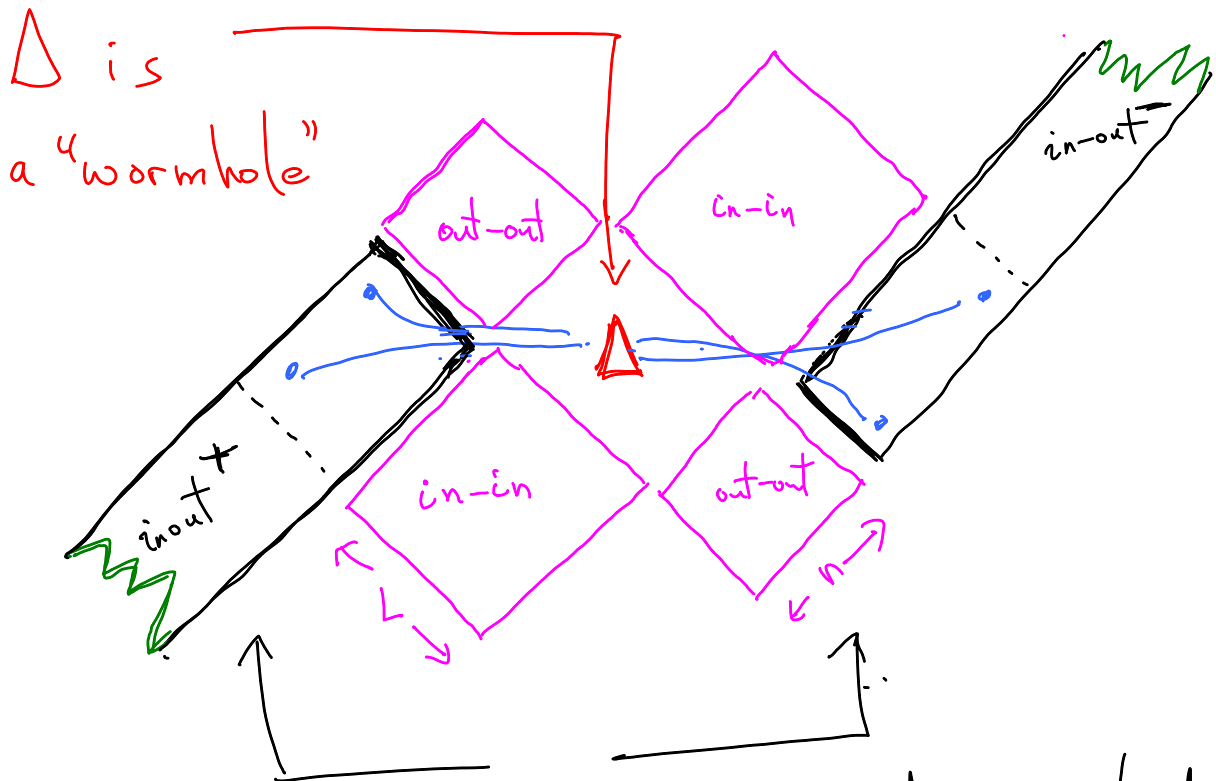


Gluing + to − winding

$$X^{+\omega_{max}}_{i\mathcal{I}} + X^{-\omega_{max}}_{i\mathcal{I}} - \cdots - \#\Delta$$

$\{$ Quite remarkable these relations exist! $\}$

The Final "Kinematical Spacetime" for $A_{n,L}$



+, − otherwise disconnected

The relations $\{R_\mu\}$ of this "Kin. Spacetime" positively generate all the rest!

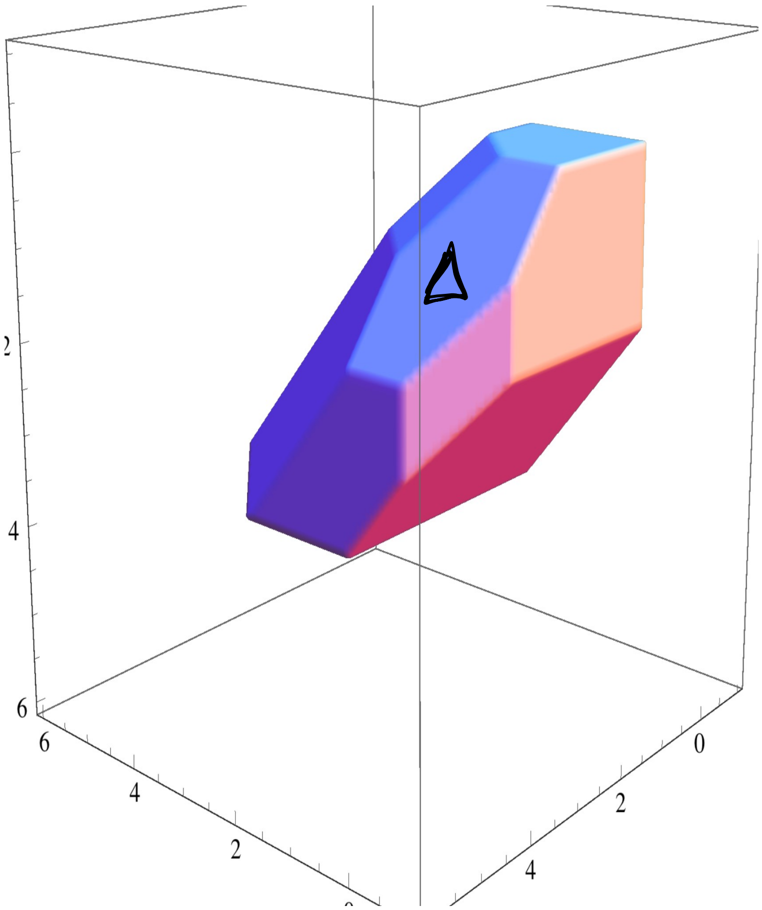
Affine Clusterhedra.

* Exactly the same ideas work for $\tilde{D}_{n+2}$ = 2-loop. So, we have clusterhedra through 2-loops and all L cut of $A_{n,L}$.

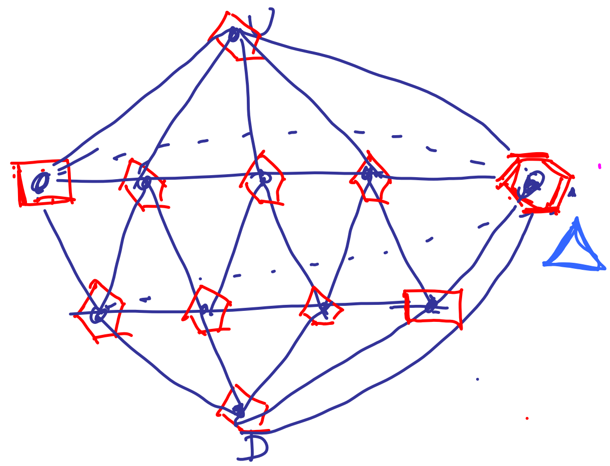
* Δ is "mutation partner" for every missing var; all vertices $\Longrightarrow$ all vertices unamb. assoc. w/ diagrams.

* All non-Δ vertices counted same # of times

* All normal facet det's involving Δ are ± 1, get all $+1$'s for amplitude upon pullback



$A_{2,1}$ Clusterhedron



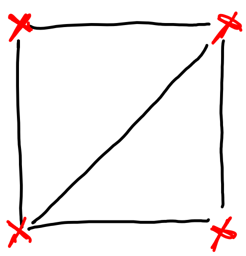
So we are done through 2-loops + some all-loop cuts. On our way to all loops. Simple Strategy:

Understand decoupled "Topological Sectors",

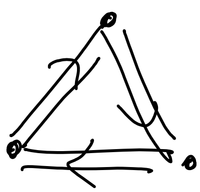
Find "Δ"'s $\longleftrightarrow$ "Wormholes" to glue them together

Novelty for $L \geq 3$ / non-planar: "Cosmological" Kinematic Space

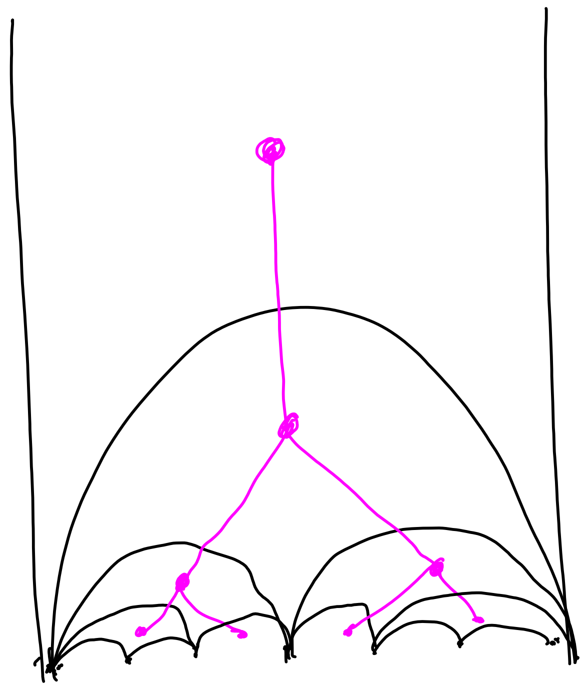
e.g. once-punctured torus



$\xrightarrow{SL(2,\mathbb{Z})}$ for gen. triang.



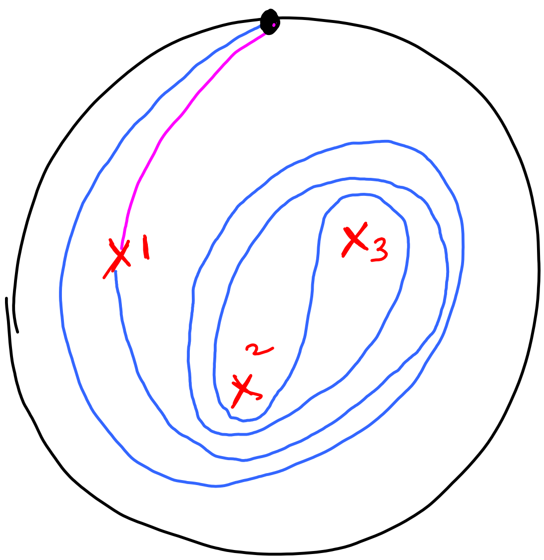
Markov



"dS" in Kin Space.

- There is a clusterhedron here, with many "Δ"'s

Novelty for $L \geq 3$/non-planar: "Cosmological" Kinematic Space

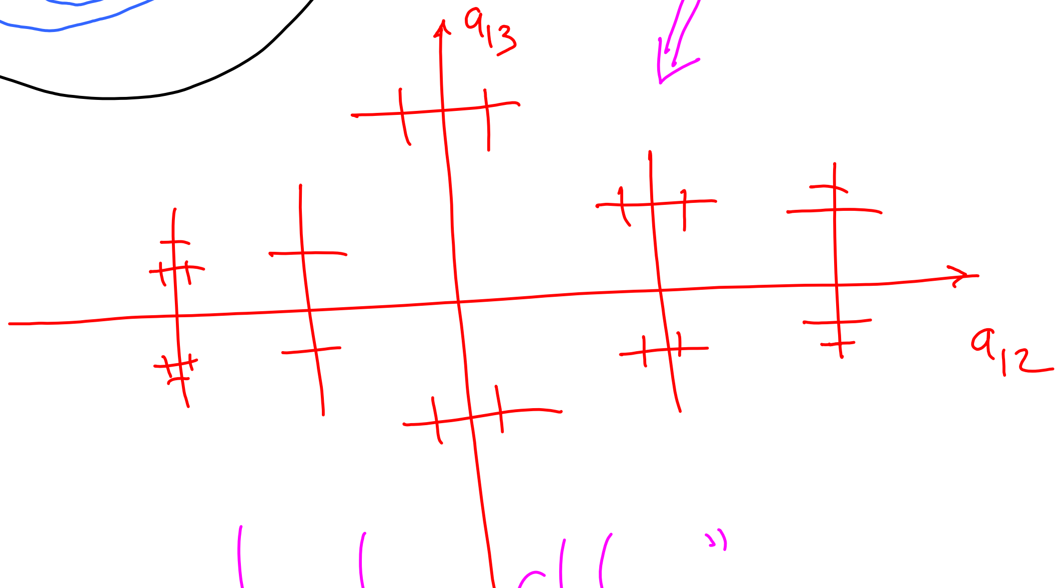


Arcs labelled by

$\mathrm{Braid}_3 / \mathrm{Braid}_2$

$=$

$\mathrm{Free}_2 (a_{12}, a_{13})$



"False vacuum eternal inflation"

... In Progress ...