Celestial Operator Products of Gluons and Gravitons

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Based on 1910.07424 with A. Raclariu, A. Strominger and E. Yuan

Introduction/Motivation

Why study scattering amplitudes?

1. To uncover fundamental structure underlying scattering amplitudes (obscured by Feynman diagrams, for example)
2. Natural observables for quantum gravity in asymptotically flat spacetimes

Central result:

- Derivation of **leading collinear limits** of gluons and gravitons from **asymptotic symmetries** (tree-level)

Strategy: **Celestial amplitudes**

- Scattering amplitudes of *boost* (not translation) eigenstates

Introduction/Motivation

Why study scattering amplitudes?

1. To uncover fundamental structure underlying scattering amplitudes (obscured by Feynman diagrams, for example)
2. Natural observables for quantum gravity in asymptotically flat spacetimes

Strategy: **Celestial amplitudes**

- Single particle states transform like primary operators in 2D CFT
- Collinear limits $\rightarrow$ operator product expansions
- *Holography*
  - Does there exist an *intrinsically defined* 2D theory whose correlation functions are 4D scattering amplitudes?

Central result:

- Derivation of **OPE coefficients** of gluons and gravitons from **asymptotic symmetries**

  $\Rightarrow$ *Symmetries provide intrinsic structure to a holographic dual*

Outline

1. Construction of Celestial Amplitudes
2. Symmetries of the $\mathcal{S}$-Matrix:
   - Soft Theorems and Asymptotic Symmetries
3. OPE Coefficients (Collinear Limits) from Symmetry

Construction of Celestial Amplitudes

Seek **basis** *in which asymptotic particles transform like* **primary operators**

Primary operators $\mathcal{O}_{\Delta,s}(z,\bar{z})$

$$\mathcal{O}_{\Delta,s}(z,\bar{z}) \to \mathcal{O}'_{\Delta,s}(z',\bar{z}') = (cz+d)^{\Delta+s}(\bar{c}\bar{z}+\bar{d})^{\Delta-s}\mathcal{O}_{\Delta,s}(z,\bar{z})$$

$$z \to z' = \frac{az+b}{cz+d}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}).$$

Conventional basis (massless particles)

- Diagonalize translations + 1 rotation ⇒ *Labels*: momentum p^μ + helicity s
- Spinor helicity variables:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \qquad \lambda \to M\lambda, \qquad z = \frac{\lambda_1}{\lambda_2} = \frac{p^1+ip^2}{p^0+p^3}.$$

  → Parametrize null momenta:

$$p^\mu(\omega,z,\bar{z}) = \frac{\omega}{\sqrt{2}}(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}).$$

- Momenta with same z are *collinear*. ($p_i \cdot p_j = -\omega_i\omega_j z_{ij}\bar{z}_{ij}$)
- Momentum eigenstates: $|p,s\rangle = |\omega,z,\bar{z},s\rangle$

↳ *Same labels as* $\mathcal{O}_{\Delta,s}(z,\bar{z})$ *apart from* $\omega \overset{?}{\leftrightarrow} \Delta$

Construction of Celestial Amplitudes

- Under Lorentz transformations,

$$p^\mu \to \Lambda^\mu{}_\nu p^\nu, \qquad \omega \to \omega|cz+d|^2.$$

$\Rightarrow$ *Seek to trade ω for $SL(2,\mathbb{C})$-invariant label Δ*

- For $p^\mu = \frac{1}{\sqrt{2}}\omega(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})$,

$$\text{Boost along } p^\mu : \quad p^\mu \to \lambda p^\mu,$$
$$\text{Generated by} : \quad K = \omega\partial_\omega.$$

- K is diagonalized by the *Mellin transform*

$$\mathcal{O}_{\Delta,s}(z,\bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1}|\omega,z,\bar{z},s\rangle$$

Celestial Amplitudes:

$$\langle \mathcal{O}_{\Delta_1,s_1}(z_1,\bar{z}_1)\ldots\mathcal{O}_{\Delta_n,s_n}(z_n,\bar{z}_n)\rangle = \left(\prod_{i=1}^n \int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1}\right)\mathcal{A}_{s_1\cdots s_n}(p_1,\ldots,p_n)$$

[Kapec, Mitra, Raclariu, & Strominger, hep-th/1609.00282;
Cheung, de la Fuente & Sundrum, hep-th/1609.00732]

Properties of Celestial Amplitudes

Celestial Amplitudes:

$$\langle \mathcal{O}_{\Delta_1,s_1}(z_1,\bar{z}_1)\ldots\mathcal{O}_{\Delta_n,s_n}(z_n,\bar{z}_n)\rangle = \Big(\prod_{i=1}^{n}\int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}\Big)\mathcal{A}_{s_1\ldots s_n}(p_1,\ldots,p_n)$$

Under $SL(2,\mathbb{C})$ transformations

$$\langle \mathcal{O}_{\Delta_1,s_1}(z_1',\bar{z}_1')\ldots\mathcal{O}_{\Delta_n,s_n}(z_n',\bar{z}_n')\rangle = \prod_{j=1}^{n}\left[(cz_j+d)^{\Delta_j+s_j}(\bar{c}\bar{z}_j+\bar{d})^{\Delta_j-s_j}\right]\langle \mathcal{O}_{\Delta_1,s_1}(z_1,\bar{z}_1)\ldots\mathcal{O}_{\Delta_n,s_n}(z_n,\bar{z}_n)\rangle$$

Comments:

- Transformation is invertible
- Collinear limit $p_i||p_j \quad \Leftrightarrow \quad z_i \to z_j$ (operator product expansion)

[Kapec, Mitra, Raclariu, & Strominger, hep-th/1609.00282; Cheung, de la Fuente & Sundrum, hep-th/1609.00732; Pasterski & Shao, hep-th/1705.01027]

Symmetries of the $\mathcal{S}$-Matrix

[He, Lysov, Mitra & Strominger, hep-th/1401.7026;
Strominger, hep-th/1703.05448]

Soft theorems $\Rightarrow$ symmetries:

- *Can always interpret soft theorems as statements of invariance of the $\mathcal{S}$-matrix under an infinite-dimensional symmetry*

Leading* soft theorem

$$\lim_{\omega\to 0} \omega \mathcal{A}_{n+1}(\omega, z, \bar{z}) = \sum_{k=1}^{n} S_k(z,\bar{z}) \mathcal{A}_n$$

- Soft factor S_k is eig.value of single particle states under operator Q_H

$$S_k(z,\bar{z})|p_k\rangle = Q_H(z,\bar{z})|p_k\rangle = \delta_{(z,\bar{z})}|p_k\rangle$$

- RHS gives transformation of single particle states under $\delta_{(z,\bar{z})}$

$$\sum_{k=1}^{n} S_k(z,\bar{z}) \mathcal{A}_n = \langle \text{out}|[Q_H, \mathcal{S}]|\text{in}\rangle$$

- Soft theorem implies the $\mathcal{S}$-matrix is invariant under the transformations $\delta_{(z,\bar{z})}$ of single particle states provided that *a soft particle is added*.

*Subleading: replace LHS with $\lim_{\omega\to 0}(1+\omega\partial_\omega)\mathcal{A}_{n+1}(\omega, z, \bar{z})$

Symmetries of the $\mathcal{S}$-Matrix

Leading soft theorem

$$\lim_{\omega\to 0}\omega\mathcal{A}_{n+1}(\omega,z,\bar{z}) = \sum_{k=1}^{n} S_k(z,\bar{z})\mathcal{A}_n = \langle \text{out}|[Q_H,\mathcal{S}]|\text{in}\rangle$$

- Denote operator which adds soft particles Q_S

$$Q_S(z,\bar{z}) \sim -\lim_{\omega\to 0}\omega\left[a(\omega,z,\bar{z}) + a^\dagger(\omega,z,\bar{z})\right]$$

- Then the LHS can be written as

$$\lim_{\omega\to 0}\omega\mathcal{A}_{n+1}(\omega,z,\bar{z}) = -\langle \text{out}|[Q_S,\mathcal{S}]|\text{in}\rangle$$

- Rearranging the soft theorem

$$\langle \text{out}|[Q,\mathcal{S}]|\text{in}\rangle = 0, \qquad Q = Q_H + Q_S.$$

$\Rightarrow$ Obtain statement of invariance under symmetry generated by Q

Soft theorems imply symmetries:

- *Can always interpret soft theorems as statements of invariance of the $\mathcal{S}$-matrix under an infinite-dimensional symmetry*

Global Symmetries of the $\mathcal{S}$-Matrix

Extracting Global Symmetries

- Parametrize Q's by functions $f(z,\bar{z})$ rather than points $(z,\bar{z})$
- Find $\mathcal{D} \sim \partial_z^m \partial_{\bar{z}}^n$ that localizes the soft factors for massless p_k

$$\mathcal{D}S_k \sim \delta^{(2)}(z - z_k)$$

- Obtain charges which act locally (only depends on f at point z_k)

$$\delta_f |p_k\rangle = Q_H[f]|p_k\rangle = \int d^2z\, f\mathcal{D}S_k|p_k\rangle$$

- By construction the soft charge is of similar form

$$Q_S[f] \sim \int d^2z\, f\mathcal{D}\lim_{\omega\to 0}\omega\left[a(\omega,z,\bar{z}) + a^\dagger(\omega,z,\bar{z})\right]$$

For some choices $\hat{f}$ of f, $\mathcal{D}\hat{f} = 0$ which implies $Q_S[\hat{f}] = 0$!

↳ *Global symmetries:* $\langle \text{out}|[Q_H[\hat{f}],\mathcal{S}]|\text{in}\rangle = 0$

Symmetry Constraints on Celestial Amplitudes

- Global symmetries of momentum space amplitudes imply global symmetries of celestial amplitudes.

$$\langle \text{out}|[Q_H[\hat{f}],\mathcal{S}]|\text{in}\rangle = 0 \qquad \Leftrightarrow \qquad \sum_{k=1}^{n}\langle \mathcal{O}_1 \ldots \delta_{\hat{f}}\mathcal{O}_k \ldots \mathcal{O}_n\rangle = 0$$

- Invariance of correlation functions implies constraints on OPE's

$$\delta(\mathcal{O}_i\mathcal{O}_j) = \delta\mathcal{O}_i\mathcal{O}_j + \mathcal{O}_i\delta\mathcal{O}_j = \sum_k C_{ijk}\delta\mathcal{O}_k$$

- Transformations of celestial operators follow from transformations of momentum eigenstates

$$\begin{aligned}\delta_{\hat{f}}\mathcal{O}_{\Delta_k,s_k}(z_k,\bar{z}_k) &= \int_0^\infty d\omega_k\, \omega_k^{\Delta_k-1}\delta_{\hat{f}}|p_k,s_k\rangle \\ &= \int_0^\infty d\omega_k\, \omega_k^{\Delta_k-1}\int d^2z\,\hat{f}\,\mathcal{D}S_k|p_k,s_k\rangle\end{aligned}$$

Global Symmetry Constraints on Celestial Amplitudes

$$\delta_{\hat{f}}\mathcal{O}_{\Delta_k,s_k}(z_k,\bar{z}_k) = \int_0^\infty d\omega_k\, \omega_k^{\Delta_k-1}\delta_{\hat{f}}|p_k,s_k\rangle = \int_0^\infty d\omega_k\, \omega_k^{\Delta_k-1}\int d^2z\,\hat{f}\,\mathcal{D}S_k|p_k,s_k\rangle$$

Example: *Leading soft graviton theorem*

[Donnay, Puhm & Strominger, 1810.05219; Stieberger & Taylor, 1812.01080]

$$S_k = \frac{\kappa}{2}\frac{\varepsilon^+_{\mu\nu}p_k^\mu p_k^\nu}{\hat{q}\cdot p_k} = -\frac{\kappa}{2}\frac{\omega_k(\bar{z}-\bar{z}_k)}{(z-z_k)}$$

- Here $\mathcal{D} = -\frac{2}{\kappa}\frac{1}{2\pi}\partial_{\bar{z}}^2$

$$\mathcal{D}S_k = \omega_k\delta^{(2)}(z-z_k)$$

- Choose $\hat{f} = 1$

$$\delta_{\hat{f}=1}|p_k,s_k\rangle = \int d^2z\,\mathcal{D}S_k|p_k,s_k\rangle = \omega_k|p_k,s_k\rangle.$$

- Perform Mellin transform

$$\delta_{\hat{f}=1}\mathcal{O}_{\Delta_k,s_k}(z_k,\bar{z}_k) = \int_0^\infty d\omega_k\, \omega_k^{\Delta_k-1}\delta_{\hat{f}=1}|p_k,s_k\rangle = \mathcal{O}_{\Delta_k+1,s_k}(z_k,\bar{z}_k)$$

⋆ *Momentum conservation captured by shifts in dimension*

Global Symmetries in Gravity

- *Subleading soft graviton theorem*

$$S_k = \frac{i\kappa}{2}\frac{\varepsilon^+_{\mu\nu}p^\nu_k\hat{q}_\rho\mathcal{J}^{\mu\rho}_k}{\hat{q}\cdot p_k} = -\frac{\kappa}{2}\frac{\bar{z}-\bar{z}_k}{z-z_k}[\omega_k\partial_{\omega_k}+s_k+(\bar{z}-\bar{z}_k)\partial_{\bar{z}_k}],$$

$$\boxed{\delta\mathcal{O}_{\Delta,s}(z,\bar{z}) = \frac{\kappa}{2}[\Delta - s + 2\bar{z}\partial_{\bar{z}}]\mathcal{O}_{\Delta,s}(z,\bar{z})}$$

Angular momentum conservation ⇔ 2D global conformal transformations

- *Subsubleading soft graviton theorem*

$$\begin{aligned} S_k &= -\frac{\kappa}{4}\frac{\varepsilon^+_{\mu\nu}\hat{q}_\rho\hat{q}_\sigma\mathcal{J}^{\mu\rho}_k\mathcal{J}^{\nu\sigma}_k}{\hat{q}\cdot p_k} \\ &= -\frac{\kappa}{4}\frac{1}{\omega_k}\frac{1}{(z-z_k)(\bar{z}-\bar{z}_k)}\left[(\bar{z}-\bar{z}_k)(\omega_k\partial_{\omega_k}+s_k)+(\bar{z}-\bar{z}_k)^2\partial_{\bar{z}_k}\right]^2, \end{aligned}$$

$$\boxed{\delta\mathcal{O}_{\Delta,s}(z,\bar{z}) = -\frac{\kappa}{4}\left[(\Delta-s-1)(\Delta-s)+4(\Delta-s)\bar{z}\partial_{\bar{z}}+3\bar{z}^2\partial^2_{\bar{z}}\right]\mathcal{O}_{\Delta-1,s}(z,\bar{z})}$$

[Kapec, Mitra, Raclariu, & Strominger, hep-th/1609.00282;
Stieberger & Taylor, 1812.01080; Adamo, Mason, & Sharma, hep-th/1905.09224;
Guevara, hep-th/1906.07810; MP, Raclariu, Strominger & Yuan, hep-th/1910.07424]

Global Symmetries in Yang-Mills

- *Leading soft gluon theorem*

$$S_k = gT_k^a \frac{\varepsilon^+ \cdot p_k}{\hat{q} \cdot p_k} = gT_k^a \frac{1}{\omega(z - z_k)},$$

$$\boxed{\delta^a O_\Delta^{\pm b}(z,\bar{z}) = -igf^{ab}{}_c O_\Delta^{\pm c}(z,\bar{z})}$$

- *Subleading soft gluon theorem*

$$S_k = gT_k^a \frac{i\varepsilon_\mu^+ \hat{q}_\nu \mathcal{J}_k^{\mu\nu}}{p_k \cdot \hat{q}} = gT_k^a \frac{1}{\omega_k(z - z_k)} \left(\omega_k \frac{\partial}{\partial \omega_k} + s_k + (\bar{z} - \bar{z}_k) \frac{\partial}{\partial \bar{z}_k} \right)$$

$$\boxed{\delta^a O_\Delta^{\pm b}(z,\bar{z}) = igf^{ab}{}_c (\Delta \mp 1 - 1 + \bar{z}\partial_{\bar{z}}) O_{\Delta-1}^{\pm c}(z,\bar{z})}$$

[He, Mitra, & Strominger, hep-th/1503.02663;
Himwich & Strominger, hep-th/1901.01622;
Adamo, Mason, & Sharma, hep-th/1905.09224;
MP, Raclariu, Strominger & Yuan, hep-th/1910.07424]

Gluon OPE Coefficients from Symmetry

Suppose* leading term in OPE between + helicity gluons takes the form

$$O^{+a}_{\Delta_1}(z_1) O^{+b}_{\Delta_2}(z_2) \sim -\frac{igf^{ab}{}_c}{z_{12}} C(\Delta_1,\Delta_2) O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

with $C(\Delta_1,\Delta_2)$ undetermined.

Invariance under global symmetry implies

$$\delta\Big(O^{+a}_{\Delta_1}(z_1) O^{+b}_{\Delta_2}(z_2)\Big) = \delta O^{+a}_{\Delta_1}(z_1) O^{+b}_{\Delta_2}(z_2) + O^{+a}_{\Delta_1}(z_1) \delta O^{+b}_{\Delta_2}(z_2)$$
$$\sim -\frac{igf^{ab}{}_c}{z_{12}} C(\Delta_1,\Delta_2)\, \delta O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

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*Amplitudes have simple poles where sum of subset of external momenta go on-shell. Collinear singularities are special case where subset is a pair (of null momenta)

$$\frac{1}{(p_1+p_2)^2} = \frac{1}{2p_1 \cdot p_2} \sim \frac{1}{z_{12}}$$

⇒ Leading singularity in z_{12} is at most a simple pole.

Gluon OPE Coefficients from Symmetry

Constrain with subleading soft gluon symmetry

$$\delta^a O^{+b}_{\Delta}(z,\bar{z}) = igf^{ab}{}_c(\Delta - 2 + \bar{z}\partial_{\bar{z}})O^{+c}_{\Delta-1}(z,\bar{z}).$$

Action on LHS

$$\delta^d\left[O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2)\right] = \delta^d O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2) + O^{+a}_{\Delta_1}(z_1)\delta^d O^{+b}_{\Delta_2}(z_2)$$

$$= igf^{da}{}_c(\Delta_1 - 2)O^{+c}_{\Delta_1-1}(z_1)O^{+b}_{\Delta_2}(z_2) + igf^{db}{}_c(\Delta_2 - 2)O^{+a}_{\Delta_1}(z_1)O^{+c}_{\Delta_2-1}(z_2) + \cdots$$

$$\sim igf^{da}{}_c(\Delta_1 - 2)\frac{-igf^{cb}{}_e}{z_{12}}C(\Delta_1 - 1, \Delta_2)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

$$+ igf^{db}{}_c(\Delta_2 - 2)\frac{-igf^{ac}{}_e}{z_{12}}C(\Delta_1, \Delta_2 - 1)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

Action on RHS

$$\delta^d\left[\frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)O^{+c}_{\Delta_1+\Delta_2-1}(z_2)\right] = \frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)\delta^d O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

$$= -\frac{igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)igf^{dc}{}_e(\Delta_1 + \Delta_2 - 3)O^{+e}_{\Delta_1+\Delta_2-2}(z_2)$$

Equating RHS & LHS → symmetry constraint

$$f^{da}{}_c f^{cb}{}_e(\Delta_1 - 2)C(\Delta_1 - 1, \Delta_2) + f^{db}{}_c f^{ac}{}_e(\Delta_2 - 2)C(\Delta_1, \Delta_2 - 1)$$

$$= f^{ab}{}_c f^{dc}{}_e C(\Delta_1,\Delta_2)(\Delta_1 + \Delta_2 - 3)$$

Gluon OPE Coefficients from Symmetry

Constrain with subleading soft gluon symmetry

$$\delta^a O^{+b}_{\Delta}(z,\bar{z}) = igf^{ab}{}_c(\Delta - 2 + \bar{z}\partial_{\bar{z}})O^{+c}_{\Delta-1}(z,\bar{z}).$$

Action on LHS

$$\delta^d\left[O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2)\right] = \delta^d O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2) + O^{+a}_{\Delta_1}(z_1)\delta^d O^{+b}_{\Delta_2}(z_2)$$

$$= igf^{da}{}_c(\Delta_1 - 2)O^{+c}_{\Delta_1-1}(z_1)O^{+b}_{\Delta_2}(z_2) + igf^{db}{}_c(\Delta_2 - 2)O^{+a}_{\Delta_1}(z_1)O^{+c}_{\Delta_2-1}(z_2) + \cdots$$

$$\sim igf^{da}{}_c(\Delta_1 - 2)\frac{-igf^{cb}{}_e}{z_{12}}C(\Delta_1 - 1, \Delta_2)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

$$+ igf^{db}{}_c(\Delta_2 - 2)\frac{-igf^{ac}{}_e}{z_{12}}C(\Delta_1, \Delta_2 - 1)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

Action on RHS

$$\delta^d\left[\frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)O^{+c}_{\Delta_1+\Delta_2-1}(z_2)\right] = \frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)\delta^d O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

$$= -\frac{igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)igf^{dc}{}_e(\Delta_1 + \Delta_2 - 3)O^{+e}_{\Delta_1+\Delta_2-2}(z_2)$$

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$$f^{da}{}_c f^{cb}{}_e(\Delta_1 - 2)C(\Delta_1 - 1, \Delta_2) + f^{db}{}_c f^{ac}{}_e(\Delta_2 - 2)C(\Delta_1, \Delta_2 - 1)$$

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$$\delta^a O^{+b}_{\Delta}(z,\bar{z}) = igf^{ab}{}_c(\Delta - 2 + \bar{z}\partial_{\bar{z}})O^{+c}_{\Delta-1}(z,\bar{z}).$$

Action on LHS

$$\delta^d\left[O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2)\right] = \delta^d O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2) + O^{+a}_{\Delta_1}(z_1)\delta^d O^{+b}_{\Delta_2}(z_2)$$

$$= igf^{da}{}_c(\Delta_1 - 2)O^{+c}_{\Delta_1-1}(z_1)O^{+b}_{\Delta_2}(z_2) + igf^{db}{}_c(\Delta_2 - 2)O^{+a}_{\Delta_1}(z_1)O^{+c}_{\Delta_2-1}(z_2) + \cdots$$

$$\sim igf^{da}{}_c(\Delta_1 - 2)\frac{-igf^{cb}{}_e}{z_{12}}C(\Delta_1 - 1,\Delta_2)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

$$+ igf^{db}{}_c(\Delta_2 - 2)\frac{-igf^{ac}{}_e}{z_{12}}C(\Delta_1,\Delta_2 - 1)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

Action on RHS

$$\delta^d\left[\frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)O^{+c}_{\Delta_1+\Delta_2-1}(z_2)\right] = \frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)\delta^d O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

$$= -\frac{igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)igf^{dc}{}_e(\Delta_1 + \Delta_2 - 3)O^{+e}_{\Delta_1+\Delta_2-2}(z_2)$$

Equating RHS & LHS → symmetry constraint

$$f^{da}{}_c f^{cb}{}_e(\Delta_1 - 2)C(\Delta_1 - 1,\Delta_2) + f^{db}{}_c f^{ac}{}_e(\Delta_2 - 2)C(\Delta_1,\Delta_2 - 1)$$

$$= f^{ab}{}_c f^{dc}{}_e C(\Delta_1,\Delta_2)(\Delta_1 + \Delta_2 - 3)$$

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Constrain with subleading soft gluon symmetry

$$\delta^a O^{+b}_{\Delta}(z,\bar{z}) = igf^{ab}{}_c(\Delta - 2 + \bar{z}\partial_{\bar{z}})O^{+c}_{\Delta-1}(z,\bar{z}).$$

Action on LHS

$$\delta^d \left[O^{+a}_{\Delta_1}(z_1) O^{+b}_{\Delta_2}(z_2) \right] = \delta^d O^{+a}_{\Delta_1}(z_1) O^{+b}_{\Delta_2}(z_2) + O^{+a}_{\Delta_1}(z_1)\delta^d O^{+b}_{\Delta_2}(z_2)$$

$$= igf^{da}{}_c(\Delta_1 - 2)O^{+c}_{\Delta_1-1}(z_1)O^{+b}_{\Delta_2}(z_2) + igf^{db}{}_c(\Delta_2 - 2)O^{+a}_{\Delta_1}(z_1)O^{+c}_{\Delta_2-1}(z_2) + \cdots$$

$$\sim igf^{da}{}_c(\Delta_1 - 2)\frac{-igf^{cb}{}_e}{z_{12}}C(\Delta_1 - 1, \Delta_2)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

$$+ igf^{db}{}_c(\Delta_2 - 2)\frac{-igf^{ac}{}_e}{z_{12}}C(\Delta_1, \Delta_2 - 1)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

Action on RHS

$$\delta^d \left[\frac{-igf^{ab}{}_c}{z_{12}} C(\Delta_1, \Delta_2) O^{+c}_{\Delta_1+\Delta_2-1}(z_2) \right] = \frac{-igf^{ab}{}_c}{z_{12}} C(\Delta_1, \Delta_2)\delta^d O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

$$= -\frac{igf^{ab}{}_c}{z_{12}} C(\Delta_1, \Delta_2) igf^{dc}{}_e(\Delta_1 + \Delta_2 - 3)O^{+e}_{\Delta_1+\Delta_2-2}(z_2)$$

Equating RHS & LHS $\rightarrow$ symmetry constraint

$$f^{da}{}_c f^{cb}{}_e(\Delta_1 - 2)C(\Delta_1 - 1, \Delta_2) + f^{db}{}_c f^{ac}{}_e(\Delta_2 - 2)C(\Delta_1, \Delta_2 - 1)$$

$$= f^{ab}{}_c f^{dc}{}_e C(\Delta_1, \Delta_2)(\Delta_1 + \Delta_2 - 3)$$

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Constrain with subleading soft gluon symmetry

$$\delta^a O^{+b}_{\Delta}(z,\bar z) = igf^{ab}{}_c(\Delta - 2 + \bar z\partial_{\bar z})O^{+c}_{\Delta-1}(z,\bar z).$$

Action on LHS

$$\delta^d\left[O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2)\right] = \delta^d O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2) + O^{+a}_{\Delta_1}(z_1)\delta^d O^{+b}_{\Delta_2}(z_2)$$

$$= igf^{da}{}_c(\Delta_1 - 2)O^{+c}_{\Delta_1-1}(z_1)O^{+b}_{\Delta_2}(z_2) + igf^{db}{}_c(\Delta_2 - 2)O^{+a}_{\Delta_1}(z_1)O^{+c}_{\Delta_2-1}(z_2) + \cdots$$

$$\sim igf^{da}{}_c(\Delta_1 - 2)\frac{-igf^{cb}{}_e}{z_{12}}C(\Delta_1 - 1,\Delta_2)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

$$+ igf^{db}{}_c(\Delta_2 - 2)\frac{-igf^{ac}{}_e}{z_{12}}C(\Delta_1,\Delta_2 - 1)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

Action on RHS

$$\delta^d\left[\frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)O^{+c}_{\Delta_1+\Delta_2-1}(z_2)\right] = \frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)\delta^d O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

$$= -\frac{igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)igf^{dc}{}_e(\Delta_1 + \Delta_2 - 3)O^{+e}_{\Delta_1+\Delta_2-2}(z_2)$$

Equating RHS & LHS → symmetry constraint

$$f^{da}{}_c f^{cb}{}_e(\Delta_1 - 2)C(\Delta_1 - 1,\Delta_2) + f^{db}{}_c f^{ac}{}_e(\Delta_2 - 2)C(\Delta_1,\Delta_2 - 1)$$

$$= f^{ab}{}_c f^{dc}{}_e C(\Delta_1,\Delta_2)(\Delta_1 + \Delta_2 - 3)$$

Gluon OPE Coefficients from Symmetry

Constrain with subleading soft gluon symmetry

$$\delta^a O^{+b}_{\Delta}(z,\bar{z}) = igf^{ab}{}_c(\Delta - 2 + \bar{z}\partial_{\bar{z}})O^{+c}_{\Delta-1}(z,\bar{z}).$$

Action on LHS

$$\delta^d\left[O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2)\right] = \delta^d O^{+a}_{\Delta_1}(z_1)O^{+b}_{\Delta_2}(z_2) + O^{+a}_{\Delta_1}(z_1)\delta^d O^{+b}_{\Delta_2}(z_2)$$

$$= igf^{da}{}_c(\Delta_1 - 2)O^{+c}_{\Delta_1-1}(z_1)O^{+b}_{\Delta_2}(z_2) + igf^{db}{}_c(\Delta_2 - 2)O^{+a}_{\Delta_1}(z_1)O^{+c}_{\Delta_2-1}(z_2) + \cdots$$

$$\sim igf^{da}{}_c(\Delta_1 - 2)\frac{-igf^{cb}{}_e}{z_{12}}C(\Delta_1 - 1,\Delta_2)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

$$+ igf^{db}{}_c(\Delta_2 - 2)\frac{-igf^{ac}{}_e}{z_{12}}C(\Delta_1,\Delta_2 - 1)O^{+e}_{\Delta+\Delta_2-2}(z_2)$$

Action on RHS

$$\delta^d\left[\frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)O^{+c}_{\Delta_1+\Delta_2-1}(z_2)\right] = \frac{-igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)\delta^d O^{+c}_{\Delta_1+\Delta_2-1}(z_2)$$

$$= -\frac{igf^{ab}{}_c}{z_{12}}C(\Delta_1,\Delta_2)igf^{dc}{}_e(\Delta_1 + \Delta_2 - 3)O^{+e}_{\Delta_1+\Delta_2-2}(z_2)$$

Equating RHS & LHS → symmetry constraint

$$f^{da}{}_c f^{cb}{}_e(\Delta_1 - 2)C(\Delta_1 - 1,\Delta_2) + f^{db}{}_c f^{ac}{}_e(\Delta_2 - 2)C(\Delta_1,\Delta_2 - 1)$$

$$= f^{ab}{}_c f^{dc}{}_e C(\Delta_1,\Delta_2)(\Delta_1 + \Delta_2 - 3)$$

Gluon OPE Coefficients from Symmetry

Applying the Jacobi identity, one finds

$$(\Delta_1 - 2)C(\Delta_1 - 1, \Delta_2) = (\Delta_1 + \Delta_2 - 3)C(\Delta_1, \Delta_2) = (\Delta_2 - 2)C(\Delta_1, \Delta_2 - 1).$$

Uniquely solved*† by Euler beta function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$\boxed{C(\Delta_1, \Delta_2) = B(\Delta_1 - 1, \Delta_2 - 1)}$$

[MP, Raclariu, Strominger & Yuan, hep-th/1910.07424]

* Overall normalization is fixed by the leading conformally soft theorem.

† With some assumptions on the boundedness and analyticity behavior of C.

Graviton OPE Coefficients from Symmetry

Consider + helicity graviton OPE of the form

$$G^{+}_{\Delta_1}(z_1)G^{+}_{\Delta_2}(z_2) \sim \frac{\bar{z}_{12}}{z_{12}} E(\Delta_1,\Delta_2) G^{+}_{\Delta_1+\Delta_2}(z_2),$$

Apply symmetry transformation from subsubleading soft graviton theorem

$$\delta\mathcal{O}_{\Delta,s}(z,\bar{z}) = -\frac{\kappa}{4}\left[(\Delta-s-1)(\Delta-s)+4(\Delta-s)\bar{z}\partial_{\bar{z}}+3\bar{z}^2\partial_{\bar{z}}^2\right]\mathcal{O}_{\Delta-1,s}(z,\bar{z}).$$

Obtain constraint

$$\begin{aligned}(\Delta_1+1)(\Delta_1-2)E(\Delta_1-1,\Delta_2)+(\Delta_2-3)(\Delta_2-2)E(\Delta_1,\Delta_2-1)\\ =(\Delta_1+\Delta_2-2)(\Delta_1+\Delta_2-3)E(\Delta_1,\Delta_2).\end{aligned}$$

Solution:

$$\boxed{E(\Delta_1,\Delta_2) = -\frac{\kappa}{2}B(\Delta_1-1,\Delta_2-1)}$$

[MP, Raclariu, Strominger & Yuan, hep-th/1910.07424]

OPE Coefficients from Asymptotic Symmetries

Summary of OPE coefficients in Einstein-Yang-Mills Theory

Graviton-Graviton OPEs

$$G^{+,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)G^{\pm,\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+1\mp 2)G^{\pm,\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2),$$

$$G^{+,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)G^{\pm,-\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\Big[B(\Delta_2+1\mp 2,1\pm 2-\Delta_1-\Delta_2)G^{\pm,\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2) + B(\Delta_1-1,1\pm 2-\Delta_1-\Delta_2)G^{\pm,-\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)\Big].$$

Gluon-Graviton OPEs

$$G^{+,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)O^{\pm a,\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+1\mp 1)O^{\pm a,\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2),$$

$$G^{+,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)O^{\pm a,-\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\Big[B(\Delta_2+1\mp 1,1\pm 1-\Delta_1-\Delta_2)O^{\pm a,\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2) - B(\Delta_1-1,1\pm 1-\Delta_1-\Delta_2)O^{\pm a,-\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)\Big].$$

[MP, Raclariu, Strominger & Yuan, hep-th/1910.07424]

Summary of OPE coefficients in EYM Theory

Gluon-Gluon OPEs

$$O^{+a,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)O^{+b,\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{-igf^{ab}{}_c}{z_{12}}\epsilon B(\Delta_1-1,\Delta_2-1)O^{+c,\epsilon}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2),$$

$$O^{+a,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)O^{+b,-\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{-igf^{ab}{}_c}{z_{12}}\epsilon\Big[-B(\Delta_2-1,3-\Delta_1-\Delta_2)O^{+c,\epsilon}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \\ +B(\Delta_1-1,3-\Delta_1-\Delta_2)O^{+c,-\epsilon}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\Big],$$

$$O^{+a,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)O^{-b,\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{-igf^{ab}{}_c}{z_{12}}\epsilon B(\Delta_1-1,\Delta_2+1)O^{-c,\epsilon}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \\ +\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\delta^{ab}B(\Delta_1,\Delta_2+2)G^{-,\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2),$$

$$O^{+a,\epsilon}_{\Delta_1}(z_1,\bar{z}_1)O^{-b,-\epsilon}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{-igf^{ab}{}_c}{z_{12}}\epsilon\Big[-B(\Delta_2+1,1-\Delta_1-\Delta_2)O^{-c,\epsilon}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \\ +B(\Delta_1-1,1-\Delta_1-\Delta_2)O^{-c,-\epsilon}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)\Big] \\ +\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\delta^{ab}\Big[B(\Delta_2+2,-1-\Delta_1-\Delta_2)G^{-,\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2) \\ +B(\Delta_1,-1-\Delta_1-\Delta_2)G^{-,-\epsilon}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)\Big].$$

[Fan, Fotopoulos & Taylor, hep-th/1903.01676;
MP, Raclariu, Strominger & Yuan, hep-th/1910.07424]

OPE Coefficients from Collinear Limits

- *Promised result:* collinear limits from asymptotic symmetries
- *Here:* momentum-space collinear limits $\to$ celestial OPE coefficients

Collinear limits

$$\lim_{z_{ij}\to 0}\mathcal{A}_{s_1\cdots s_n}(p_1,\cdots,p_n)\longrightarrow\sum_s \text{Split}^s_{s_is_j}(p_i,p_j)\mathcal{A}_{s_1\cdots s\cdots s_n}(p_1,\cdots,P,\cdots,p_n)$$

where $P^\mu = p_i^\mu + p_j^\mu$ and $\omega_P = \omega_i + \omega_j$.

OPE follows from Mellin transform

$$\lim_{z_i\to z_j}\mathcal{O}_{\Delta_i,s_i}(z_i)\mathcal{O}_{\Delta_j,s_j}(z_j) = \int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}\int_0^\infty d\omega_i\,\omega_j^{\Delta_j-1}\text{Split}^s_{s_is_j}(p_i,p_j)|P,s\rangle$$
$$+\text{ subleading in } z_{ij}$$

[Fan, Fotopoulos & Taylor, hep-th/1903.01676;
MP, Raclariu, Strominger & Yuan, hep-th/1910.07424]

OPE Coefficients from Collinear Limits

OPE follows from Mellin transform

$$\mathcal{O}_{\Delta_i,s_i}(z_i)\mathcal{O}_{\Delta_j,s_j}(z_j) \sim \int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1}\int_0^\infty d\omega_i\, \omega_j^{\Delta_j-1}\mathrm{Split}^s_{s_is_j}(p_i,p_j)|P,s\rangle$$

Example: *Positive Helicity Gravitons*

$$\mathrm{Split}^2_{22}(p_i,p_j) = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\frac{(\omega_i+\omega_j)^2}{\omega_i\omega_j}$$

$$G^+_{\Delta_i}(z_i)G^+_{\Delta_j}(z_j) \sim -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1}\int_0^\infty d\omega_i\, \omega_j^{\Delta_j-1}\frac{(\omega_i+\omega_j)^2}{\omega_i\omega_j}|\omega_i+\omega_j,z_j,\bar{z}_j,2\rangle$$

Simplify with change-of-variables

$$\omega_i = t\omega_P, \qquad \omega_j = (1-t)\omega_P.$$

$$G^+_{\Delta_i}(z_i)G^+_{\Delta_j}(z_j) \sim -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\underbrace{\int_0^1 dt\, t^{\Delta_i-2}(1-t)^{\Delta_j-2}}_{B(\Delta_i-1,\Delta_j-1)}\underbrace{\int_0^\infty d\omega_P\, \omega_P^{\Delta_i+\Delta_j-1}|\omega_P,z_j,\bar{z}_j,2\rangle}_{G^+_{\Delta_i+\Delta_j}(z_j)}$$

The result *precisely* matches the OPE coefficient derived from symmetry!

Summary and Discussion

Summary:

- Reviewed the construction of celestial amplitudes.
- Derived global symmetry action from soft theorems.
- Obtained leading OPE coefficients of gluons and gravitons from symmetry constraints.
- Demonstrated equivalence of symmetry-derived OPE coefficients with those from collinear splitting functions.

Outlook/Future Directions:

- *Takeaway lesson:*
  Asymptotic symmetries place non-trivial constraints on the structure of celestial amplitudes.
- *Possible extensions:* Subleading OPE terms, strings, loops, etc.