Symmetries of Celestial Amplitudes

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based on:

A. Fotopoulos, Bin Zhu, St.St., T.R. Taylor:

- **BMS Algebra from Soft and Collinear Limits**

  arXiv:1912.10973

  JHEP 03 (2020) 130

- Wei Fan, A. Fotopoulos, St.St., T.R. Taylor

  to appear

see also:

St.St., T.R. Taylor:

- **Strings on Celestial Sphere**

  arXiv:1806.05688

  Nucl. Phys. B935 (2018) 388-411

- **Symmetries of Celestial Amplitudes**

  arXiv:1812.01080

  Phys. Lett. B793 (2019) 141-143

Traditional momentum space

$$p_k^\mu, \quad k = 1, \ldots, n$$
$$p_k^2 = -m_k^2$$

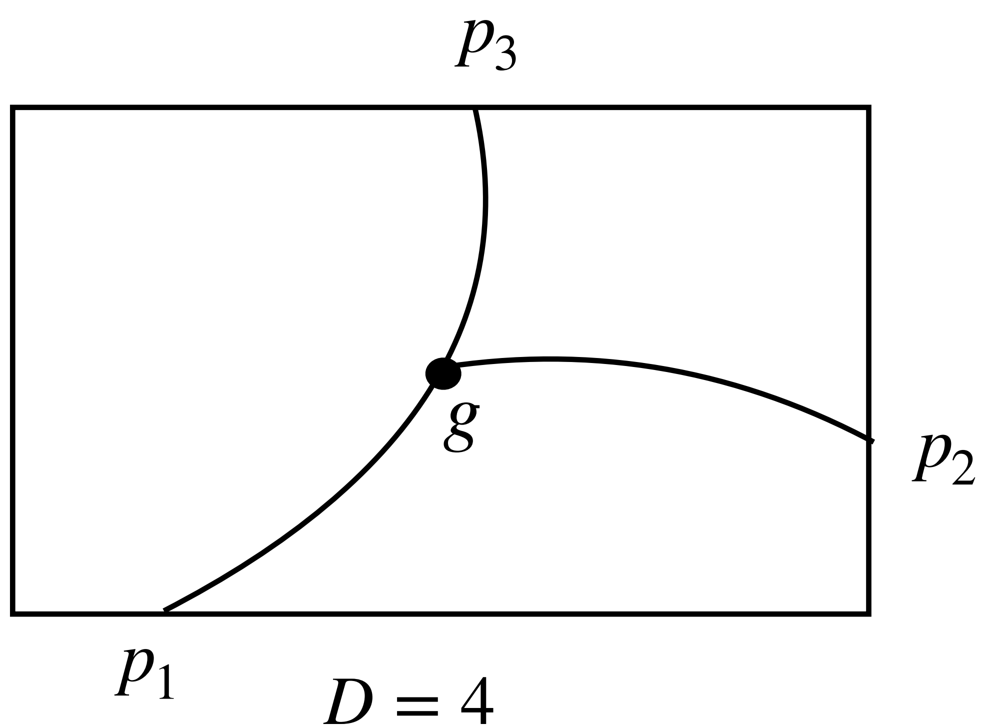
- amplitudes specified by asymptotic wave functions, which transform simply under space-time translations
- with manifest translation symmetry
- traditional amplitudes describe transitions between momentum eigenstates

D=4 Minkowski space probably not the right space
to see **all** symmetries
of scattering amplitudes

Scattering amplitudes in D=4
have interpretation
as Euklidian **D=2** conformal correlators

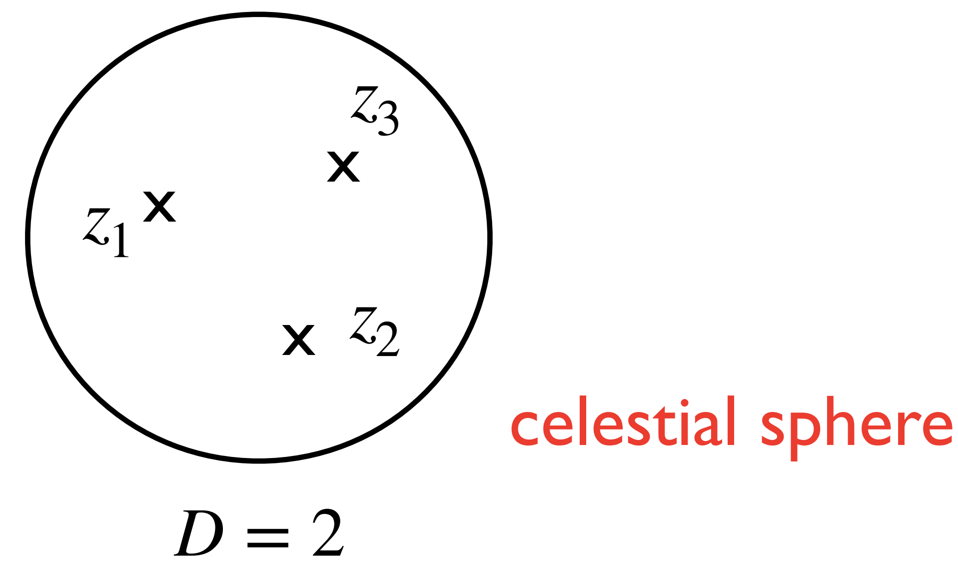
Basic Idea

Amplitudes = conformal correlators of primary fields on celestial sphere



$$z_k = \frac{p_k^1 + i p_k^2}{p_k^0 + p_k^3}$$

$$=$$



$$\sim \frac{g}{|z_1 - z_2|^{h_1+h_2-h_3} \; |z_2 - z_3|^{h_2+h_3-h_1} \; |z_1 - z_3|^{h_1+h_3-h_2}}$$

D=4 space-time QFT correlators

D=2 Euklidian CFT correlators

Lorentz symmetry

$SO(1,3) \simeq SL(2,\mathbf{C})$

$$z_i \to \frac{a z_i + b}{c z_i + d}$$

global conformal symmetry on CS^2

Why ?

- Constrain S-matrix and understand amplitude relations

From studying scattering amplitudes:
deep connections between
gravity and gauge interactions
e.g.: KLT, BCJ, EYM (double-copy-construction)

- scattering amplitudes in both gauge and gravity theories suggest a deeper connection

- indication for the existence of some gauge structure in quantum gravity

- New way of looking at quantum field theory and quantum gravity

- flat space-time holography

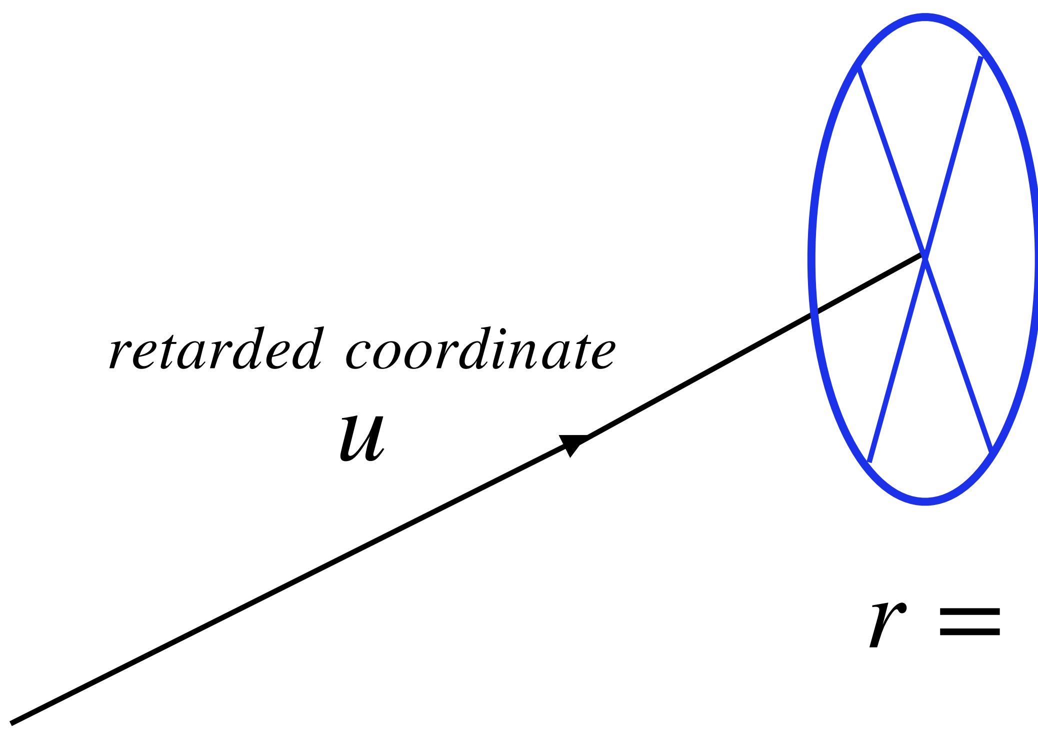
$$ds^2 = -dt^2 + d\vec{x}^2$$

Flat Minkowski metric in retarded (or Bondi) coordinates $(u, r, z, \bar{z})$

$$\begin{cases} x^0 = u + r \\ x^1 = \dfrac{r\,(z+\bar{z})}{1+|z|^2} \\ x^2 = -i\dfrac{r\,(z-\bar{z})}{1+|z|^2} \\ x^3 = \dfrac{r(1-|z|^2)}{1+|z|^2} \end{cases}$$

$$r^2 = \vec{x}^2$$

$$ds^2 = -du^2 - 2\,dudr + \underbrace{\frac{4r^2}{(1+|z|^2)^2}}_{S^2}\,dzd\bar{z}$$



Null infinity $\mathscr{I}^+$

$S^2 \times \mathbf{R}$

$(u, z, \bar{z})$

natural arena
for in and out states

Massless particle on celestial sphere

described by
- the point $z \in CS^2$ at which it enters or exits the celestial sphere
- SL(2,C) Lorentz quantum numbers $(h, \bar{h})$

$$z \in CS^2 \quad \Longrightarrow \quad p^\mu = \frac{\omega}{1+|z|^2} q^\mu(z,\bar{z})$$

Null vector q

$$q_\mu q^\mu = 0$$

with: $$q^\mu = (1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2)$$

$$\omega = E$$

invert: $$z = \frac{p^1 + ip^2}{p^0 + p^3}$$

$$(\vec{p})^2 = (p^0)^2 \quad , \quad E = p^0$$

$$\boxed{p^\mu \longrightarrow (\omega, z, \bar{z})}$$

plane waves in Minkowski: $\exp\{\pm i p_\mu x^\mu\}$

boost eigenstates: $\exp\{\pm iEu\}$

Particles <-> Operators

in momentum basis: plane waves with momentum $p = \omega q(z)$

in conformal basis: conformal primary wave functions Φ

"state operator correspondence"

$$\Phi_{h,\bar{h}}\left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = (cz+d)^{2h}\,(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}\,\Phi_{h,\bar{h}}(z,\bar{z})$$

with:

$$\left.\begin{array}{ll} h+\bar{h}=\Delta & \text{dimension} \\ h-\bar{h}=J & \text{spin} \end{array}\right\} \quad (h,\bar{h}) = \frac{1}{2}(\Delta+J, \Delta-J)$$

In the massless case, with or without spin,
transition from momentum space to conformal primary wavefunctions
with conformal dimension Δ_j
is implemented by Mellin transform:

$$\tilde{\phi}(\Delta) = \int_0^\infty d\omega \; \omega^{\Delta-1} \phi(\omega)$$

E.g.: plane wave $\exp\{\pm i p_\mu x^\mu\}$

$$\phi_\Delta^\pm(x, z, \bar{z}) = \int_0^\infty d\omega \; \omega^{\Delta-1} \; \exp\left\{ \pm i\omega q_\mu x^\mu - \epsilon\omega \right\}$$

$$= \left\{ x^\mu q_\mu(z, \bar{z}) \mp i\epsilon \right\}^{-\Delta}$$

solves D=4
Klein-Gordon equation

scalar: J=0 $\quad h = \bar{h} = \dfrac{\Delta}{2}$ $\quad \Delta = 1 + i\lambda, \; \lambda \in \mathbf{R}$

Pasterski, Shao (2017)

n-point amplitude on celestial sphere

$$\mathscr{A}(\{p_i, \epsilon_j\}) = i(2\pi)^4\ \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^{n} p_k\right)\ A(\{p_i, \epsilon_j\})$$

with:

$$\langle ij \rangle = 2\ (\omega_i \omega_j)^{1/2}\ (z_i - z_j)$$
$$[ij] = 2\ (\omega_i \omega_j)^{1/2}\ (\bar{z}_i - \bar{z}_j)$$

$$\epsilon^\mu(q)_\pm = \frac{1}{\sqrt{2}} \begin{cases} \partial_z q^\mu = (\bar{z}, 1, -i, -\bar{z}) \\ \partial_{\bar{z}} q^\mu = (z, 1, \ i, -z) \end{cases}$$

Celestial amplitudes $\tilde{\mathscr{A}}$ of massless particles are obtained
from momentum-space amplitudes $\mathscr{A}$
by Mellin transforms w.r.t. particle energies $\Delta_j = 1 + i\lambda_j$

$$\boxed{\tilde{\mathscr{A}}_{\{\Delta_l\}}(z_l, \bar{z}_l) = \left(\prod_{l=1}^{n} \int_0^\infty \omega_l^{\Delta_l - 1}\ d\omega_l\right)\ \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N} \omega_k q_k) \times A(\omega_n, z_n, \bar{z}_n)}$$

D=2 CFT correlators involve conformal wave packets

Gauge Amplitudes

four-gluon amplitude:

$$\boxed{\begin{aligned}\tilde{\mathcal{A}}_4(-,-,+,+) = 8\pi\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)\\ \times\, r^{\frac{5}{3}}\,(r-1)^{\frac{2}{3}}\,\delta\left(-4+\sum_{i=1}^{4}\Delta_i\right)\end{aligned}}$$

$$r = \frac{z_{12}\,z_{34}}{z_{23}\,z_{41}}$$

conformal invariant cross-ratio on CS^2

$$r^{-1} = \sin^2\left(\frac{\theta}{2}\right)$$

Pasterski, Shao, Strominger (2017)

$$h_1 = \tfrac{i}{2}\lambda_1,\;\; h_2 = \tfrac{i}{2}\lambda_2,\;\; h_3 = 1+\tfrac{i}{2}\lambda_3\;,\, h_4 = 1+\tfrac{i}{2}\lambda_4$$

$$\bar{h}_1 = 1+\tfrac{i}{2}\lambda_1,\;\; \bar{h}_2 = 1+\tfrac{i}{2}\lambda_2,\;\; \bar{h}_3 = \tfrac{i}{2}\lambda_3\;,\, \bar{h}_4 = \tfrac{i}{2}\lambda_4$$

higher-point: involve Gaussian hypergeometric functions like string amplitudes

Schreiber, Volovich, Zlotnikov (2017)

Graviton Amplitudes

four-graviton amplitude:

$$\tilde{\mathscr{A}}_4(--,--,++,++) = 2\pi\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$
$$\times\, r^{\frac{11}{3}-\frac{\beta}{3}}\,(r-1)^{-\frac{1}{3}-\frac{\beta}{3}}\,\delta\left(-2+\sum_{i=1}^{4}\Delta_i\right)$$

St.St., Taylor (2018)

$$h_1 = -\frac{1}{2}+\frac{i}{2}\lambda_1,\;\; h_2 = -\frac{1}{2}+\frac{i}{2}\lambda_2,\;\; h_3 = \frac{3}{2}+\frac{i}{2}\lambda_3\;, h_4 = \frac{3}{2}+\frac{i}{2}\lambda_4$$

$$\bar{h}_1 = \frac{3}{2}+\frac{i}{2}\lambda_1,\;\; \bar{h}_2 = \frac{3}{2}+\frac{i}{2}\lambda_2,\;\; \bar{h}_3 = -\frac{1}{2}+\frac{i}{2}\lambda_3\;, \bar{h}_4 = -\frac{1}{2}+\frac{i}{2}\lambda_4$$

- *first calculation of graviton amplitude in the conformal basis*
- *important for the soft graviton theorems* $\Delta \to 1,0,\ldots$ *in celestial basis*

no holomorphic factorization (due to supertranslation operator P)

Operator product expansion

Celestial conformal field theory (CCFT)

$$
\begin{aligned}
\mathcal{O}^a_{\Delta_1,-1}(z,\bar z)\mathcal{O}^b_{\Delta_2,+1}(w,\bar w) = {} & \frac{C_{(-,+)-}(\Delta_1,\Delta_2)}{z-w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),-1}(w,\bar w) \\
& + \frac{C_{(-+)+}(\Delta_1,\Delta_2)}{\bar z-\bar w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),+1}(w,\bar w) \\
& + C_{(-+)--}(\Delta_1,\Delta_2)\,\frac{\bar z-\bar w}{z-w}\,\delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),-2}(w,\bar w) \\
& + C_{(-+)++}(\Delta_1,\Delta_2)\,\frac{z-w}{\bar z-\bar w}\,\delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),+2}(w,\bar w)\ + reg\,.
\end{aligned}
$$

Derive from collinear limits of D=4 EYM amplitudes

Fan, Fotopoulos, St.St., Taylor, Zhu (2019)

D=4 S-matrix constrains OPE
or vice versa

Derive from first principles and consistency conditions

Pate, Raclariu, Strominger, Yuan (2019)

extended
BMS
symmetry

Symmetries

At null infinity $\mathscr{I}^{\pm}$ more (hidden) symmetries present
to constrain S-matrix

→ non-trivial consistency on amplitudes

$$z_i \to \frac{az_i + b}{cz_i + d}$$

$$\boxed{SL(2,Z)_{z_i}: \ \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow (cz_i + d)^{\Delta_i + J_i}\, (\bar{c}\bar{z}_i + \bar{d})^{\Delta_i - J_i}\, \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\})}$$

$$P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2} = P^0 + P^3$$

St.St., Taylor (2018)

$$\boxed{P^{(j)}_{-1/2,-1/2}: \ \tilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow \tilde{\mathscr{A}}_n(\{\Delta_j + 1, J_i\})}$$

comprises into translation operator P^{μ} *shifts conformal dimension* Δ_j

celestial gravitational amplitudes appear
as gauge amplitudes translated in space-time

In usual QFT soft theorems $E_i \to 0$ play an important role in consistency and structure of amplitudes $(\omega_i \to 0)$

in Mellin space "soft-limits" reproduce Weinberg's soft theorem

Kapec, Mitra, Raclariu, Strominger (2016)

Donnay, Puhm, Strominger (2018)

$$\Delta \to 0,1,\ldots$$

Strominger (2013)

also relate Ward identities and BMS symmetries:

explicit field realization

(i) energy-momentum tensor T(z):

soft-graviton $\Delta \to 0$

$$T(z) := \tilde{\mathcal{O}}_{\Delta=2,J=+2}(z,\bar{z}) = \frac{3}{\pi}\int d^2w\ (z-w)^{-4}\ \mathcal{O}_{\Delta=0,J=-2}(w,\bar{w})$$

$(h,\bar{h}) = (2,0)$

shadow transformation *Fotopoulos, Taylor (2019)*

then:

$$\boxed{\langle T(z)\ \prod_{i=1}^{n} O_{\Delta_i}(z_i,\bar{z}_i)\rangle = \sum_{i=1}^{n}\left(\frac{h_{O_i}}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i}\right)\langle\prod_{i=1}^{n} O_{\Delta_i}(z_i,\bar{z}_i)\rangle}$$

OPE:

$$\boxed{\begin{aligned} T(w)T(z) &= \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \ldots \\ T(w)\overline{T}(z) &= reg\,. \end{aligned}}$$

Fotopoulos, St.St., Taylor, Zhu (2019)

$c = 0$

$\longrightarrow L_n, \bar{L}_m$

<u>(ii) supertranslation operator P(z):</u>

soft-graviton $\Delta \to 1$

$$P(z,\bar{z}) := \partial_{\bar{z}} \mathcal{O}_{\Delta=1,J=+2}(z,\bar{z})$$

$(h,\bar{h}) = (\frac{3}{2},\frac{1}{2})$

then:

$$\boxed{\left\langle P(z_0)\prod_{j=1}^{n}\mathcal{O}_{\Delta_j,l_j}(z_j,\bar{z}_j)\right\rangle = \frac{1}{4}\sum_{i=1}^{n}\frac{c_i(\Delta_i)}{c_i(\Delta_i+1)}\frac{1}{z_0-z_i}\left\langle\prod_{n=1}^{n}\mathcal{O}_{\Delta_j,l_j}(z_j,\bar{z}_j)\right\rangle\Bigg|_{\Delta_i\to\Delta_i+1}}$$

OPE:

$$T(w)P(z) = \frac{3}{2(w-z)^2}P(z) + \frac{1}{w-z}\partial_z P(z) + reg\,.$$

In addition to Virasoro symmetry, we construct all supertranslation generators acting on primary fields

Fotopoulos, St.St., Taylor, Zhu (2019)

construct:

$$P_{n-\frac{1}{2},-\frac{1}{2}} = \frac{1}{i\pi(n+1)} \oint dw\, w^{n+1} \left[T(w), P_{-\frac{1}{2},-\frac{1}{2}}\right]$$

$$P_{n-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{i\pi(m+1)} \oint d\bar{w}\, \bar{w}^{m+1} \left[\bar{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}\right]$$

$$P_{-1/2,-1/2} = e^{(\partial_h+\partial_{\bar{h}})/2} = P^0 + P^3$$

we find:

$$\boxed{\left[P_{n-\frac{1}{2},m-\frac{1}{2}}, \phi^{h,\bar{h}}(z,\bar{z})\right] = z^n \bar{z}^m \phi^{h+\frac{1}{2},\bar{h}+\frac{1}{2}}(z,\bar{z})}$$

$\longrightarrow$ $P_{k,l}, \bar{P}_{k,l}$

$\longrightarrow$ local (or extended) BMS algebra:

$$m,n \in \mathbf{Z},\ i,j,k,l \in \mathbf{Z}+\frac{1}{2}$$

$$\boxed{\begin{aligned} &[P_{ij}, P_{k,l}] = 0 \ , \\ &[L_n, P_{k,l}] = \left(\frac{1}{2}n - k\right) P_{n+k,l} + n(n^2-1)\ C_{n,k} \ , \\ &[\bar{L}_n, P_{k,l}] = \left(\frac{1}{2}n - l\right) P_{k,n+l} + n(n^2-1)\ \bar{C}_{n,l} \ . \end{aligned}}$$

Barnich (2017)

Conformal soft-theorems $\longleftrightarrow$ Ward identities $\longleftrightarrow$ BMS algebra

BMS$^{\pm}$ group = symmetry of asymptotically flat D=4 space-time at null infinity $\mathscr{I}^{\pm}$

global BMS symmetry on celestial sphere	local BMS symmetry on celestial sphere
Lorentz group: global conformal transformations on celestial sphere SL(2,C) $z \to \frac{az+b}{cz+d}$ $L_{-1} = \partial$ $L_0 = z\partial + h$ $L_1 = z^2\partial + 2hz$	local conformal transformations = superrotations T(z) $[L_m, L_n] = (m-n)\ L_{m+n}$ $[\bar{L}_m, \bar{L}_n] = (m-n)\ \bar{L}_{m+n}$
global space-time translation: Abelian subgroup of supertranslations $P_{-1/2,-1/2} = e^{(\partial_h+\partial_{\bar{h}})/2}$ $\quad P_{1/2,1/2} = z\ e^{(\partial_h+\partial_{\bar{h}})/2}$ $P_{-1/2,1/2} = \bar{z}\ e^{(\partial_h+\partial_{\bar{h}})/2}$ $\quad P_{-1/2,1/2} = \|z\|^2\ e^{(\partial_h+\partial_{\bar{h}})/2}$	local space-time translations =supertranslations P(z) $P_{n-\frac{1}{2},m-\frac{1}{2}} \quad n, m \in \mathbf{Z}$

Symmetries of the celestial OPEs and correlators
S-matrix (non-trivial consistency)

Further Directions

- understand Virasoro central charge (-one-loop ?)

- establish double-copy structure
  (elaborate on gauge/gravity connections)

- high-energy (large λ) limit: string world-sheet = celestial sphere
  celestial $CFT_2 \simeq$ string world-sheet CFT_2

- understanding the nature of 2D CFT on celestial sphere would enable a holographic description of flat spacetime

- uplift AdS_3/CFT_2 holography to $\mathscr{M}_4$
  towards flat space-time holography