Symmetries of Celestial Amplitudes

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based on:

A. Fotopoulos, Bin Zhu, St.St., T.R. Taylor:

- **BMS Algebra from Soft and Collinear Limits**
  
arXiv:1912.10973
  
  JHEP 03 (2020) 130

- Wei Fan, A. Fotopoulos, St.St., T.R. Taylor
  
to appear

see also:

St.St., T.R. Taylor:

- **Strings on Celestial Sphere**
  
arXiv:1806.05688
  

- **Symmetries of Celestial Amplitudes**
  
arXiv:1812.01080
  
Traditional momentum space

\[ p_k^\mu, \quad k = 1, \ldots, n \]
\[ p_k^2 = -m_k^2 \]

- amplitudes specified by asymptotic wave functions, which transform simply under space-time translations
- with manifest translation symmetry
- traditional amplitudes describe transitions between momentum eigenstates

\[ \text{D}=4 \text{ Minkowski space probably not the right space to see all symmetries of scattering amplitudes} \]

Scattering amplitudes in \( \text{D}=4 \) have interpretation as Euclidean \( \text{D}=2 \) conformal correlators
Basic Idea

Amplitudes = conformal correlators of primary fields on celestial sphere

\[ z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3} \]

D=4 space-time QFT correlators

Lorentz symmetry

\[ SO(1,3) \simeq SL(2,\mathbb{C}) \]

D=2 Euclidean CFT correlators

global conformal symmetry on \( CS^2 \)
Why?

• **Constrain S-matrix and understand amplitude relations**

  *From studying scattering amplitudes:*

  **deep connections between**
  **gravity and gauge interactions**
  e.g.: KLT, BCJ, EYM (double-copy-construction)

- scattering amplitudes in both gauge and gravity theories suggest a deeper connection

- indication for the existence of some gauge structure in quantum gravity

• **New way of looking at quantum field theory and quantum gravity**

  - flat space-time holography
Flat Minkowski metric in \textit{retarded} (or Bondi) coordinates \((u, r, z, \bar{z})\)

\[
ds^2 = -dt^2 + d\vec{x}^2
\]

\[
ds^2 = -du^2 - 2\, dudr + \frac{4r^2}{(1 + |z|^2)^2} \, dzd\bar{z}
\]

\[
S^2
\]

Null infinity \(\mathcal{I}^+\)

\[
S^2 \times \mathbb{R}
\]

\((u, z, \bar{z})\)

\[
r^2 = \bar{x}^2
\]

\[
x^0 = u + r
\]

\[
x^1 = \frac{r(z + \bar{z})}{1 + |z|^2}
\]

\[
x^2 = -i\frac{r(z - \bar{z})}{1 + |z|^2}
\]

\[
x^3 = \frac{r(1 - |z|^2)}{1 + |z|^2}
\]

\[
r = \infty
\]

natural arena

for in and out states
Massless particle on celestial sphere

described by \{ 
  \begin{align*}
    \text{• the point } z \in \mathbb{C}S^2 \text{ at which it enters or exits the celestial sphere} \\
    \text{• SL}(2,\mathbb{C}) \text{ Lorentz quantum numbers } (h, \bar{h})
  \end{align*}
\}

\[ z \in \mathbb{C}S^2 \quad \Longrightarrow \quad p^\mu = \frac{\omega}{1 + |z|^2} \quad q^\mu(z, \bar{z}) \]

Null vector \( q \):
\[ q_\mu q^{\mu} = 0 \]

with:
\[ q^\mu = (1 + |z|^2, z + \bar{z}, -i(z - \bar{z}), 1 - |z|^2) \]

\[ \omega = E \]

invert:
\[ z = \frac{p^1 + ip^2}{p^0 + p^3} \]

\[ (\vec{p})^2 = (p^0)^2, \quad E = p^0 \]

\[ p^\mu \longrightarrow (\omega, z, \bar{z}) \]

plane waves in Minkowski:
\[ \exp\{\pm ip_{\mu}x^\mu\} \]

boost eigenstates:
\[ \exp\{\pm iE u\} \]
Particles <-> Operators

in momentum basis: plane waves with momentum $p = \omega q(z)$

in conformal basis: conformal primary wave functions $\Phi$

“state operator correspondence”

$$\Phi_{h,\bar{h}}\left(\begin{array}{c}
az + b \\
\frac{cz + d}{c\bar{z} + \bar{d}}
\end{array}, \begin{array}{c}
\bar{a}\bar{z} + \bar{b} \\
\frac{\bar{c}\bar{z} + \bar{d}}{c\bar{z} + \bar{d}}
\end{array}\right) = (cz + d)^{2h} (\bar{c}\bar{z} + \bar{d})^{2\bar{h}} \Phi_{h,\bar{h}}(z, \bar{z})$$

with:

\[
\begin{align*}
h + \bar{h} &= \Delta \\
h - \bar{h} &= J
\end{align*}
\]

$\Delta$ dimension $\quad J$ spin

\[
(h, \bar{h}) = \frac{1}{2}(\Delta + J, \Delta - J)
\]
In the massless case, with or without spin, transition from momentum space to conformal primary wavefunctions with conformal dimension $\Delta_j$ is implemented by Mellin transform:

$$\tilde{\phi}(\Delta) = \int_0^\infty d\omega \ \omega^{\Delta - 1} \phi(\omega)$$

E.g.: plane wave $\exp\{\pm ip_\mu x^\mu\}$

$$\phi_{\Delta}^{\pm}(x, z, \bar{z}) = \int_0^\infty d\omega \ \omega^{\Delta - 1} \exp\left\{\pm i\omega q_\mu x^\mu - \epsilon\omega\right\}$$

$$= \left\{x^\mu q_\mu(z, \bar{z}) \mp i\epsilon\right\}^{-\Delta}$$

solves D=4 Klein-Gordon equation

scalar: $J=0$ \hspace{1cm} $h = \bar{h} = \frac{\Delta}{2}$

$$\Delta = 1 + i\lambda, \ \lambda \in \mathbb{R}$$

Pasterski, Shao (2017)
n-point amplitude on celestial sphere

\[ \mathcal{A}(\{p_i, e_j\}) = i(2\pi)^4 \delta^{(4)} \left( p_1 + p_2 - \sum_{k=3}^{n} p_k \right) A(\{p_i, e_j\}) \]

with:
\[ \langle ij \rangle = 2 \left( \omega_i \omega_j \right)^{1/2} (z_i - z_j) \]
\[ [ij] = 2 \left( \omega_i \omega_j \right)^{1/2} (\bar{z}_i - \bar{z}_j) \]
\[ e^\mu(q)_\pm = \frac{1}{\sqrt{2}} \left\{ \partial_{z} q^\mu = (\bar{z},1,-i,-\bar{z}) \right\} \]
\[ \partial_{\bar{z}} q^\mu = (z,1,i,-z) \]

Celestial amplitudes \( \tilde{A} \) of massless particles are obtained from momentum-space amplitudes \( A \) by Mellin transforms w.r.t. particle energies \( \Delta_j = 1 + i \lambda_j \)

\[ \tilde{A}_{\{\Delta_l\}}(z_l, \bar{z}_l) = \left( \prod_{l=1}^{n} \int_{0}^{\infty} \omega_l^{\Delta_l - 1} d\omega_l \right) \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N} \omega_k q_k) \]
\[ \times A(\omega_n, z_n, \bar{z}_n) \]

D=2 CFT correlators involve conformal wave packets
Gauge Amplitudes

four-gluon amplitude:

\[ \tilde{A}_4(-, -, +, +) = 8\pi \delta(r - \bar{r}) \theta(r - 1) \left( \prod_{i<j} z_{ij}^{\bar{h}_i - h_j - \bar{h}_j} \right) \times r^\frac{5}{3} (r - 1)^{\frac{2}{3}} \delta \left( -4 + \sum_{i=1}^{4} \Delta_i \right) \]

\[ r = \frac{z_{12} z_{34}}{z_{23} z_{41}} \]

conformal invariant cross-ratio on \( CS^2 \)

\[ r^{-1} = \sin^2 \left( \frac{\theta}{2} \right) \]

\[ h_1 = \frac{i}{2} \lambda_1, \quad h_2 = \frac{i}{2} \lambda_2, \quad h_3 = 1 + \frac{i}{2} \lambda_3, \quad h_4 = 1 + \frac{i}{2} \lambda_4 \]

\[ \bar{h}_1 = 1 + \frac{i}{2} \lambda_1, \quad \bar{h}_2 = 1 + \frac{i}{2} \lambda_2, \quad \bar{h}_3 = \frac{i}{2} \lambda_3, \quad \bar{h}_4 = \frac{i}{2} \lambda_4 \]

higher-point: involve Gaussian hypergeometric functions like string amplitudes

\[ \text{Pasterski, Shao, Strominger (2017)} \]

\[ \text{Schreiber, Volovich, Zlotnikov (2017)} \]
Graviton Amplitudes

four-graviton amplitude:

\[ \tilde{A}_4(\ldots,\ldots,+++,++) = 2\pi \delta(r-\bar{r}) \theta(r-1) \left( \prod_{i<j}^{4} \frac{h_i-h_j}{z_{ij}^{3/2}} \frac{\bar{h}_i-\bar{h}_j}{\bar{z}_{ij}^{3/2}} \right) \times r^{1/3-\beta/3} (r-1)^{-1/3-\beta/3} \delta \left( -2 + \sum_{i=1}^{4} \Delta_i \right) \]

\[
\begin{align*}
    h_1 &= -\frac{1}{2} + \frac{i}{2} \lambda_1, \quad h_2 = -\frac{1}{2} + \frac{i}{2} \lambda_2, \quad h_3 = \frac{3}{2} + \frac{i}{2} \lambda_3, \quad h_4 = \frac{3}{2} + \frac{i}{2} \lambda_4 \\
    \bar{h}_1 &= \frac{3}{2} + \frac{i}{2} \lambda_1, \quad \bar{h}_2 = \frac{3}{2} + \frac{i}{2} \lambda_2, \quad \bar{h}_3 = -\frac{1}{2} + \frac{i}{2} \lambda_3, \quad \bar{h}_4 = -\frac{1}{2} + \frac{i}{2} \lambda_4
\end{align*}
\]

- first calculation of graviton amplitude in the conformal basis
- important for the soft graviton theorems \( \Delta \to 1,0,\ldots \) in celestial basis

no holomorphic factorization (due to supertranslation operator \( P \))
Operator product expansion

Celestial conformal field theory (CCFT)

\[
\begin{align*}
\mathcal{O}^a_{\Delta_1, -1}(z, \bar{z}) \mathcal{O}^b_{\Delta_2, +1}(w, \bar{w}) &= \frac{C_{(-,+)}(\Delta_1, \Delta_2)}{z - w} \sum_c f^{abc} \mathcal{O}^c_{(\Delta_1 + \Delta_2 - 1), -1}(w, \bar{w}) \\
&+ \frac{C_{(-+)}(\Delta_1, \Delta_2)}{\bar{z} - \bar{w}} \sum_c f^{abc} \mathcal{O}^c_{(\Delta_1 + \Delta_2 - 1), +1}(w, \bar{w}) \\
&+ C_{(-+)}(\Delta_1, \Delta_2) \frac{\bar{z} - \bar{w}}{\bar{z} - \bar{w}} \delta^{ab} \mathcal{O}_{(\Delta_1 + \Delta_2), -2}(w, \bar{w}) \\
&+ C_{(-+)}(\Delta_1, \Delta_2) \frac{z - w}{z - w} \delta^{ab} \mathcal{O}_{(\Delta_1 + \Delta_2), +2}(w, \bar{w}) + \text{reg}.
\end{align*}
\]

Derive from collinear limits of D=4 EYM amplitudes

Fan, Fotopoulos, St.St., Taylor, Zhu (2019)

Derive from first principles and consistency conditions

Pate, Raclariu, Strominger, Yuan (2019)

\{ extended BMS symmetry

D=4 S-matrix constrains OPE or vice versa
Symmetries

At null infinity $\mathcal{I}^\pm$ more (hidden) symmetries present to constrain $S$-matrix

$\rightarrow$ non-trivial consistency on amplitudes

$z_i \rightarrow \frac{az_i + b}{cz_i + d}$

$SL(2,Z)_{z_i} : \tilde{A}_n(\{\Delta_i, J_i\}) \rightarrow (cz_i + d)^{\Delta_i + J_i} (\bar{c} \bar{z}_i + \bar{d})^{\Delta_i - J_i} \tilde{A}_n(\{\Delta_i, J_i\})$

$P_{-1/2,-1/2} = e^{(\partial + \bar{\partial})/2} = P^0 + P^3$

$P^{(j)}_{-1/2,-1/2} : \tilde{A}_n(\{\Delta_i, J_i\}) \rightarrow \tilde{A}_n(\{\Delta_j + 1, J_i\})$

comprises into translation operator $P^\mu$ shifts conformal dimension $\Delta_j$

celestial gravitational amplitudes appear as gauge amplitudes translated in space-time

St.St., Taylor (2018)
In usual QFT soft theorems $E_i \to 0$ play an important role in consistency and structure of amplitudes $(\omega_i \to 0)$ in Mellin space “soft-limits” reproduce Weinberg’s soft theorem

$$\Delta \to 0, 1, \ldots$$

also relate Ward identities and BMS symmetries:

explicit field realization

(i) energy-momentum tensor $T(z)$:

$$T(z) := \tilde{\mathcal{O}}_{\Delta=2, J=+2}(z, \bar{z}) = \frac{3}{\pi} \int d^2w \ (z - w)^{-4} \mathcal{O}_{\Delta=0, J=-2}(w, \bar{w})$$

$(h, \bar{h}) = (2, 0)$

shadow transformation $\text{Fotopoulos, Taylor (2019)}$

then:

$$\langle T(z) \prod_{i=1}^{n} O_{\Delta_i}(z_i, \bar{z}_i) \rangle = \sum_{i=1}^{n} \left( \frac{h_{O_i}}{(z - z_i)^2} + \frac{\partial_{z_i}}{z - z_i} \right) \langle \prod_{i=1}^{n} O_{\Delta_i}(z_i, \bar{z}_i) \rangle$$
OPE:

\[
T(w)T(z) = \frac{2T(z)}{(w - z)^2} + \frac{\partial_z T(z)}{w - z} + \ldots
\]

\[
T(w)\overline{T}(z) = \text{reg}.
\]

(ii) supertranslation operator \( P(z) \):

\[
P(z, \bar{z}) := \partial_{\bar{z}} \mathcal{O}_{\Delta=1,J=+2}(z, \bar{z})
\]

\((h, \bar{h}) = (\frac{3}{2}, \frac{1}{2})\)

then:

\[
\left\langle P(z_0) \prod_{j=1}^{n} \mathcal{O}_{\Delta_j,l_j}(z_j, \bar{z}_j) \right\rangle = \frac{1}{4} \sum_{i=1}^{n} \frac{c_i(\Delta_i)}{c_i(\Delta_i + 1)} \frac{1}{z_0 - z_i} \left\langle \prod_{n=1}^{n} \mathcal{O}_{\Delta_j,l_j}(z_j, \bar{z}_j) \right\rangle \bigg|_{\Delta_i \to \Delta_i + 1}
\]

OPE:

\[
T(w)P(z) = \frac{3}{2(w - z)^2} P(z) + \frac{1}{w - z} \partial_z P(z) + \text{reg}.
\]
In addition to Virasoro symmetry, we construct all supertranslation generators acting on primary fields

Fotopoulos, St.St., Taylor, Zhu (2019)

\[
P_{n-\frac{1}{2}, m-\frac{1}{2}} = \frac{1}{i\pi(n+1)} \int dw \, w^{n+1} [T(w), P_{n-\frac{1}{2}, m-\frac{1}{2}}]
\]

we find:

\[
\left[ P_{n-\frac{1}{2}, m-\frac{1}{2}}, \phi^{h,\bar{h}}(z, \bar{z}) \right] = z^n \bar{z}^m \phi^{h+\frac{1}{2}, \bar{h}+\frac{1}{2}}(z, \bar{z})
\]

local (or extended) BMS algebra:

\[
[P_{ij}, P_{k,l}] = 0 ,
\]

\[
[L_n, P_{k,l}] = \left( \frac{1}{2} n - k \right) P_{n+k,l} + n(n^2 - 1) C_{n,k} ,
\]

\[
[\bar{L}_n, P_{k,l}] = \left( \frac{1}{2} n - l \right) P_{k,n+l} + n(n^2 - 1) \bar{C}_{n,l} .
\]

Barnich (2017)

Conformal soft-theorems ↔ Ward identities ↔ BMS algebra
**BMS± group = symmetry of asymptotically flat D=4 space-time at null infinity \( \mathcal{J}^\pm \)**

<table>
<thead>
<tr>
<th><strong>Global BMS Symmetry on Celestial Sphere</strong></th>
<th><strong>Local BMS Symmetry on Celestial Sphere</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Lorentz Group:</strong> global conformal transformations on celestial sphere ( \text{SL}(2,\mathbb{C}) )</td>
<td>local conformal transformations = superrotations ( T(z) )</td>
</tr>
</tbody>
</table>
| \( z \rightarrow \frac{az+b}{cz+d} \) | \[ [L_m, L_n] = (m-n) \ L_{m+n} \]
| \( L_{-1} = \partial \) \ | \[ [\bar{L}_m, \bar{L}_n] = (m-n) \ \bar{L}_{m+n} \]
| \( L_0 = z\partial + h \) \ | \( L_1 = z^2\partial + 2hz \) |

**Global Space-time Translation:**
Abelian subgroup of supertranslations

| \( P_{-1/2,-1/2} = e^{(\partial_h+\partial_{\bar{h}})/2} \) | \( P_{1/2,1/2} = z \ e^{(\partial_h+\partial_{\bar{h}})/2} \) |
| \( P_{-1/2,1/2} = \bar{z} \ e^{(\partial_h+\partial_{\bar{h}})/2} \) | \( P_{-1/2,1/2} = |z|^2 \ e^{(\partial_h+\partial_{\bar{h}})/2} \) |

**Symmetries of the Celestial OPEs and Correlators S-Matrix (Non-trivial Consistency)**
Further Directions

• understand Virasoro central charge (-one-loop ?)

• establish double-copy structure
  (elaborate on gauge/gravity connections)

• high-energy (large $\lambda$) limit: string world-sheet = celestial sphere
  celestial $CFT_2 \simeq$ string world-sheet $CFT_2$

• understanding the nature of 2D CFT on celestial sphere
  would enable a holographic description of flat spacetime

• uplift $AdS_3/CFT_2$ holography to $M_4$
  towards flat space-time holography