Calabi-Yaus for scattering amplitudes from twistor geometry

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Work to appear, in collaboration with Matthias Volk.
Building on previous work with Andrew McLeod, Jacob Bourjaily, Matt von Hippel, Hui-He and Matthias Wilhelm.

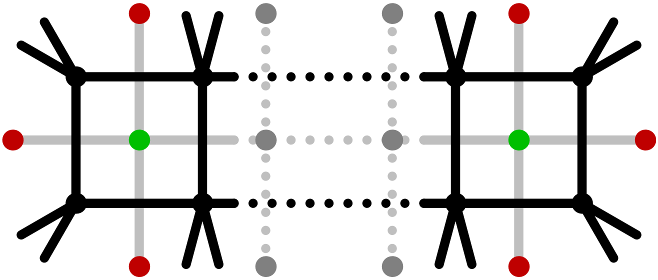
Context

- Iterated integrals have been ubiquitous in loop integral calculations so far. They are very interesting number-theoretically and are relatively easy to work with computationally. They can be integrated to special functions called (generalized) polylogarithms.
- We are not yet at a level where the calculations can be completely automated, due to various algebraic complications. During direct integration algebraic numbers appear and they satisfy non-trivial factorization identities, which makes automated simplification challenging.
- However, by any reasonable measure, most integrals at higher loops will not be of iterated type. Their geometry contains genus one curves or higher-dimensional Calabi-Yau varieties.

I will try to provide some answers to the following questions

1. When do genus one curves or Calabi-Yau varieties appear?
2. Why are they Calabi-Yau?
3. What can we say about their topology?
4. Supersymmetrization.

For definiteness, I will focus on a subclass of integrals in $\mathcal{N} = 4$ theory, called train-track integrals, but the methods apply more generally. This class of integrals was studied by Bourjaily, He, McLeod, von Hippel, Wilhelm.



There are two ways to check if a given integral has Calabi-Yau geometry. The hard way is to do direct integration until one obtains

$$\int_{\Gamma} \Omega \times \text{Polylogarithms}, \tag{1}$$

where Γ is some contour and Ω is a top holomorphic form on some interesting algebraic variety. For example, for a genus one curve of equation $y^2 = P_4(x)$ we have $\Omega = \frac{dx}{y}$.

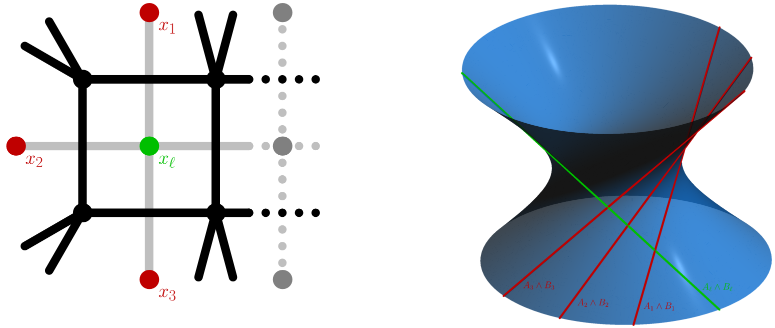
The easy way is to compute residues around poles where propagators vanish until this is not possible anymore (sometimes Jacobians are generated and one may take residues around their poles too). If, after applying this procedure, one obtains a *constant* function (zero-form), then the initial integral is called *pure* and, in all known examples, can be expressed in terms of generalized polylogarithms.

Momentum twistors are a way to make the dual conformal symmetry manifest (Hodges).

dual space	momentum twistor space $\mathbb{P}^3$
point x	line L_x
$(x-y)^2 = 0$	intersecting lines L_x, L_y

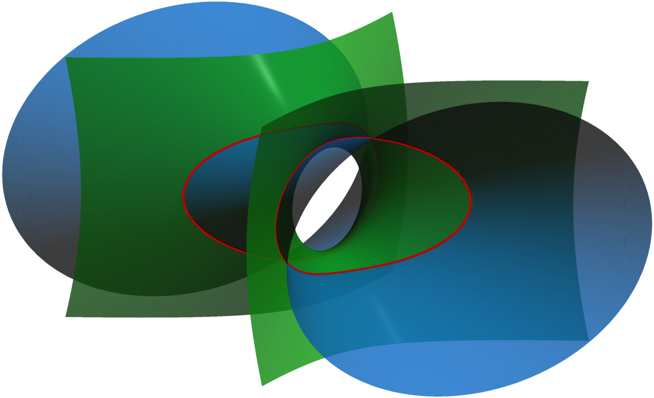
Table: Correspondence

In momentum twistor space the locus where propagators go on-shell has a simple geometrical interpretation in terms of intersections of lines.



x_1
x_2
x_ℓ
x_3
$A_3 \wedge B_3$
$A_2 \wedge B_2$
$A_1 \wedge B_1$
$A_\ell \wedge B_\ell$

The intersection of two quadrics in $\mathbb{P}^3$ is a genus one curve.



Twistor construction for the two-loop train-track

What is the leading singularity locus (and the holomorphic one-form) in twistor language? We can build a quadric Q_l from the left three lines and a quadric Q_r from the right three lines. These quadrics intersect in a curve $C = Q_l \cap Q_r$. Given a point $p \in C$, through p passes a line intersecting the three defining lines of Q_l and a line intersecting the three defining lines of Q_r.
The holomorphic one-form can be found by taking two Poincaré residues

$$\omega_C = \mathrm{Res}_{Q_l} \mathrm{Res}_{Q_r} \frac{\omega_{\mathbb{P}^3}}{Q_l Q_r}, \tag{2}$$

where $\omega_{\mathbb{P}^3} = x_0 dx_1 \wedge dx_2 \wedge dx_3 - x_1 dx_0 dx_2 dx_3 + \cdots$ is the $\mathrm{PGL}(4)$-invariant weight four form on $\mathbb{P}^3$.

Comparing genus one curves

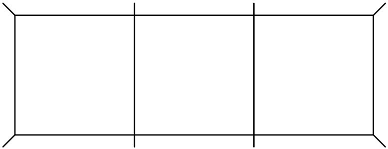
The curve C can be characterized by the complex structure modulus τ or by the j-invariant. The computation of τ involves integrals, while j can be defined algebraically.
The curve C is the intersection of a pencil of quadrics $\mu_0 Q_l + \mu_1 Q_r$. A member of this pencil becomes singular at four points.[1] From these four points in $\mathbb{P}^1$ with coordinates $[\mu_0 : \mu_1]$ we can build a cross-ratio λ. Then the j-invariant is

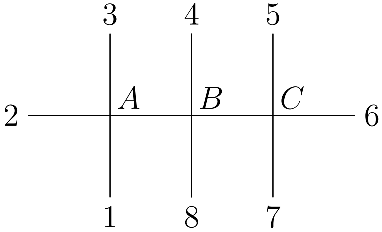
$$j = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}. \tag{3}$$

The j-invariant can also be calculated by doing the integrals using Feynman parametrization. This calculation looks very different but the j-invariants agree.

[1] A quadric in $\mathbb{P}^3$ can be thought as a 4×4 matrix which becomes singular when its determinant vanishes. This determinant is of degree four in λ_0, λ_1.

Three-loop train-track

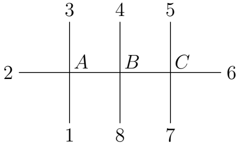




Consider next the three-loop train-track diagram. Its leading singularity locus has been studied by Bourjaily, He, McLeod, von Hippel, Wilhelm by some laborious procedure (using Feynman parametrization and involving computer calculations using `Macaulay2`).

We can instead do this analysis in momentum twistor space.

Three-loop train-track twistor geometry



Here are the steps of the geometric construction:

1. A point on L_4 and another on L_8 define a line L_B.
2. The lines L_1, L_2 and L_3 determine a quadric Q_l.
3. The lines L_5, L_6 and L_7 determine a quadric Q_r.
4. The line L_B generically intersects[2] Q_l in two points and Q_r in two points (Bézout).
5. The condition that the line L_B is tangent to Q_l is an equation of bidegree $2, 2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ (which is a genus one curve).
6. The K3 surface is then a branched cover over $\mathbb{P}^1 \times \mathbb{P}^1$.

[2] We take the line L_B not to be contained in Q_l or Q_r.

Three-loop train-track twistor geometry

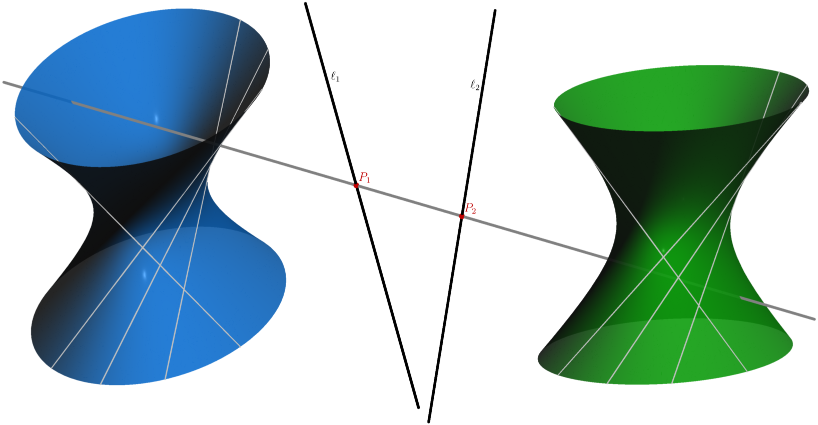


Figure: K3 twistor geometry.

Leading singularity as a branched cover

The leading singularity locus is a four-fold cover over a generic point in $\mathbb{P}^1 \times \mathbb{P}^1$ (two intersections with Q_l and two intersections with Q_r). It is a double cover over the genus one curve C_l (corresponding to a tangent to Q_l and two intersections with Q_r). It is also a double cover over C_r (tangent to Q_r and two intersections with Q_l). Finally, there is no branching over the eight intersection points of $C_l \cap C_r$.
This is an analog of the construction of a genus one curve as a double cover branched over four points on $\mathbb{P}^1$. From these four points we can compute a cross-ratio and a j-invariant. What is the analog for K3?

Euler characteristic

We use surgery. We have

- four copies of the points $\mathbb{P}^1 \times \mathbb{P}^1 - C_l \cup C_r$
- two copies of the points $C_l \cup C_r - C_l \cap C_l$
- one copy of the points $C_l \cap C_r$

We also know that

- $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = \chi(\mathbb{P}^1)^2$.
- $\chi(\mathbb{P}^1) = 2$ since $\mathbb{P}^1$ is a two-sphere.
- $\chi(C_l) = \chi(C_r) = 0$ since C_l and C_r are tori.
- $\chi(pt) = 1$.
- inclusion-exclusion $\chi(C_l \cup C_r) = \chi(C_l) + \chi(C_r) - \chi(C_l \cap C_r)$.

Euler characteristic

Then,

$$\chi(S) = 4(\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(C_l \cup C_r))+ \\ 2(\chi(C_l \cup C_r) - \chi(C_l \cap C_r)) + \chi(C_l \cap C_r) = \\ 4\chi(\mathbb{P}^1 \times \mathbb{P}^1) - 2\chi(C_l \cup C_r) - \chi(C_l \cap C_r) = \\ 4 \times 2 \times 2 - 2 \times (-8) - 8 = 24. \quad (4)$$

The Hodge diamond of K3 is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad (5)$$

We find that the generic Picard rank is 9 and a moduli space of dimension 11.

The three-fold

One can also build the three-fold (corresponding to a four-loop train-track) as a toric CICY.

$$\mathbb{P}\left(\begin{smallmatrix} 1&1&0&0&1&0&0&0&0&0\\ 0&0&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&1\\ 0&0&0&0&0&0&0&1&1&1 \end{smallmatrix}\right) \quad \left[\begin{array}{c|c|c} 2&0&1\\ 2&0&1\\ 0&2&1\\ 0&2&1 \end{array}\right]^{12,28}_{-32} \tag{6}$$

We have $h_{11} = 12$, $h_{2,1} = 28$ and $\chi = -32$. This result was obtained using the `nef.x` computer program by Kreutzer et al. The complete intersections become high codimension which makes them hard to analyze.

Why Calabi-Yau?

Plausibility arguments:

- We start in compact, complex, Kähler manifold such as momentum twistor or several copies thereof.
- From the beginning we have a nowhere vanishing $\mathrm{PGL}(4)$-invariant differential form on the embedding space.
- We take Poincaré residues until it's not possible anymore, so we have a holomorphic form.

Some caveats:

- For some values of the kinematics there may be singularities and it becomes possible to take further residues.
- There may be *several* holomorphic top forms. For $\mathcal{N}$ they should fit in a representation of $\mathrm{PGL}(4)$.

Supersymmetry

All the constructions involve lines and incidence relations. Can be expressed using delta functions

- $\delta^3_{\mathbb{P}^3}(P_1; P_2)$
- $\delta^2_{\mathbb{P}^3}(L; P) = \int \omega_{\mathbb{P}^1}(\alpha)\delta_{\mathbb{P}^3}(\alpha_0 P_0 + \alpha_1 P_1; P)$
- $\delta_{\mathbb{P}^3}(L_1; L_2) = \int \omega_{\mathbb{P}^1}(\alpha)\delta_{\mathbb{P}^3}(L_1; \alpha_0 P_0 + \alpha_1 P_1)$
- $\delta_{\mathbb{P}^3}(Q; P) = \int \mu(L)\delta(L; L_1)\delta(L; L_2)\delta(L; L_3)\delta^2(L; P)$.

Can supersymmetrize the delta functions [Mason & Skinner] so we have for example $\delta^{1|8}_{\mathbb{P}^{3|4}}(\mathcal{Q}; \mathcal{P})$.

Supersymmetrization

"Intersection" of two superquadrics $\mathcal{Q}_l$ and $\mathcal{Q}_r$. *Not* a super-Riemann surface!
We can define a $1|12$-form

$$\omega_{\mathcal{C}}^{1|12} = \int \omega_{\mathbb{P}^{3|4}}(\mathcal{P}) \delta_{\mathbb{P}^{3|4}}^{1|8}(\mathcal{Q}_l; \mathcal{P}) \delta_{\mathbb{P}^{3|4}}^{1|8}(\mathcal{Q}_r; \mathcal{P}). \qquad (7)$$

The construction of supersymmetrization generalizes straightforwardly to other cases.
What restrictions does the existence of this extra structure (which is mysterious from a mathematical point of view) imply?

Open questions

- Mirror symmetry (*à la* Bloch, Kerr, Vanhove)?
- Topology at higher loops.
- Non-planar integrals?
- Explicit computation?

The End