Renormalization and Scale Evolution of the Soft-Quark Soft Function

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June 8, 2020

World SCET 2020

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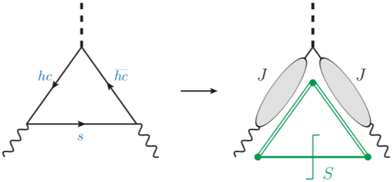
Based on arXiv:2005.03013

Motivation

- Soft functions are generic in SCET $\Rightarrow$ large distance Physics
  - LO in power counting: soft gauge bosons only (eikonal)
  - sub-LO in power counting: soft fermions as well
    (Moult, Stewart, Vita; Moult, Stewart,Vita,Zhu;
    Beneke, Broggio,Garny,Jaskiewicz, Szafron, Vernazza,Wang;...)
- Soft-quark soft function appears in the factorization of $H \to \gamma\gamma$ with light quark loops (Liu, Neubert: 1912.08818)



$$T_3 = H_3^{(0)} \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} J^{(0)}(M_h\ell_-)\, J^{(0)}(-M_h\ell_+)\, S^{(0)}(\ell_+\ell_-)$$

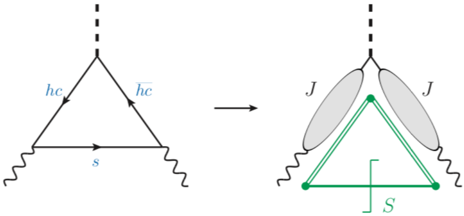
Leading double logs in T_3 originate from soft terms!

Definition

Soft-quark soft function is defined from the discontinuity of the soft quark propagator dressed by two finite-length Wilson lines:

$$-\frac{(4\pi)^{1-\epsilon}}{N_c}\, e^{\epsilon\gamma_E}\, \mu^{2\epsilon}\, \langle 0|\, T\, \mathrm{Tr}\, S_{\bar{n}}(0, r_1\bar{n})\, q_s(r_1\bar{n})\, \bar{q}_s(r_2 n)\, S_n(r_2 n, 0)\, |0\rangle$$

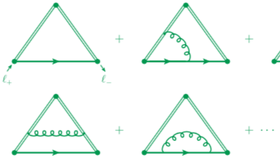
$$S(w) = \frac{1}{2\pi i}\left[\mathcal{S}(w+i0) - \mathcal{S}(w-i0)\right]$$



Bare soft function at one loop

$$S^{(0)}(w) = m_{b,0}\,\mu^{2\epsilon}\left[S_a^{(0)}(w)\,\theta(w-m_{b,0}^2) + S_b^{(0)}(w)\,\theta(m_{b,0}^2-w)\right]$$

Some of the diagrams that contribute at one loop:



$$S_a^{(0)}(w) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(w-m_{b,0}^2\right)^{-\epsilon}\left[1+\epsilon\,\frac{C_F\alpha_{s,0}}{4\pi}\,2e^{\epsilon\gamma_E}\,\frac{3-2\epsilon}{1-2\epsilon}\,\Gamma(\epsilon)\,\frac{\left(m_{b,0}^2\right)^{1-\epsilon}}{w-m_{b,0}^2}\right]$$

$$+\frac{C_F\alpha_{s,0}}{4\pi}\left[\left(-\frac{2}{\epsilon^2}+\frac{6}{\epsilon}+\frac{2}{\epsilon}\,\ln\!\left(1-\frac{1}{\hat{w}_0}\right)+12-\frac{\pi^2}{3}\right)\left(w-m_{b,0}^2\right)^{-2\epsilon}\right.$$

$$\left.-2\,\mathsf{Li}_2\!\left(\frac{1}{\hat{w}_0}\right)-2\left(\ln\hat{w}_0-1\right)\ln\!\left(1-\frac{1}{\hat{w}_0}\right)-3\ln^2\!\left(1-\frac{1}{\hat{w}_0}\right)+\mathcal{O}(\epsilon)\right],$$

$$S_b^{(0)}(w) = \frac{C_F\alpha_{s,0}}{4\pi}\left(m_{b,0}^2\right)^{-2\epsilon}\left[-\frac{4}{\epsilon}\,\ln(1-\hat{w}_0)+6\ln^2(1-\hat{w}_0)+\mathcal{O}(\epsilon)\right]$$

Renormalization at one loop

Renormalizing only the b-quark mass and α_s in the bare soft function **does not** remove all the $\frac{1}{\epsilon^n}$ poles.
Operator renormalization is required.

$$S(w,\mu) = \int_0^\infty dw' \, Z_S(w,w';\mu) \, S^{(0)}(w')$$

How?

$$T_3 = H_3^{(0)} \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \, J^{(0)}(M_h \ell_-) \, J^{(0)}(-M_h \ell_+) \, S^{(0)}(\ell_+ \ell_-)$$

Derive Z_s from the known RGE's of the jet functions $J^{(0)}$ and the hard function $H_3^{(0)}$ with the conjecture that T_3 is scale invariant.

$$Z_S(w,w';\mu) = \left[1 + \frac{C_F \alpha_s}{4\pi} \left(\frac{2}{\epsilon^2} - \frac{2}{\epsilon} \ln \frac{w}{\mu^2} - \frac{3}{\epsilon}\right)\right] \delta(w - w') - \frac{C_F \alpha_s}{\pi \epsilon} \, w \, \Gamma(w,w') + \mathcal{O}(\alpha_s^2)$$

$$\Gamma(\omega,\omega') = \left[\frac{\theta(\omega-\omega')}{\omega(\omega-\omega')} + \frac{\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}\right]_+$$

The renormalized $S(w,\mu)$ is free of $\frac{1}{\epsilon^n}$ poles.

Renormalized Soft Function

$$S(w,\mu) = m_b \left[S_a(w,\mu)\, \theta(w - m_b^2) + S_b(w,\mu)\, \theta(m_b^2 - w) \right]$$

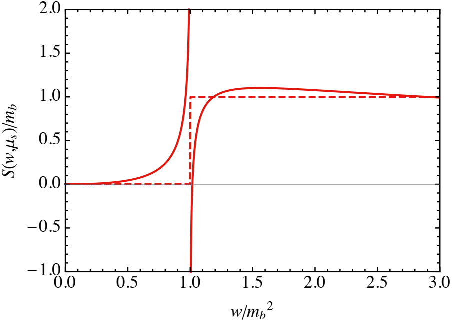
with

$$S_a(w,\mu) = 1 + \frac{C_F \alpha_s}{4\pi} \Big[- L_w^2 - 6L_w + 3L_m + 8 - \frac{\pi^2}{2} + 2\text{Li}_2\Big(\frac{1}{\hat{w}}\Big) - 4\ln\Big(1 - \frac{1}{\hat{w}}\Big) \Big(L_m + 1 + \ln\Big(1 - \frac{1}{\hat{w}}\Big) + \frac{3}{2}\ln \hat{w} \Big) \Big],$$

$$S_b(w,\mu) = \frac{C_F \alpha_s}{\pi} \ln(1 - \hat{w}) \left[L_m + \ln(1 - \hat{w}) \right]$$

$$L_w = \ln(w/\mu^2), \quad L_m = \ln(m_b^2/\mu^2)$$

Renormalized soft function $S(w,\mu)/m_b$ for $\mu = m_b$ at tree level (dashed) and one-loop order (solid).



Renormalization Group Equation

Remember: The renormalized soft function is part of T_3 in the physical amplitude for $H \to \gamma\gamma$. $\Rightarrow$ It should be free of large logs. Resummation on $S(w,\mu)$ required!

From the RG invariance of T_3 and the known two loop RGE's of J and H_3 one can write the two loop RGE for the soft function.

(Two loop RGE of jet function: Liu, Neubert; 2003.03393)

$$\frac{d}{d\ln\mu} S(w,\mu) = -\int_0^\infty dw'\, \gamma_S(w,w';\mu)\, S(w',\mu)$$

$$\gamma_S(w,w';\mu) = -\left[\Gamma_{\text{cusp}}(\alpha_s)\, \ln\frac{w}{\mu^2} - \gamma_s(\alpha_s)\right]\delta(w-w') - 2\Gamma_{\text{cusp}}(\alpha_s)\, w\, \Gamma(w,w') - 2C_F\left(\frac{\alpha_s}{2\pi}\right)^2 \frac{w\,\theta(w'-w)}{w'(w'-w)}\, h\left(\frac{w}{w'}\right) + \mathcal{O}(\alpha_s^3)$$

$\star$ It is possible to build a *diagonal space* where the RG evolution of the soft function becomes local in w at all orders. Similarly to the *dual space* for LCDA for $B-$mesons.

(Bell,Feldmann,Wang,Yip:1308.6114)

Solution of the RGE

To find a solution to the RGE the key observation is that:

$$\mathcal{F}(a) \equiv \int_0^\infty dw'\, w\, \Gamma(w,w') \left(\frac{w'}{w}\right)^a = \left[H(a) + H(-a)\right]$$

$$\mathcal{H}(-a) \equiv \frac{1}{\beta_0}\int_0^\infty dw'\, \frac{w\,\theta(w'-w)}{w'(w'-w)}\, h\!\left(\frac{w}{w'}\right)\left(\frac{w'}{w}\right)^a = \frac{1}{\beta_0}\int_0^1 \frac{dx}{1-x}\, h(x)\, x^{-a}$$

$\Rightarrow$ Look for a solution that has a power law in w: Use the Laplace transform of the soft function wrt $\ln(w/m_b^2)$: $S(w,\mu) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\eta\, \tilde{S}(\eta,\mu)\left(\frac{w}{m_b^2}\right)^\eta$

Then the following ansatz gives a solution to the RGE:

$$\left(\frac{w}{\mu_s^2}\right)^{\eta - a_\Gamma(\mu_s,\mu)} \exp\left[2S(\mu_s,\mu) + a_{\gamma_s}(\mu_s,\mu)\right] \exp\left[2\int\limits_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha\, \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}\, \mathcal{F}\big(\eta - a_\Gamma(\mu_s,\mu_\alpha)\big)\right]$$

$$\times \exp\left[\int\limits_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\left[2C_F\left(\frac{\alpha}{2\pi}\right)^2 \beta_0\, \mathcal{H}\big(a_\Gamma(\mu_s,\mu_\alpha) - \eta\big) + \mathcal{O}(\alpha^3)\right]\right];$$

$$S(\mu_s,\mu) = -\int\limits_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha\, \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)} \int\limits_{\alpha_s(\mu_s)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}$$

$$a_\Gamma(\mu_s,\mu) = -\int\limits_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha\, \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}$$

Solution to the RGE

The exact solution of the RGE of the soft function at NLO:

$$S(w,\mu) = U_S(w;\mu,\mu_s)\int_0^\infty \frac{dw'}{w'}\, S(w',\mu_s)\, \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} d\eta \left(\frac{w}{w'}\right)^{\eta} \frac{\Gamma^2\big(1-\eta+a_\Gamma(\mu_s,\mu)\big)\,\Gamma^2(1+\eta)}{\Gamma^2\big(1+\eta-a_\Gamma(\mu_s,\mu)\big)\,\Gamma^2(1-\eta)}$$

$$\times\left[1-\frac{C_F}{\beta_0\pi}\int\limits_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha \int_0^1 \frac{dx}{1-x}\, h(x)\, x^{a_\Gamma(\mu_s,\mu_\alpha)-\eta} + \mathcal{O}(\alpha_s^2)\right]$$

Evolution kernel:

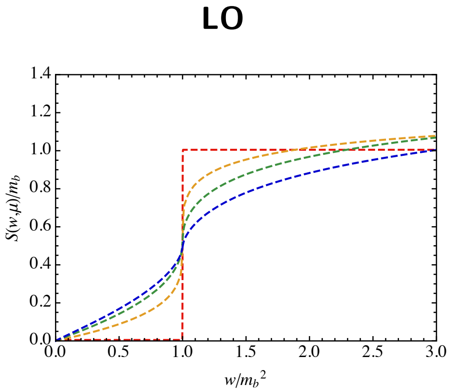
$$U_S(w;\mu,\mu_s) = \left(\frac{we^{-4\gamma_E}}{\mu_s^2}\right)^{-a_\Gamma(\mu_s,\mu)} \exp\left[2S(\mu_s,\mu)+a_{\gamma_s}(\mu_s,\mu)\right]$$

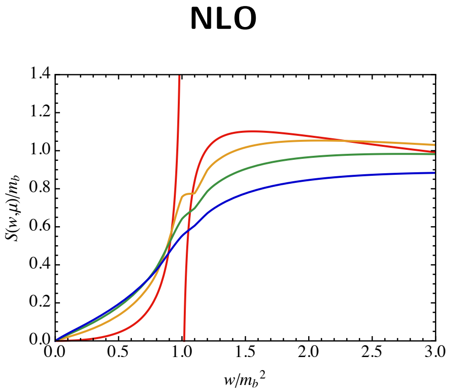
It is possible to analytically integrate over η and obtain an η-independent result in terms of Meijer-G functions:

$$S(w,\mu) = U_s(w;\mu,\mu_s)\int_0^\infty \frac{dx}{x} S(w/x,\mu_s) G_{4,4}^{2,2}\left(\begin{matrix} -a,-a,1-a,1-a \\ 1,1,0,0 \end{matrix}\,\middle|\, \frac{1}{x}\right)$$

$$-\,m_b\frac{C_F\alpha_s(\mu_s)}{\pi}\int_0^1 \frac{dx}{1-x}\frac{h(x)}{\beta_0}\frac{r^{1+\frac{2C_F}{\beta_0}\ln x}-1}{1+\frac{2C_F}{\beta_0}\ln x} G_{4,4}^{2,2}\left(\begin{matrix} -a,-a,1-a,1-a \\ 0,1,0,0 \end{matrix}\,\middle|\, \frac{xm_b^2}{w}\right)$$

Evolution at LO and NLO in RG-improved PT





$\mu = m_b$, $\mu = 10\,GeV$, $\mu = 20\,GeV$, $\mu = 40\,GeV$.

Matching scale set to $\mu_s = m_b$.

$\star$The discontinuity is smoothed out after evolution.

Back to the initial motivation: T_3 convolution

$$T_3 = H_3(\mu) \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} J(M_h\ell_-,\mu) J(-M_h\ell_+,\mu) S(\ell_+\ell-,\mu)$$

There are two things concerning the T_3 convolution:

1. It is not well defined because of its end-point divergences
(see Ze Long Liu's talk)

$$T_{3,\mathrm{LO}} = H_3(\mu)\, \exp\Big[-4S(\mu_j,\mu) - 2a_{\gamma'}(\mu_j,\mu)\Big] \frac{\Gamma^2\big(1-a_\Gamma(\mu_j,\mu)\big)}{\Gamma^2\big(1+a_\Gamma(\mu_j,\mu)\big)} \int_0^\infty \frac{dw}{w}\, S(w,\mu) \left(\frac{-M_h^2\, we^{-4\gamma_E}}{\mu_j^4}\right)^{a_\Gamma(\mu_j,\mu)} \int_0^\infty \frac{d\ell_-}{\ell_-}$$

⋆ Resummation can only cure the divergence in $w \Rightarrow$ rapidity regulators needed

2. In deriving the Z_S we assumed that T_3 is scale invariant.

× In momentum space generic rapidity regulators break T_3 scale invariance!

RG invariance and rapidity regularization

- **Analytic regulator**

$$\int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \to \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \left(\frac{M_h(\ell_+ - \ell_-) - i0}{\nu^2}\right)^{-2\delta}$$

- **Cut off scheme**

$$\int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \to \lim_{\sigma \to -1} \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+}$$

Even at $\mathcal{O}(\alpha_s)$ the scale invariance is broken for both schemes:

$$\frac{dT_3^{\text{analytic}}}{d\ln\mu} = \frac{N_c \alpha_b}{\pi} \frac{y_b}{\sqrt{2}} m_b \frac{C_F \alpha_s}{4\pi} \left(\frac{-M_h^2 m_b^2}{\nu^4}\right)^{-\delta} \frac{8\Gamma^2(\delta)}{\Gamma(1+2\delta)} \left[-H(\delta) - H(-\delta)\right] + \mathcal{O}(\alpha_s^2)$$

$$\frac{dT_3^{\text{cutoff}}}{d\ln\mu} = \frac{N_c \alpha_b}{\pi} \frac{y_b}{\sqrt{2}} m_b \frac{C_F \alpha_s}{4\pi} 16\zeta_3 + \mathcal{O}(\alpha_s^2)$$

Cut off on the rapidity $y = \frac{1}{2} \ln \frac{\ell_+}{\ell_-}$? It will still break the RG invariance starting at $\mathcal{O}(\alpha_s^2)$.

How about the initial conjecture on the RG inariance of T_3?

- In the diagonal space the T_3 integrand is local

- Rapidity regularization can be imposed and have a *point-wise* RG invariance in (ℓ_+, ℓ_-) plane.

$$T_3^{diag} = H_3(\mu) \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \, j(M_h \ell_-, \mu) \, j(-M_h \ell_+, \mu) \, s(\ell_+ \ell_-, \mu)$$

This would justify the fact that Z_s derived based on this conjecture cancels all the $1/\epsilon^n$ poles of bare soft function at one-loop.

Conclusions

- The Z_S derived at one loop for the soft-quark soft function cancels all the remaining $1/\epsilon^n$ poles.
- We have seen an exact solution to the RGE of the soft quark soft function in momentum space.
- Regularizing the rapidity divergences in the convolution integral T_3 is generically incompatible with its RG invariance.
- The convolution T_3 can be regularized with rapidity regulators and remain locally scale invariant, in the diagonal space.

Thank you!

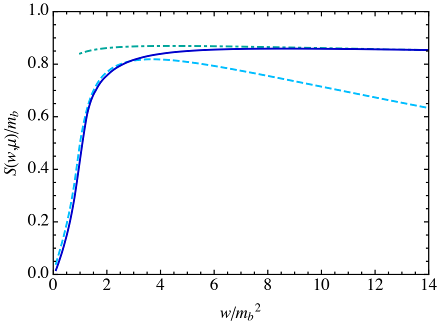
Back up slides

Dynamical scale setting

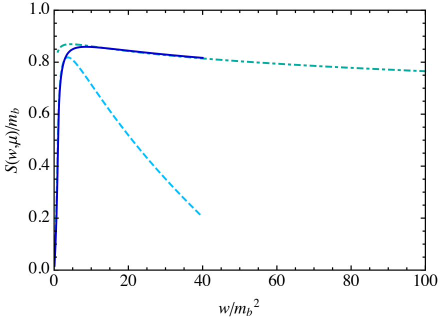
In the asymptotic limit $\omega \gg m_b^2$ the solution of the RGE takes a simplified form

$$S_\infty(\omega,\mu) = m_b(\mu_s) U_s(\omega;\mu,\mu_s) \mathcal{S}_\infty(\partial,\mu_s) \left(\frac{\omega}{\mu_s}\right)^\eta \frac{\Gamma^2(1-\eta+a_\Gamma)\Gamma^2(1+\eta)}{\Gamma^2(1+\eta-a_\Gamma)\Gamma^2(1-\eta)} \times \exp\left[-C_F \int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} \frac{d\alpha}{\pi} \mathcal{H}\left(a_\Gamma(\mu_s,\mu_\alpha)-\eta\right) + \mathcal{O}(\alpha_s^2)\right]_{|\eta=0}$$

On the other hand the study of the renormalized soft function at a broader range of scaling requires a dynamical scale setting $\mu_s^2 = r_s\omega$.



1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
$S(w,\mu)/m_b$
$w/m_b{}^2$



1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
$S(w,\mu)/m_b$
$w/m_b{}^2$

Diagonal Space

If we look closer at the solution in Laplace space:

$$\tilde{S}(\eta,\mu) = U_s(m_b^2;\mu,\mu_s)\frac{\Gamma^2(1+\eta+a_\Gamma)\Gamma^2(1-\eta)}{\Gamma^2(1-\eta-a_\Gamma)\Gamma^2(1+\eta)}\exp\left[-C_F\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)}\frac{d\alpha}{\pi}\mathcal{H}(-\eta-a_\Gamma(\mu_\alpha,\mu))\right]\tilde{S}(\eta+a_\Gamma,\mu_s)$$

we can split the integration boundary at some scale ρ where $\alpha_s = \alpha_s(\rho)$ and define a new function s.t

$$s(\omega,\mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta\tilde{S}(\eta,\mu)\frac{\Gamma^2(1+\eta)}{\Gamma^2(1-\eta)}\exp\left[-C_F\int_{\alpha_s(\mu)}^{\alpha_s'}\frac{d\alpha}{\pi}\mathcal{H}(-\eta-a_\Gamma(\mu_\alpha,\mu))\right]\left(\frac{\omega}{m_b^2}\right)^\eta$$

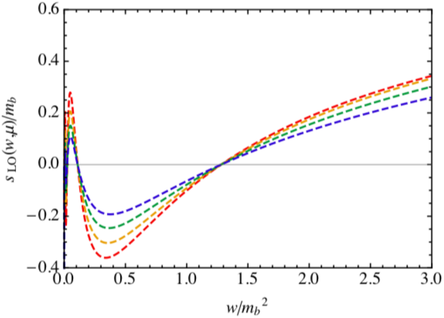
Then it is not difficult to show that the $s(\omega,\mu)$ evolves locally.

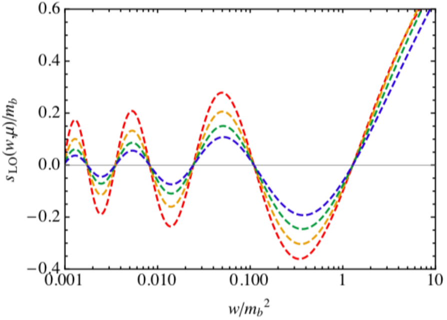
$$s(\omega,\mu) = U_s(\omega;\mu,\mu_s)s(\omega,\mu_s)$$

This is equivalent to finding a basis where the γ_s is diagonal to all orders.

Also the jet function appearing in T_3 can be written in the diagonal space.

LO soft function in the diagonal space for different scales





Diagonal space

The soft function in the dual space is not independent on the scale ρ:

$$\frac{d}{d\ln\rho}s(\omega;\mu,\rho) = \left(2C_F\left(\frac{\alpha_s(\rho)}{2\pi}\right)^2\int_0^1\frac{dx}{1-x}h(x)x^{a_\Gamma(\mu,\rho)} + \mathcal{O}(\alpha_s^3)\right)s(\eta,\omega/x,\rho)$$

Momentum Space $\Leftrightarrow$ Diagonal space:

There is a well defined way to transform the soft function $s(\omega,\mu)$ from the diagonal space to the soft function $S(\omega,\mu)$ in momentum space, via a *transfer* function F_s:

$$s(\omega,\mu) = \int_0^\infty\frac{dx}{\sqrt{x}}F_s(x)S(x\omega,\mu) \qquad S(\omega,\mu) = \int_0^\infty\frac{dx}{\sqrt{x}}s(\omega/x,\mu)F_s^{-1}(x), \quad x = \frac{\omega'}{\omega}$$

$$F_s(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\eta\frac{\Gamma^2(1+\eta)}{\Gamma^2(1-\eta)}\frac{x^{-\eta}}{\sqrt{x}}\exp\left[-C_F\int_{\alpha_s(\mu)}^{\alpha_s'}\frac{d\alpha}{\pi}\mathcal{H}(-\eta-a_\Gamma(\mu_\alpha,\mu))\right]$$

$$F_s^{-1}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\eta\frac{\Gamma^2(1+\eta)}{\Gamma^2(1-\eta)}\frac{x^{-\eta}}{\sqrt{x}}\exp\left[C_F\int_{\alpha_s(\mu)}^{\alpha_s'}\frac{d\alpha}{\pi}\mathcal{H}(\eta-a_\Gamma(\mu_\alpha,\mu))\right]$$