A Finite S-Matrix

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arXiv: 1906.03271 with M. Schwartz[1]
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The S-matrix

- Leads to predictions for colliders.
  - Standard model observables computed to N3LO.
- Properties extensively studied.
  - What is the best way to **encode its content** (spinors, twistors, amplituhedron)?
  - What are its **symmetries** (dual conformal invariance, Steinmann relations)?
- Despite this interest, **the *S*-matrix does not exist**.
  - S-matrix elements are divergent in perturbation theory and zero non-perturbatively in theories with massless particles.
- What is the **fundamental object of interest**?

Outline

- Observations about amplitudes in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory (SYM).
  - **Exponentiation** of 4 and 5-point amplitudes.
  - Symmetries of **remainder functions** for higher point amplitudes.
  - Can we use SCET to explain observations in $\mathcal{N} = 4$?
- Definition of "hard" S-matrix: $\boldsymbol{S_H}$
  - Use **universality** of asymptotic interactions to extract interesting part of scattering amplitudes.
  - Interpretations of S_H:
    a. Wilson coefficients in SCET
    b. Remainder functions in $\mathcal{N} = 4$ SYM
    c. Coherent states

Introduction - Scattering Amplitudes in $\mathcal{N} = 4$

- Recently, amplitudes in $\mathcal{N} = 4$ SYM have been **bootstrapped** to 6 and 7 loops [1].
- IR divergences obscure symmetries and simplicity of $\mathcal{N} = 4$ amplitudes.
- Compute **remainder functions R** instead.
  - **Remainder function**: A **ratio** of full amplitude and an exponentiation of 1-loop divergences.
- Remainder functions sometimes obey **Steinmann relations** and **dual conformal invariance** used in bootstrapping.

Can SCET help with understanding why remainder functions in $\mathcal{N} = 4$ simplify and are interesting?

4-Point Amplitude in $\mathcal{N} = 4$

- 4-point 1-loop amplitude in $\mathcal{N} = 4$ is complicated

$$M_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} + \frac{1}{\epsilon} M_4^{(1)}(\epsilon^{-1}) + M_4^{(1)}(\epsilon^0) + \cdots$$

$$\begin{aligned}
M_4^{(1)}(\epsilon^{-1}) &= -\ln\frac{\mu^2}{-s} - \ln\frac{\mu^2}{-t} \\
M_4^{(1)}(\epsilon^{0}) &= -\ln\frac{\mu^2}{-t}\ln\frac{\mu^2}{-s} + \frac{2\pi^2}{3} \\
M_4^{(1)}(\epsilon^{1}) &= -\frac{\pi^2}{2}\ln\frac{-s}{u} - \frac{1}{3}\ln^3\frac{-s}{u} + \frac{\pi^2}{12}\ln\frac{\mu^2}{-s} - \frac{1}{6}\ln^3\frac{\mu^2}{-s} + \frac{\pi^2}{4}\ln\frac{\mu^2}{u} \\
&\quad + \frac{1}{2}\ln^2\frac{-s}{u}\ln\frac{\mu^2}{u} - \frac{1}{2}\ln\frac{-s}{u}\ln\frac{-t}{u}\ln\frac{\mu^2}{u} - \ln\frac{-s}{u}\mathrm{Li}_2\frac{-s}{u} + \mathrm{Li}_3\frac{-s}{u} + \frac{7}{3}\zeta_3 + (s \leftrightarrow t) \\
M_4^{(1)}(\epsilon^{2}) &= \frac{5\pi^2}{24}\ln^2\frac{-s}{u} + \frac{1}{8}\ln^4\frac{-s}{u} + \frac{3}{8}\ln\frac{-s}{u}\ln\frac{-t}{u} + \frac{1}{6}\ln^3\frac{-s}{u}\ln\frac{-t}{u} \\
&\quad - \frac{1}{4}\ln^2\frac{-s}{u}\ln^2\frac{-t}{u} + \frac{\pi^2}{24}\ln^2\frac{\mu^2}{-s} - \frac{1}{24}\ln^4\frac{\mu^2}{s} - \frac{\pi^2}{2}\ln\frac{-s}{u}\ln\frac{\mu^2}{u} - \frac{1}{3}\ln^3\frac{-s}{u}\ln\frac{\mu^2}{u} \\
&\quad + \frac{\pi^2}{8}\ln^2\frac{\mu^2}{u} + \frac{1}{4}\ln^2\frac{-s}{u}\ln^2\frac{\mu^2}{u} - \frac{1}{4}\ln\frac{-s}{u}\ln\frac{-t}{u}\ln^2\frac{\mu^2}{u} + \frac{7}{3}\zeta_3\ln^2\frac{\mu^2}{-s} + \frac{1}{2}\ln^2\frac{-s}{u}\mathrm{Li}_2\frac{-s}{u} \\
&\quad - \ln\frac{-s}{u}\ln\frac{\mu^2}{u}\mathrm{Li}_2\frac{-s}{u} + \ln\frac{\mu^2}{u}\mathrm{Li}_3\frac{-s}{u} - \ln\frac{-s}{u}\mathrm{Li}_3\frac{-t}{u} - \mathrm{Li}_4\frac{-s}{u} + \frac{49\pi^4}{720} + (s \leftrightarrow t)
\end{aligned}$$

Use universality of IR divergences to simplify.

$\mathcal{N} = 4$ Remainder Functions

- In $\mathcal{N} = 4$ SYM, in large N_c limit, n-point MHV amplitude can be written using **BDS ansatz** [2, 3] as:

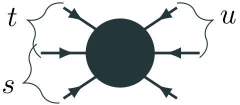
$$\mathcal{M}_n^{\text{BDS}} \sim \exp\left[\sum_L \lambda^L M_n^{1-\text{loop}} + \text{const.}\right]$$

  - Inspired by understanding of IR divergences by Catani [4].
- BDS ansatz holds for $n = 4, 5$.
- **Remainder function:** Ratio of $\mathcal{M}_n$ to $\mathcal{M}_n^{\text{BDS}}$:

$$R_n^{\text{BDS}} = \ln\left[\frac{\mathcal{M}_n}{\mathcal{M}_n^{\text{BDS}}}\right]$$

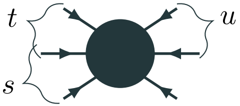
$\mathcal{N} = 4$ amplitudes simplify by taking ratios.

$\mathcal{N} = 4$: Dual Conformal Invariance of R^{BDS}



- $\mathcal{N} = 4$ is **dual conformally invariant**.
    - Dual conformal invariance (DCI): Amplitude depends only on DCI cross-ratios of Mandelstams.
- Amplitudes (e.g. $\mathcal{M}_4$) violate DCI at 1-loop.
- Remainder functions R_n^{BDS} are IR finite and DCI.
    - DCI is restored when computing the remainder functions instead of the amplitude $\mathcal{M}$.

$\mathcal{N} = 4$: Steinmann Relations



- Statement: **$\mathcal{M}$ cannot have sequential discontinuities in partially overlapping channels.**
  - $\mathcal{M}$ cannot contain $\ln s \ln t$ but can contain $\ln s \ln u$.
  - Old proof in S-matrix theory [5, 6].
  - New proof in time-ordered perturbation theory [7].
- Modified "BDS like" remainder function

$$R^{\text{BDS like}} = \ln\left[\frac{\mathcal{M}}{\mathcal{M}^{\text{BDS like}}}\right]$$

  is IR finite and may obey the **Steinmann Relations**.
  - Important for bootstrapping amplitudes.

Questions

- We have explored:
  - 4 point amplitude in $\mathcal{N} = 4$ is complicated, but exponentiates.
  - Remainder functions of higher point amplitudes in $\mathcal{N} = 4$ may be dual conformally invariant or obey Steinmann relations.
  - Use BDS subtraction schemes inspired by Catani's understanding of IR divergences.
- *Why are **remainder functions** R interesting in $\mathcal{N} = 4$?*
- *Why does R obey **symmetries** such as DCI?*
- *Why should R obey the **Steinmann relations**?*

Questions

What is the *fundamental object* we should calculate?

Can we exploit *universality of IR divergences* more consistently to expose the symmetric core of the S-matrix?

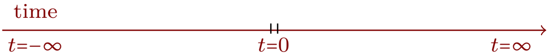
Step back: What are scattering amplitudes?

- Cross sections, decay rates, and other observables are proportional to squares of S-matrix elements:

$$\sigma \propto \int |\langle f| S |i\rangle|^2 d\Pi_f, \quad \Gamma \propto \int |\langle f| S |i\rangle|^2 d\Pi_f$$

- Intuitively: S-matrix gives the **probability amplitude** for an initial state $|i\rangle$ at $t = -\infty$ to transform into a final state $|f\rangle$ at $t = +\infty$.
  - Idea: $S = \lim_{t\to\infty} e^{-iHt}$? Gives infinitely oscillating phases when acting on energy eigenstates.
  - Resolution: Project onto **free states** at $t = \pm\infty$.

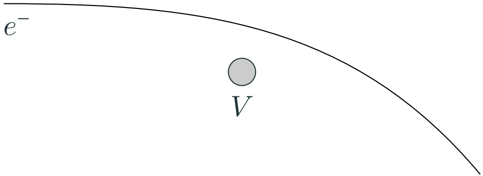
What are scattering amplitudes?



$e^{-iH_0 t}\,|i\rangle$ ↓ freely evolving state in far past

e^{-iHt} ↓ scattering, evolution with H

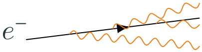
$e^{-iH_0 t}\,|f\rangle$ ↓ freely evolving state in far future



$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f|\, e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-}\, |i\rangle$$

Modify S-matrix to S_H

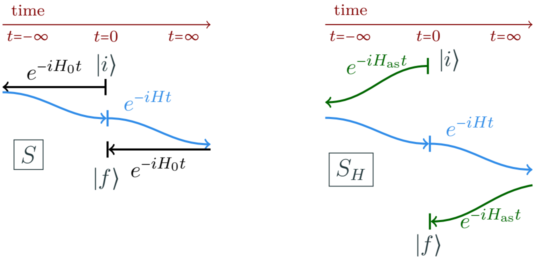
Interactions do not vanish as $t \to \pm\infty$ in theories with massless particles.



Redefine S-matrix in theories with long range interactions:

$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f | e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-} | i \rangle$$

$$\to S^H_{fi} = \lim_{t_\pm \to \pm\infty} \langle f | e^{iH_{\text{as}} t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_{\text{as}} t_-} | i \rangle$$



Choice of H_{as}

Pick H_{as} using **factorization**, and techniques from **SCET**:

$$H_{\text{as}} = H_{\text{SCET}}$$

$$S^H_{fi} = \lim_{t_\pm \to \pm\infty} \langle f| e^{iH_{\text{as}}t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_{\text{as}}t_-} |i\rangle$$

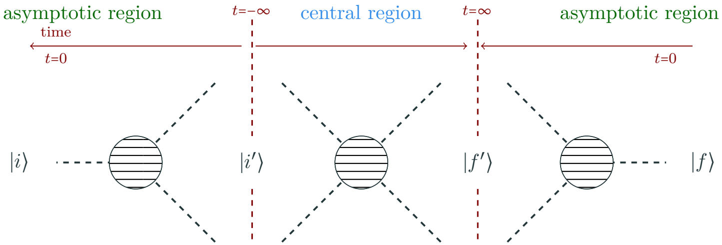
- IR finite by construction because of **universality of IR divergences** in gauge theories.
- States **evolve independently of how they scatter**.
- No scales, most integrals are zero in dim reg.
- New UV divergences dealt with using renormalization.

Three part calculation

Calculation trick in perturbation theory:

$$S^H_{fi} = \int d\Pi'_f \int d\Pi'_i \underbrace{\langle f|\, \Omega^{\rm as}_+ \,|f'\rangle}_{\text{TOPT rules}} \underbrace{\langle f'|\, S \,|i'\rangle}_{\text{usual Feynman rules}} \underbrace{\langle i'|\, \Omega^{\rm as}_+ \,|i\rangle}_{\text{TOPT rules}}$$

Calculations split into three parts:



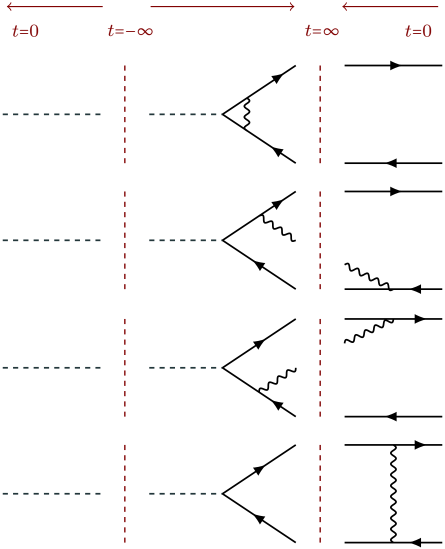
Asymptotic evolution **dresses** the states.

Example: $Z \to e^+ e^-$ for $H_{\text{as}} = H_{\text{SCET}}$

$m_e=0,\ L=\ln\frac{-E_{CM}^2}{\mu^2}$

$\mathcal{M}_0$: LO matrix element

Dim reg, CM frame



$$=\mathcal{M}_0\,\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon_{\text{UV}}}-\frac{2}{\epsilon_{\text{IR}}^2}-\frac{4+2L}{\epsilon_{\text{IR}}}-8+\frac{\pi^2}{6}-L^2+3L\right]$$

$$=\mathcal{M}_0\,\frac{\alpha}{4\pi}\left[\frac{2}{\epsilon_{\text{IR}}^2}+\frac{4+2L}{\epsilon_{\text{IR}}}-\frac{2}{\epsilon_{\text{UV}}^2}-\frac{4+2L}{\epsilon_{\text{UV}}}\right]$$

$$\langle e^+e^-|S_H|Z\rangle^{\overline{\text{MS}}}=\mathcal{M}_0+\mathcal{M}_0\,\frac{\alpha}{4\pi}\left[-8+\frac{\pi^2}{6}-L^2+3L\right]$$

Example: $Z \to e^+ e^-$ for $H_{\text{as}} = H_{\text{SCET}}$

$$\langle e^+ e^- | \, S_H \, | Z \rangle^{\overline{\text{MS}}} = \mathcal{M}_0 + \mathcal{M}_0 \frac{\alpha}{4\pi} \left[-8 + \frac{\pi^2}{6} - L^2 + 3L \right]$$

- S_H amplitudes in $\overline{\text{MS}}$ are the same as Wilson coefficients in matching to SCET.
  - Usually: difference of matrix elements in different theories (different fields, quantum numbers, degrees of freedom).
  - Here: matrix elements of a **single operator** using different Hamiltonians but **same fields**.
- S_H extracts the **interesting, non-universal** part of S.

Interpretation of S_H

a. Wilson coefficients in SCET
   - Encode hard dynamics.

b. Remainder functions in $\mathcal{N} = 4$ SYM
   - UV counterterm fixed by 1-loop S_H amplitude makes 2-loop S_H amplitude take a simple form in $\mathcal{N} = 4$.

c. Dressed states / Coherent states

a. Wilson Coefficients

- Calculations of S_H are identical to calculations of **Wilson coefficients** in SCET.
- S_H computes amplitudes with universal asymptotic evolution removed → encodes **hard dynamics**.
- Properties of Wilson coefficients identical to those of S_H.
  - What are the **analytic properties** of S_H?
  - What can we learn from **bootstrapping** S_H?
  - What are the **symmetry properties** of S_H?
  - Does S_H obey the **Steinmann relations**?

b. $\mathcal{N} = 4$ Remainder Functions

S_H *approach*:

- **Subtract** divergent amplitudes instead of taking ratios.
- Well-defined operator, no ad hoc subtractions.
- S_H amplitude for 4-points, 1-loop after renormalization:

$$\widehat{M}_4^{\mathrm{BDS},(1)} = -\ln\frac{\mu^2}{-t}\ln\frac{\mu^2}{-s} + \frac{5\pi^2}{6}$$

- S_H amplitude for 4-points, 2-loop after renormalization:

$$\widehat{M}_4^{\mathrm{BDS},(2)} = \frac{1}{2}\left[\widehat{M}_4^{\mathrm{BDS},(1)} - \frac{\pi^2}{6}\right]^2$$

- Finite parts exponentiate.

b. $\mathcal{N} = 4$ Remainder Functions

- UV counterterm fixed by 1-loop amplitude $\widehat{M}_4^{\mathrm{BDS},(1)}$ makes 2-loop amplitude $\widehat{M}_4^{\mathrm{BDS},(2)}$ take a simple form.
  - Cross terms between counterterm and $\widehat{M}_4^{\mathrm{BDS},(1)}$ simplify $\widehat{M}_4^{\mathrm{BDS},(2)}$.
- Different choices of remainder functions (BDS, BDS like, . . .) correspond to different **subtraction schemes** in SCET.

SCET explains why $\mathcal{N} = 4$ remainder functions take a simple form.

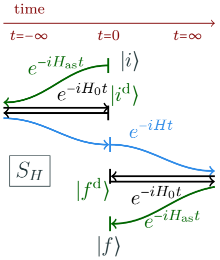
c. Coherent States

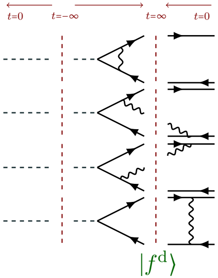
- Arise as intermediate steps in S_H calculations:

$$S^H_{fi} = \sum_{f'} \sum_{i'} \underbrace{\langle f | \Omega_+^{\rm as} | f' \rangle \langle f' |}_{\langle f^{\rm d} |} S \underbrace{| i' \rangle \langle i' | \Omega_+^{\rm as} | i \rangle}_{| i^{\rm d} \rangle}$$

- IR divergent states → IR problem moved from S to states:

$$|e^{\,\rm d}\rangle = |e(p)\rangle + e \sum_{\epsilon} \int \frac{d^{d-1}k}{(2\pi)^{d-1} 2\omega_k\, 2\omega_p} \frac{2\,p\cdot\epsilon(k)}{\omega_k - \frac{\vec{p}\cdot\vec{k}}{\omega_p} - i\varepsilon} |e(p-k), \gamma(k)\rangle + \cdots$$





Conclusions

$\boldsymbol{S_H}$: "hard" S-matrix defined using SCET
in theories with massless particles.

- Extracts the **fundamental** and **interesting** part of the S-matrix.
- Encodes hard dynamics of scattering processes.
- Interpretations:
  a. Wilson coefficients
     – *Future direction: Use **bootstrapping methods** to learn about **Wilson coefficients**?*
  b. $\mathcal{N} = 4$ remainder functions
     – ***SCET explains** why $\mathcal{N} = 4$ remainder functions take a simple form.*
  c. Coherent states

References i

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