Analytic Continuation and Reciprocity for Splitting Functions in QCD

Tong-Zhi Yang

Zhejiang University

with Hao Chen, Ming-xing Luo, Hua Xing Zhu and Yu Jiao Zhu

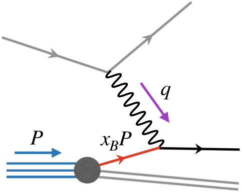
2006.xxxxx

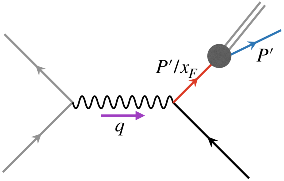
June 12, 2020
SCET 2020 Workshop

PDFs and FFs

$$\frac{d f_{i/N}}{d \ln \mu} = 2 \sum_k P^S_{ik} \otimes f_{k/N}$$

$$\frac{d\, d_{N'/i}}{d \ln \mu} = 2 \sum_k d_{N'/k} \otimes P^T_{ki}$$





$$x_B = -q^2/(2P \cdot q)$$

$$x_F = 2P' \cdot q/q^2$$

$$f_{q/N}(x_B) = \int \frac{db^-}{2\pi} e^{-ib^- x_B P^+} \times \langle N(P)| \bar{\chi}_i(b^-) \frac{\not{n}}{2} \chi_i(0) |N(P)\rangle$$

$$d_{\overline{N}/\bar{q}}(x_F) = \sum_X x_F{}^{1-2\epsilon} \int \frac{dt}{2\pi} e^{ib^- P'^+/x_F} \times \langle 0| \chi_i(b^-) \left|\overline{N}(P'), X\right\rangle \frac{\not{n}}{2} \left\langle \overline{N}(P'), X\right| \bar{\chi}_i(0) |0\rangle$$

Kinematic relation $P \to -P'$ $\quad x_B \to 1/x_F$

Significant efforts in calculation of splitting functions

- One-loop results $P^{(0)}$[Altarelli and Parisi, 1977]
- Two-loop results $P^{(1)}$[Furmanski and Petronzio, 1980; Curci, Furmanski and Petronzio, 1980]
- Three-loop results for space-like splitting functions
  - Non-singlet[Moch, Vermaseren and Vogt, 2004]
  - Singlet[Vogt, Moch and Vermaseren, 2004]
- Partial four-loop results[Moch, Ruijl, Ueda, Vermaseren and Vogt,2017]

Three-loop time-like splitting functions $P^{T(2)}$

Continuation between bare DIS and e^+e^- structure functions

- Non-singlet $P_{ns}^{(2)\,T}$ and diagonal elements of singlet[Mitov, Moch and Vogt, 2006; Moch and Vogt, 2007]
  - Problems in coefficient $C^3_{F/A}P^{(0)}_{qq/gg}\zeta_2\ln^2 z$ for $P^{(2)\,T}_{ns/gg}(z)$
  - Corrected by sum rules
- Off-diagonal elements of singlet[Almasy, Moch and Vogt, 2011]
  - $P^{(2)\,T}_{gq}$ determined
  - an uncertainty remaining for $P^{(2)\,T}_{qg}$

$$\delta P^{(2)\,T}_{\rm qg}(z) = \pm 2\zeta_2\beta_0\,(C_A - C_F)\,(11 + 24\ln z)P^{(0)\,T}_{\rm qg}(z)$$

Important to have an independent derivation of singlet non-diagonal elements

Crossing relation

- Gribov-Lipatov relation

$$P^{T(0)}(z) = P^{S(0)}(z) \equiv P^{(0)}(z)$$

- Drell-Levi-Yan relation

$$P^{T(0)}(z) = -z\, P^{S(0)}(x)\Big|_{x\to 1/z}$$

- Both relations break down beyond one-loop order, for example:

$$-zP^{S(1)}_{qq'}(1/z) - P^{T(1)}_{q'q}(z) = 8C_F T_F\Bigg(\frac{-(1-z)\left(38z^2+47z+38\right)}{9z} - \frac{(z+1)\left(4z^2+11z+4\right)}{3z}\ln(z)\Bigg) \neq 0$$

$$P^{T(1)}_{q'q}(z) - P^{S(1)}_{qq'}(z) = 8C_F T_F\Bigg(-\frac{(1-z)\left(56z^2+47z+20\right)}{9z} + \frac{1}{3}\left(-8z^2-21z-9\right)\ln(z) + (z+1)\ln^2(z)\Bigg) \neq 0$$

Problems with direct analytic continuation

- Mismatch of helicity sum and average in CDR, such as

$$P_{gq}^{T}(z) \sim (1-\epsilon) P_{qg}^{S}(1/z)$$

- Additional $z^{-2\epsilon}$ factor from phase space when going from parton frame to hadron frame in FFs

To solve the above two problems, using bare quantities for analytic continuation

- Splitting functions are extracted from squared amplitude quantities

$$P_{ji} \sim \int dP.S. \left| \overline{Sp}_{jX \leftarrow i} \right|^2$$

Squared amplitudes are not analytic functions of external momentum, careful analysis are needed.

Outline

- Splitting functions from TMD distributions
- Analytic continuation of (generalized) splitting amplitude
- Reparameterization III invariance in SCET, loop and phase space integrals, rapidity divergence
- Prescriptions for analytic continuation of TMDs and 3-loop time-like splitting functions
- Evidence of reciprocity relation in singlet sector in QCD

TMD definitions

Two fields separated with a transverse displacement $b_\perp$

- TMD beam function

$$\mathcal{B}_{q/N}(x_B, b_\perp) = \int \frac{db^-}{2\pi} e^{-ix_B b^- P^+} \times \langle N(P)|\bar{\chi}_n(0, b^-, b_\perp)|X_n\rangle \frac{\bar{\not{n}}}{2} \langle X_n|\chi_n(0)|N(P)\rangle$$

- TMD fragmentation function

$$\mathcal{F}_{\overline{N}/\bar{q}}(x_F, b_\perp) = \sum_X x_F^{1-2\epsilon} \int \frac{db^-}{2\pi} e^{ib^- P'^+/x_F} \times \langle 0|\bar{\chi}_n(0, b^-, b_\perp)|\overline{N}(P'), X_n\rangle \frac{\bar{\not{n}}}{2} \langle \overline{N}(P'), X_n|\chi_n(0)|0\rangle$$

- Outgoing quark Wilson line

$$\chi_n = W_n^\dagger \xi_n \,, W_n^\dagger(x) = \mathcal{P} \exp\left(ig_s \int_0^\infty ds \bar{n} \cdot A_n(x + s\bar{n}) e^{-\varepsilon s} \right)$$

Extracting splitting functions from poles of TMDPDF/FF

- Zero-bin subtraction and operator product expansion

$$\frac{\mathcal{B}_{ij}^{\text{bare}}}{\mathcal{S}_{0b}} = Z^B \mathcal{B}_{ij} = Z^B \sum_k \mathcal{I}_{ik} \otimes f_{k/j} \quad \frac{\mathcal{F}_{ij}^{\text{bare}}}{\mathcal{S}_{0b}} = Z^B \mathcal{F}_{ij} = Z^B \sum_k d_{i/k} \otimes \mathcal{C}_{kj}$$

  - In exponential regularization, $\mathcal{S}_{0b}$ is bare TMD soft function, known to three-loop Ref.[Ye Li and Hua Xing Zhu, 2016]
  - Z^B is the multiplicative renormalization factor
  - $\mathcal{I}_{ik}$ and $\mathcal{C}_{kj}$ is the matching kernel (finite in ϵ)

- Single pole of partonic collinear PDFs/FFs in terms of splitting functions

$$f_{i/j}^{(n)} \sim \frac{P_{ij}^{S,(n)}}{n\epsilon}, \quad d_{i/j}^{(n)} \sim \frac{P_{ij}^{T,(n)}}{n\epsilon}$$

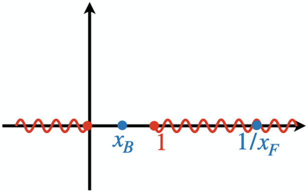
General structure of TMD distribution functions

- Full results for TMD beam function to three-loop[Luo,Yang, Zhu,Zhu,2019; Ebert,Mistlberger, Vita,2020] (see also Gherardo's tallk)
- All results are in terms of harmonic polylogrithms(HPLs) only

$$H_{-1,\vec{\omega}}(x_B) = \int_0^{x_B} \frac{dt}{1+t} H_{\vec{\omega}}(t) \quad H_{0,\vec{\omega}}(x_B) = \int_0^{x_B} \frac{dt}{t} H_{\vec{\omega}}(t)$$

$$H_{1,\vec{\omega}}(x_B) = \int_0^{x_B} \frac{dt}{1-t} H_{\vec{\omega}}(t)$$

- It's important that we don't get squared root alphabets in TMDs
- Branch cut structure



Need to work out the prescription

TMDs from splitting amplitude and phase space integrals

- Beam and fragmentation functions can be calculated from splitting amplitude[Waalewijn,Ritzmann,2014]

$$\mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i} = \langle X_n|\chi_n(0)|V^{i}_{P_l}(P_r)\rangle\,, \quad \mathbf{Sp}^{\mathrm{T}}_{X_n \bar{\imath}\leftarrow \bar{q}} = \langle X_n, V^{\bar{\imath}}_{P_l}(P_r)|\chi_n(0)|0\rangle$$

$|V^{i}_{P_l}(P_r)\rangle$ is incoming hadron state with momentum $P = P_l + P_r$

- All order amplitude representation of TMD beam and FFs

$$\mathcal{B}_{q/i} = \int d\mathrm{PS}_{X_n} e^{-iK_\perp\cdot b_\perp}\delta(K^+ - (1-x_B)P^+)\left|\overline{\mathbf{Sp}}^{\mathrm{S}}_{X_n q\leftarrow i}\right|^2$$

$$\mathcal{F}_{\bar{\imath}/\bar{q}} = x_F^{1-2\epsilon}\int d\mathrm{PS}_{X_n} e^{-iK_\perp\cdot b_\perp}\delta\left(K^+ - (1/x_F - 1)\,P'^+\right)\left|\overline{\mathbf{Sp}}^{\mathrm{T}}_{X_n \bar{\imath}\leftarrow \bar{q}}\right|^2$$

where K^μ is the total momentum of $|X_n\rangle$.

- Main challenge: understand the crossing relation for amplitude, and analytic structure of loop and phase space integrals

Prescription for Analytic Continuation from causality

Space-like amplitude: $\langle X_n|\chi_n(0)|V^i_{P_l}(P_r)\rangle\,,\quad \chi_n(x) = W^\dagger_n(x)\xi_n(x)$

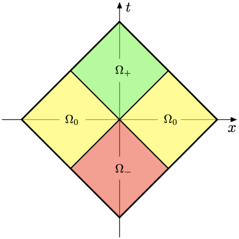
$W^\dagger_n(x) = \sum_{\text{perm}} \exp\left(-\frac{g}{\bar{n}\cdot\mathcal{P}}\bar{n}\cdot A_n(x)\right)$, local operator in residual coordinate

$$\xrightarrow{\text{LSZ}} \int d^d x e^{-iP_r\cdot x}\langle X_n|T\{\chi_n(0)J_{P_l}(x)\}|0\rangle\,,\; J_{P_l}(x) = i(i\mathcal{P}_l + \partial_x)^2 V^i_{P_l}(x)$$

$$\begin{aligned} T\{\chi_n(0)J_{P_l}(x)\} &= \pm\theta(x^0)J_{P_l}(x)\chi_n(0) + \theta(-x^0)\chi_n(0)J_{P_l}(x) \\ &= \pm J_{P_l}(x)\chi_n(0) + \theta(-x^0)\left[\chi_n(0), J_{P_l}(x)\right]_\mp \end{aligned}$$

$$\xrightarrow{\text{causality}} \underbrace{\int d^d x}_{x\in\Omega_-}\; e^{-iP_r x}\langle X_n|\left[\chi_n(0), J_{P_l}(x)\right]_\mp|0\rangle$$

$$\boxed{P_r \to P_r + i\varepsilon q_I\,,\; q_I \in \Omega^+}$$



Penrose diagram in Minkowski space

Prescription for Analytic Continuation from causality

Time-like amplitude

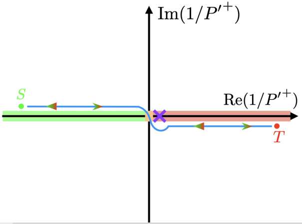
$$\langle X_n, V_{P'_l}(P'_r)|\chi_n(0)|0\rangle = \underbrace{\int d^d x}_{x\in\Omega^+}\, e^{iP'_r x}\langle X_n, V_{P'_l}(P'_r)|\left[J_{P'_l}(x), \chi_n(0)\right]_{\mp}|0\rangle$$

$$\boxed{P'_r \to P'_r + i\varepsilon q_I\,, q_I \in \Omega^+} \Rightarrow \boxed{P'^+ \to P'^+ + i0_+}$$

$$P'^+ + i0_+ \xrightarrow{\text{crossing}} -P^+ + i0_+ = P^+ e^{+i\pi} + i0_+ = P^+ e^{+i\pi + i0_+}$$

Crossing relation for amplitude

$$\mathbf{Sp}^{\mathrm{T}}_{X_n\bar{\imath}\leftarrow\bar{q}}(P^+ e^{+i\pi+i0_+}, \cdots) = \mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i}(P^+ e^{i0_+}, \cdots)\,,$$
$$\mathbf{Sp}^{\mathrm{S}}_{X_n q\leftarrow i}(P'^+ e^{-i\pi+i0_+}, \cdots) = \mathbf{Sp}^{\mathrm{T}}_{X_n\bar{\imath}\leftarrow\bar{q}}(P'^+ e^{i0_+}, \cdots)$$

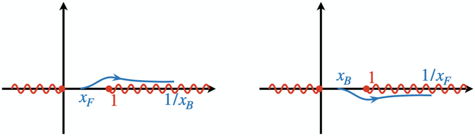


Prescription in x_F and x_B

$$\boxed{P'^{+} \to P'^{+} + i0_{+}}$$

Prescription in x_F and x_B

$$x_F = \frac{P'^{+}}{P'^{+} + K^{+}} \Rightarrow x_F \to x_F + i0_{+}$$



Prescription determined, the remaining complication is to determine the origin of $(1-x)^{\epsilon}$ and $\ln(1-x)$ terms

Reparameterization III invariance in $\mathrm{SCET_I}$

We consider threshold limit($x \to 1$), the corresponding theory is $\mathrm{SCET_I}$

$$\mathcal{L}_{\mathrm{SCET_I}} = \mathcal{L}_{n\xi} + \mathcal{L}_{ng} + \mathcal{L}_{us}$$

The loop momentum can be either in collinear mode or ultra-soft mode

$$\text{c}: \quad l \sim (1, \lambda^2, \lambda)$$
$$\text{us}: \quad l \sim (\lambda^2, \lambda^2, \lambda^2)$$

Reparameterization III invariance(RPI-3) in $\mathrm{SCET_I}$

$$n_\mu \to e^{\alpha} n^{\mu}$$
$$\bar{n}_\mu \to e^{-\alpha} \bar{n}_\mu$$

Analytic structure of TMD FFs in threshold limit

Decomposition of TMD FFs

$$\mathcal{F}^{(L)} = \sum_{n=1}^{L} \sum_{m_L=0}^{L-1} \sum_{m_R=0}^{L-1} \sigma^{T,n,m_L,m_R}, \quad n + m_L + m_R = L$$

$$\sigma^{T,n,m_L,m_R} = x_F^{1-2\epsilon} \int [d^d K] \int d\mathrm{PS}_{X_n} \delta^{(d)}(K - \sum_{r=1}^{n} k_r) \overline{\mathbf{Sp}}^{T,n,m_L} \overline{\mathbf{Sp}}^{*T,n,m_R}$$

$$\int [d^d K] = \int d^d K e^{-iK_\perp \cdot b_\perp} \delta\left(K^+ - (1/x_F - 1) P'^+\right)$$

Threshold limit

$$\lim_{x_F \to 1} \sigma^{T,n,m_L,m_R} \sim e^{i(m_L - m_R)\pi\epsilon} \sum_{m_1=0}^{m_L} C_{m_1} (1 - x_F)^{m_1 \epsilon} \sum_{m_2=0}^{m_R} C^*_{m_2} (1 - x^*_F)^{m_2 \epsilon}$$

C_{m_1} and $C^*_{m_2}$ are rational functions about x_F and ϵ

Predicting behavior of σ^{T,n,m_L,m_R}(consider m_R =0 case)

Decompose the loop momentum in collinear mode and ultra-soft mode

collinear mode : $l_1 \cdots l_{m_1}$

ultra-soft mode : $l_{m_1+1} \cdots l_{m_L}$ and $k_1 \cdots k_{n-1}, K$

$$\lim_{x_F \to 1} \sigma^{T,n,m_L,0} = \sum_{m_1=0}^{m_L} \hat{\sigma}^{T,n,m_1,0}$$

$$\hat{\sigma}^{T,n,m_1,0} = \underbrace{\int [d^d K]}_{\text{rapidity divergence}} \underbrace{\int dPS_{X_n} \delta(K - \sum_{r=1}^{n} k_r) \int \prod_{j=m_1+1}^{m_L} d^d l_j}_{\text{ultra-soft loop and phase space}} \underbrace{\int \prod_{i=1}^{m_1} d^d l_i}_{\text{collinear loop}} \mathcal{J}$$

$\mathcal{J}$ is the squared amplitude which can be predicted from $\mathrm{SCET_I}$, uniform in scaling λ

$\mathcal{J} = \mathcal{J}_{\text{u.s.}} \times \mathcal{J}_c$ $\quad \mathcal{J}_{\text{u.s.}}$ doesn't depend on collinear loop momentum

Collinear and ultra-soft integrals

- Collinear loop integrals

$$\int \prod_{i=1}^{m_1} d^d l_i \mathcal{J}_c \sim (2P' \cdot K)^{-m_1\epsilon} \mathcal{J}'_{\text{u.s.}}(\frac{n\cdot l_{m+1}}{n\cdot K}, \cdots \frac{n\cdot l_{m_L}}{n\cdot K}, \frac{n\cdot k_1}{n\cdot K}, \cdots \frac{n\cdot k_{n-1}}{n\cdot K})$$

  - $(2P' \cdot K)^{-m_1\epsilon}$ is determined by RPI-3 and dimensional analysis
  - We have replaced P' in $\mathcal{J}'_{\text{u.s.}}$ using $P' = \bar{n}\cdot P' n^\mu/2$
- Ultra-soft loop and phase-space integrals

$$(2P' \cdot K)^{-m_1\epsilon} \int dPS_{X_n} \delta(K - \sum_{r=1}^{n} k_r) \int \prod_{j=m_1+1}^{m_L} d^d l_j \mathcal{J}_{\text{u.s.}} \times \mathcal{J}'_{\text{u.s.}}$$

$$\sim (2P' \cdot K)^{-m_1\epsilon} (\bar{n}\cdot K n\cdot K)^{-(m-m_1+n-1)\epsilon} \times f\left(\frac{K^2}{\bar{n}\cdot K n\cdot K}\right) \sim (1-x_F)^{m_1\epsilon}$$

$$\vec{K}_T^2 = \bar{n}\cdot K n\cdot K - K^2 \sim (1-x_F)^0 \sim 1$$

Every collinear loop will generate a $(1-x_F)^\epsilon$

Every ultra-soft loop or phase-space will generate a $(1-x_F)^0 \sim 1$

Integrating over K with transverse momentum fixed

Subtlety from rapidity divergence

$$\int_0^\infty \frac{d\,\bar{n}\cdot K}{\bar{n}\cdot K} \text{ is not well defined}$$

All rapidity regulators explicitly violate RPI-3, we use exponential regulator $\exp(-\tau n\cdot K)$[Li,Neill,Zhu,2016]

- $\ln(1-x_F)$ from $\exp(-\tau n\cdot K)$ when integrating over K
- Combination $\ln(\tau n\cdot K) \sim L_\tau = \ln\left[\frac{\tau}{(1-x_F)\bar{n}\cdot P'}\right]$
- Well-defined in $x_F \to 1/x_B\,, P' \to -P$

$$L_\tau \to \ln\left[\frac{x_B\,\tau}{(1-x_B)\bar{n}\cdot P}\right] \equiv L_{\tau,B}$$

Analytic structure of TMD FF at 3-loop

General formula

$$\int dP.S.\overline{\mathbf{Sp}}^{T,n,m_L}\overline{\mathbf{Sp}}^{*T,n,m_R}$$

$$\sim e^{i(m_L-m_R)\pi\epsilon}\sum_{m_1=0}^{m_L} C_{m_1}(1-x_F)^{m_1\epsilon}\sum_{m_2=0}^{m_R} C^*_{m_2}(1-x_F^*)^{m_2\epsilon}\times[L_\tau \text{ terms from R.D.}]$$

Building blocks	TMDFF(3-loop)
VV*R	$C_c C_c^*(1-x_F)^\epsilon(1-x_F^*)^\epsilon + C_c C_s^*(1-x_F)^\epsilon + C_s C_c^*(1-x_F^*)^\epsilon + C_s C_s^*$
VVR + c.c.	$e^{2i\pi\epsilon}\left(C_{c,c}(1-x_F)^{2\epsilon} + C_{c,s}(1-x_F)^\epsilon + C_{s,s}\right) + \text{c.c.}$
VRR + c.c.	$e^{i\pi\epsilon}\left(\left[C_c^0 + C_c^1 L_\tau\right](1-x_F)^\epsilon + \left[C_s^0 + C_s^1 L_\tau\right]\right) + \text{c.c.}$
RRR	$\left[C_0^0(\epsilon) + C_0^1 L_\tau + C_0^2 L_\tau^2\right]$

Analytic structure of TMD beam functions at 3-loop

Time-like to space-like

$$(1-x_F) \to (1-x_B)e^{-i\pi}\,, \quad (1-x_F^*) \to (1-x_B^*)e^{i\pi}\,, \quad L_\tau \to L_{\tau,B}$$

Building blocks	TMD beam functions
$\mathrm{VV^*R}$	$\frac{1}{2}C_c C_c^* (1-x_B)^\epsilon (1-x_B^*)^\epsilon + C_c C_s^* e^{-i\pi\epsilon}(1-x_B)^\epsilon + \frac{1}{2}C_s C_s^* + \mathrm{c.c.}$
$\mathrm{VVR+c.c.}$	$C_{c,c}(1-x_B)^{2\epsilon} + C_{c,s}e^{i\pi\epsilon}(1-x_B)^\epsilon + C_{s,s}e^{2i\pi\epsilon} + \mathrm{c.c.}$
$\mathrm{VRR+c.c.}$	$\left[C_c^0 + C_c^1 L_{\tau,B}\right](1-x_B)^\epsilon + \left[C_s^0 + C_s^1 L_{\tau,B}\right] e^{i\pi\epsilon} + \mathrm{c.c.}$
RRR	$\left[C_0^0(\epsilon) + C_0^1 L_{\tau,B} + C_0^2 L_{\tau,B}^2\right]$

Space-like to time-like

$$(1-x_B) \to (1-x_F)e^{i\pi}\,, \quad (1-x_B^*) \to (1-x_F^*)e^{-i\pi}\,, \quad L_{\tau,B} \to L_\tau$$

Results and basic checks

- Satisfying sum rules

$$\int_0^1 dz\, z\left(2N_f P^T_{q_i g}(z) + P^T_{gg}(z)\right) = 0\,, \int_0^1 dz\, z\left(P^T_{gq}(z) + P^T_{qq}(z)\right) = 0 \quad \checkmark$$

- Agree with results in [Mitov, Moch and Vogt, 2006; Moch and Vogt, 2007; Almasy, Moch and Vogt, 2011] except $P_{qg}^{T(2)}$
- The difference with Ref.[Almasy, Moch and Vogt, 2011] is

$$P_{qg}^{T,(2)}\big|_{\text{CLYZZ}} - P_{qg}^{T,(2)}\big|_{\text{AMV}} = \\ -2\zeta_2(C_A - C_F)\beta_0\big[-4 + 8z + z^2 + 6(1 - 2z + 2z^2)\ln z\big]$$

  - This term also satisfies sum rules ✓
  - Vanishes in $\mathcal{N} = 1$ SYMs ($C_A = C_F = N_f$) ✓

This term can not be detected by sum rules or $\mathcal{N} = 1$ SYMs

Application of our results to the reciprocity relation

Splitting function & Anomalous dimension

Melin transformation

$$\bar{f}_a(N) = -\int_0^1 dx\, x^{N-1} f_a(x) \quad \gamma_{ab}(N) = -\int_0^1 dx\, x^{N-1} P_{a\leftarrow b}(x)$$

DGLAP in moment space: $\frac{d}{d\ln\mu^2}\bar{f}_a(N,\mu^2) = -\gamma^S_{ab}(N)\,\bar{f}_b(N,\mu^2)$

Light-cone expansion: $\bar{f}_a(N) \sim \langle N(P)|\,\bar{\psi}_a \bar{n}\cdot\gamma(i\bar{n}\cdot D)^{N-1}\psi_a\,|N(P)\rangle$

N-moment of the $P^S \longleftrightarrow$ anomalous dimension of Twist-2 Spin-N local operator

History of reciprocity relation

- At small N, $2\gamma_S(N) = 2\gamma_T(N + 2\gamma_S(N))$ [Mueller, 1983; Neill, Ringer,2020](See also Duff Neill's talk)
- At large N, proposed[Dokshitzer, Marchesini,Salam, 2006]

$$2\gamma_\sigma(N) = F(N - \sigma\,\gamma_\sigma(N)) \quad \text{SL: } \sigma = -1 \quad \text{TL: } \sigma = 1$$

- In CFT, [Basso, Korchemsky, 2006]

$$2\gamma_S(N) = F(N + \gamma_S(N))$$

$$\text{Assumption: } 2\gamma_T(N) = F(N - \gamma_T(N))$$

$$\Rightarrow \boxed{2\gamma_S(N) = 2\gamma_T(N + 2\gamma_S(N))}$$

Time-like process controlled by space-like data

Jet function & EEC calculation in CFT[Dixon, Moult, Zhu, 2019; Korchemsky, 2019; Caron-Hout; Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov, 2019; Chen,Moult,Zhang,Zhu,2020]

$$\frac{dJ(\ln\frac{zQ^2}{\mu^2},\mu^2)}{d\ln\mu^2} = \int_0^1 y^{N-1} J(\ln\frac{zy^2Q^2}{\mu^2},\mu^2) P_T(y,\mu^2)$$

Power-law solution for fixed coupling: $J(zQ^2,\mu^2) = C(\alpha_s)\left(\frac{zQ^2}{\mu^2}\right)^{\gamma_J}$

γ_J satisfies the relation $2\gamma_J(N,\alpha_s) = 2\gamma_T(N+2\gamma_J,\alpha_s)$

$$\text{If } \gamma_J = \gamma_S \quad \Rightarrow \boxed{2\gamma_S(N) = 2\gamma_T(N+2\gamma_S(N))}$$

- Reciprocity in CFT $\Rightarrow$ time-like fragmentation controlled by A.D. of local operator
- It would be very interesting to have such relation in QCD

Reciprocity for matrix-valued splitting function

For non-singlet, the reciprocity relation has been established[Dokshitzer, Marchesini,Salam, 2006]
For singlet sector(2×2 matrix), such relation has not been established

$$\frac{d\vec{J}(\ln \frac{zQ^2}{\mu^2}, \mu^2)}{d \ln \mu^2} = \int_0^1 y^{N-1} \vec{J}(\ln \frac{zy^2 Q^2}{\mu^2}, \mu^2) \cdot \widehat{P}_T(y, \mu^2)$$

Fourier transform at fixed coupling

$$\widetilde{J}(\omega, \alpha_s) \cdot \underbrace{(i\omega \mathbb{1} - \widehat{\gamma}_T(N + 2i\omega))}_{\text{eigenvalue problem}} = 0 \quad i\omega \sim \gamma_S$$

We expect the reciprocity to be valid in the eigenvalue sense

$$2\gamma_S^{(i)}(N, \alpha_s(\mu)) = 2\gamma_T^{(i)}\left(N + 2\gamma_S^{(i)}(N, \alpha_s(\mu)), \alpha_s(\mu)\right)$$

$\gamma^{(i)}, i = 1, 2$ represents two eigenvalues of 2 by 2 matrix of singlet sector γ

Non-trivial evidence of reciprocity in QCD@3-loop

- With the newly determined $P_{qg}^{T,(2)}$, we provide first evidence that reciprocity also holds in singlet sector

$$\left[2\gamma_S^{(i)}(N) - 2\gamma_T^{(i)}(N + 2\gamma_S^{(i)}(N))\right]_{\text{CLYZZ}} = 0$$

- Remarkably, the reciprocity relation works for all N
- Hints on hidden structure beyond small N and large N
- On the other hand, using the results from Ref.[Almasy, Moch and Vogt, 2011]

$$\left[2\gamma_S^{(i)}(N) - 2\gamma_T^{(i)}(N + 2\gamma_S^{(i)}(N))\right]_{\text{AMV}} =$$
$$\pm \left(\frac{\alpha_s}{4\pi}\right)^3 C_F(C_A - C_F)(11C_A - 2N_f)N_f 8\pi^2$$
$$\times \frac{(N-2)(N^2+N+2)(5N^2+24N^3+53N^2+50N+12)}{9(N-1)N^3(N+1)^3(N+2)^2\sqrt{\left[\text{Tr}(\gamma^{(0)}(N))\right]^2 - 4\text{det}(\gamma^{(0)}(N))}} \neq 0$$

Summary

- Extracting splitting functions from TMDs
- Analytic continuation of splitting amplitude
- RPI-3, collinear and ultra-soft loop integrals, and phase-space integral, rapidity divergence
- Determine three-loop time-like splitting functions completely
- First evidence for reciprocity relation in singlet sector in QCD

Thanks for your listen!