# Symmetry, Gauge Theory, and Asymptotic Freedom

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- What is symmetry in physics?
- Focus on a particular symmetry: gauge symmetry
- Local gauge symmetry generates fundamental interactions
- Asymptotic freedom

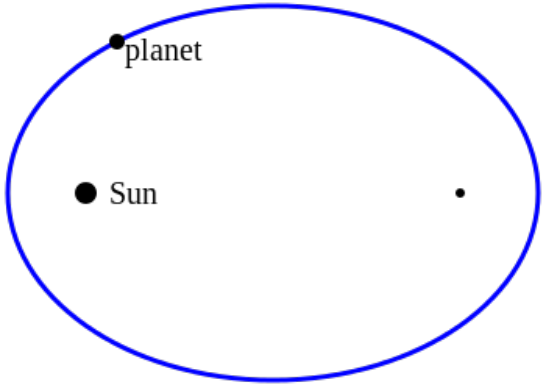
SYMMETRY: A similarity of form or arrangement, on either side of a dividing line.

We first become aware of *symmetries of shape:*

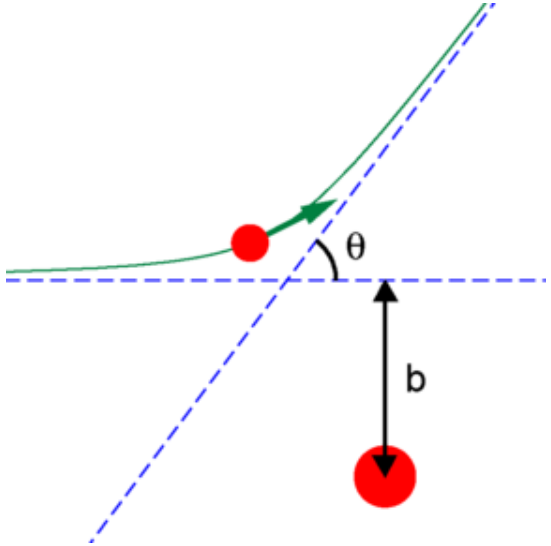
or

These are static. Now consider *symmetries of motion:*

planetary orbits




and particle trajectories



*Generalize to symmetries of sets of all possible motions:* equations of motion – laws of fundamental forces.

For example,

$$F_{grav} = G\frac{m_1 m_2}{r^2} \qquad \text{or} \qquad \mathrm{F}_{EM} = K\frac{q_1 q_2}{r^2}$$

The more axes of symmetry you could draw, the more symmetrical a system is. Notice that the law of motion can have more symmetry (e.g. spherical) than any particular realization of it (e.g. elliptical orbit).

**Maybe the deepest truths about nature are NOT the physical laws, which have certain symmetries among their properties.**

***Maybe the deepest truths are the symmetries themselves, which, in order to be realized in our universe, <u>generate</u> the forms of the physical laws.***

**Noether's Theorem: For every continuous transformation under which a Lagrangian is invariant, there exists a conserved current.**

*Every kind of transformation that leaves the laws of physics unchanged implies the existence of a conservation law.*

Here is *an example,* to show that spatial translation symmetry implies conservation of linear momentum:

**Intuitively:** we expect that space is homogeneous, meaning that the results of your experiment should not depend on where you do it (excluding effects of local forces). Thus if something in your experiment has momentum $p$ in one location, and it experiences no force, then the momentum $p$ elsewhere should be identical.

Translate this into math:

Recall Lagrange's Equation: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

Consider a non-relativistic, non-interacting (force-free) particle: $$L=\frac{p^2}{2m}$$

*This system (homogeneous space) is invariant with respect to a parameter (translation x).*

The Lagrangian does not contain that parameter, $x$.

So: $$\frac{\partial L}{\partial x} = 0.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\frac{d}{dt}(p) = 0$$

$p =$ constant, momentum is conserved.

*Other examples:*

| **Symmetry** | **Conservation Law** |
|---|---|
| Rotation (Spatial isotropy) | Angular momentum |
| Translation in time | Energy |
| Gauge | Charge |

Focus on the last one, **gauge invariance**. We'll follow these steps to explore it:

1. Recall gauge invariance in classical E&M.

2. Recall the Hamiltonian for the system of a charged particle in an EM field.

3. Phase invariance in quantum mechanics.

4. The connection between gauges and phases.

5. **Maybe the requirement of local gauge invariance is a deep truth about nature that determines what the fundamental forces (strong, electroweak, gravity) can be like....**

## 1. Recall gauge invariance in classical E&M:

Combine:

$$\left.\begin{array}{l}\text{Maxwell's Eq.}\quad \vec{\nabla}\cdot\vec{B}=0\\ +\\ \text{Vector calculus } \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0 \text{ [any } \vec{A}\text{]}\end{array}\right\}\Rightarrow \vec{B}=\vec{\nabla}\times\vec{A}$$

$$\left.\begin{array}{l}\vec{B}=\vec{\nabla}\times\vec{A}\\ +\\ \text{Vector calculus } \vec{\nabla}\times\left(\vec{\nabla}\Lambda\right)=0 \text{ [any } \Lambda\text{]}\end{array}\right\}\Rightarrow \left\{\begin{array}{l}\text{For a } \vec{B} \text{ and } \vec{A}_0\text{, if}\\ \vec{B}=\vec{\nabla}\times\vec{A}_0\text{, then}\\ \vec{B}=\vec{\nabla}\times\left(\vec{A}_0+\vec{\nabla}\Lambda\right)\\ \textit{also} \text{ works, for any } \Lambda.\end{array}\right\}$$

So we can pick any $\Lambda$, as long as it satisfies the rest of Maxwell's Equations. Consider these next...

Combine:

$$\left.\begin{array}{l}\text{Maxwell's Eq.}\quad \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\\ +\\ \vec{B}=\vec{\nabla}\times\vec{A}\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\vec{\nabla}\times\vec{E}=-\dfrac{\partial\left(\vec{\nabla}\times\vec{A}\right)}{\partial t},\ \text{or}\\ \vec{\nabla}\times\left(\vec{E}+\dfrac{\partial\vec{A}}{\partial t}\right)=0\end{array}\right\}$$

$$\left.\begin{array}{l}\vec{\nabla}\times\left(\vec{E}+\dfrac{\partial\vec{A}}{\partial t}\right)=0\\ +\\ \text{Vector calculus}\ \vec{\nabla}\times\left(\vec{\nabla}\Lambda\right)=0\ [\text{any }\Lambda]\\ \text{Call this particular }\Lambda=-\phi\end{array}\right\}\Rightarrow\left\{\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\phi\right\}$$

$$\left.\begin{array}{l}\vec{E}+\dfrac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\phi\\ +\\ \vec{A}\rightarrow\vec{A}^{\dagger}=\vec{A}+\vec{\nabla}\Lambda\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\vec{E}+\dfrac{\partial\left(\vec{A}^{\dagger}-\vec{\nabla}\Lambda\right)}{\partial t}=-\vec{\nabla}\phi,\ \text{or}\\ \vec{E}+\dfrac{\partial\vec{A}^{\dagger}}{\partial t}=-\vec{\nabla}\underbrace{\left(\phi-\dfrac{\partial\Lambda}{\partial t}\right)}_{\text{Call this }\phi^{\dagger}}\end{array}\right\}$$

.

**Conclusion:** **Maxwell's Laws are unchanged by the simultaneous transformations:**

$$\left\{\begin{array}{l} \vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda \\ \phi \rightarrow \phi - \dfrac{\partial \Lambda}{\partial t} \end{array}\right\} \text{ for any scalar } \Lambda.$$

This combined choice of A and $\phi$ is the <u>choice of gauge</u>.

So Maxwell's Equations have *gauge symmetry*.

2. Recall: **the Hamiltonian* for a charged particle in an EM field** is:

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 - e\phi$$

Compare this to:

*the Hamiltonian for a free particle:*

$$H = \frac{p^2}{2m}$$

So adding the EM field $\left(\phi, \vec{A}\right)$ appears the same as:

- offsetting the particle's energy by $e\,\phi$
- changing its momentum $\vec{p} \rightarrow \vec{p} - e\vec{A}$

*This is derived at page 575 of "Classical Electrodynamics," by J.D. Jackson.

## 3. Phase invariance in Quantum Mechanics

Recall: a QM prediction about a physical process requires a $\psi^*\psi$ combination (e.g., expectation values, probability densities).

So the transformation $\psi \rightarrow \psi \cdot e^{i\theta}$ does not affect predictions.

If $\theta$ is NOT a function of coordinates, we say *"$\psi$ has global phase invariance."*

*Consider the implications if the phase θ DOES depend on coordinates...*

so $\boldsymbol{\theta = \theta(x)}$ [$x$ can be space or time]

This is a "local phase"

....so the phase of ψ can be different at <u>every</u> point in space and time.

It may seem unlikely that any physical law could be invariant with respect to local phase...

but (surprisingly!) **if we require that quantum mechanics be invariant with respect to *local phase*, we find that** the electromagnetic force is *required to exist.*

"Quantum Mechanics": the Schroedinger and Dirac Equations, with their $\partial^2/\partial x^2$ terms.

Furthermore,

Global phase invariance leads to charge conservation.
-and-
Local phase invariance restricts the photon to be massless.

*Show how a demand for local phase invariance requires the electromagnetic force to exist:*

Suppose nature insists that quantum mechanics be invariant with respect to the local phase of ψ.

Recall that the Schroedinger Equation is non-relativistic. Let's be more general and use instead the relativistic equation that applies to particles like electrons and quarks.

This is the **Dirac Equation:**

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0$$ for a *free* electron.

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi(x)=0$$

where $\mu=(0,1,2,3)$

The $\gamma^{\mu}$ are 4x4 matrices:

$$\gamma^{0}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ \gamma^{1}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\ \gamma^{2}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},\ \gamma^{3}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}$$

The $\partial_{\mu}$ are derivatives:

$$\partial_0=\frac{1}{c}\frac{\partial}{\partial t},\ \partial_1=\frac{\partial}{\partial x},\ \partial_2=\frac{\partial}{\partial y},\ \partial_3=\frac{\partial}{\partial z}$$

$m$ is the mass of the particle involved,

and a summation over $\mu$ is implied.

What we mean by "invariance of the Dirac Equation" is preservation of the equation's form.

Suppose that nature insists:

"If $\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi(x)=0,$

then $\left(i\gamma^{\mu}\partial_{\mu}-m\right)e^{i\theta(x)}\psi(x)=0$ must be true too."

Can this work?

Consider the term $\partial_{\mu}e^{i\theta(x)}\psi(x)$:

by the chain rule of calculus, this is

$$\partial_{\mu}e^{i\theta(x)}\psi(x)=e^{i\theta(x)}\left[\partial_{\mu}\psi(x)+i\left(\partial_{\mu}\theta(x)\right)\psi(x)\right]$$

which will NOT reproduce the form of the Dirac Equation.

So $\psi\rightarrow e^{i\theta(x)}\psi$ *alone* does not work.

Nature has to try a little harder to make the local phase invariance thing work.

Try the *combination* :

$$\left\{\begin{array}{l} \psi \to e^{i\theta(x)}\psi \\ \partial_\mu \to \partial_\mu - ieA_\mu \\ A_\mu \to A_\mu + \dfrac{1}{e}\partial_\mu \theta(x) \end{array}\right\}$$

$A_\mu$ is for now an arbirary 4-dimensional vector.

We will find that this combination works:

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi(x)\Rightarrow\underbrace{\left[i\gamma^{\mu}\left\{\partial_{\mu}-ie\left(A_{\mu}+\frac{1}{e}\partial_{\mu}\theta(x)\right)\right\}-m\right]e^{i\theta(x)}\psi(x)}$$

$$\partial_{\mu}-ieA_{\mu} \qquad e^{i\theta(x)}\psi$$

$$A_{\mu}+\frac{1}{e}\partial_{\mu}\theta(x)$$

$$e^{i\theta(x)}\left[i\gamma^{\mu}\underbrace{\left(\partial_{\mu}-ieA_{\mu}\right)}-m\right]\psi(x)$$

If we call this "$D_{\mu}$", the "gauge-covariant derivative," then the form of the Dirac Equation is preserved as:

$$e^{i\theta(x)}\left[i\gamma^{\mu}D_{\mu}-m\right]\psi(x)=0$$

**It takes simultaneous changes of ψ and $\partial_\mu$ to make local phase invariance work.**

***Why is this important?***

Imposing the requirement of local phase invariance upon the Dirac Equation *mandates* that

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$$

**What is this $A_\mu$?**

Recall that quantum mechanics identifies the derivative operator with momentum:

$$p_{op} = -i\vec{\nabla} \qquad [\hbar=1]$$

so (using the 4-vector notation),

$$p_\mu = -i\partial_\mu$$

where $p_\mu=(E, p_x, p_y, p_z)$

So another way to write $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$ is:

$$ip_\mu \to ip_\mu - ieA_\mu$$

$$p_\mu \to p_\mu - eA_\mu$$

But this is exactly the way the momentum operator transforms in the presence of an EM field $(\phi, \vec{A})$!

So the $A_\mu$ here is NOT arbitrary, but is the usual pair of potentials $(\phi, \vec{A})$ of the EM field.

**Requiring local gauge symmetry calls into existence the electromagnetic interaction, $A_\mu$, "hidden," in the derivative operator.**

Because the requirement of local gauge symmetry makes the force (e.g., electromagnetism) exist, we call the carriers of the force (e.g., photons) ***gauge bosons.***

## 4. The connection between gauges and phases.

Begin with the Schroedinger Equation:

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Plug in the EM Hamiltonian: $H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + e\vec{A}\right)^2 - e\phi$:

$$\left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} + e\vec{A}\right)^2 - e\phi\right]\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad \text{"Eq 1"}$$

Now do either of 2 things:

Make the gauge transformation upon the operators:

$$\left\{\begin{array}{l}\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \\ \phi \to \phi - \frac{\partial\Lambda}{\partial t}\end{array}\right\}$$

Make the phase transformation upon the wave function:

$$\psi \to e^{ie\Lambda/\hbar}\psi$$

Plug either into Eq 1. They produce identical equations.

So "gauge = phase"

Now we could have 2 points of view:

*View #1:* "The principle of local gauge invariance generates the EM interaction"

-or-

*View #2:* "The EM interaction exists, and it manifests local gauge invariance as one of its properties.

**Which point of view is more fundamental?**

The principle of gauge invariance (View #1) can ALSO:

- predict charge conservation
- predict the masslessness of the photon

whereas View #2 (the description of the EM field) cannot.

*So perhaps View #1 is more fundamental?*

Optional slide on how global gauge invariance generates charge conservation

**It's related to energy conservation.**

*Propose the converse:*

- Suppose charge q can be created (requiring work $W_1$) and destroyed (requiring work $W_2$).
- Suppose $W \neq W(\phi)$. This is global gauge invariance.
- Let q be moved from one place ($\phi_1$) to another ($\phi_2$), requiring energy change $\Delta E = q(\phi_1 - \phi_2)$.
- Then the total energy change associated with

  creation + move + destruction is

  $\Delta E = W_1 + q(\phi_1 - \phi_2). + W_2.$
- If W were a function of $\phi$, it would be possible to adjust the W's and $\phi$'s to make $\Delta E = 0$ (energy conservation preserved) while maintaining non-zero q.
- But if $W \neq W(\phi)$ [global gauge invariance], then $W_1$ must equal $W_2$, so the only way to ensure $\Delta E = 0$ is to require q=0 [charge conservation].

## 5. Requiring local gauge invariance generates the strong force too

We showed that:

$$\text{allowing } \psi \to e^{i\theta(x)}\psi \text{ calls into existence } \left(\phi, \vec{A}\right), \text{ the EM potential.}$$

We treated the $\theta(x)$ as a scalar.

Generalize: **let the $\theta(x)$ be matrices.**

Suppose θ(x) is 3x3 and unitary.

Expand it in a basis $T$:

$$\theta(x) = \sum_i \alpha_i T^i$$

Again require that $\psi(x) \rightarrow e^{i\theta(x)}\psi(x)$ maintains the form of the Dirac Equation.

**You are then forced to create a new covariant derivative:**

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + igT_i G^i_\mu$$

$G_\mu$ is a field, like $A_\mu$

$g$ is a charge like $e$

The $G^i_\mu$ are the gluons. There are 8 of them.

**And this is the strong force.**

*An extra complication –*

Recall that a gauge transformation requires that the derivative ($\partial_\mu$) and the field ($A_\mu$ or $G_\mu$) transform together. **The transformation of $G_\mu$ is different from the transformation of $A_\mu$ – *that's good, as it makes the strong force different from the electromagnetic one.***

Transformations:

**EM**

$$\partial_\mu \to \partial_\mu + ieA_\mu$$

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \theta(x)$$

**Strong**

$$\partial_\mu \to \partial_\mu + igT_i G_\mu^i$$

$$G_\mu^a \to G_\mu^a - \frac{1}{g}\partial_\mu \alpha^a - f_{abc}\alpha^b G_\mu^c$$

Because the T's are non-commutating matrices, this extra term arises. The $f_{abc}$ are the structure constants associated with the group, SU(3), represented by those matrices.

**What physics does the extra term ($-f_{abc}\,\alpha^b\,G_\mu{}^c$) give to the strong force that is not present in the electromagnetic?**

Gluons (the carriers of the strong force) can interact with each other; photons (the carriers of EM) cannot.

This produces **asymptotic freedom**, the property that the strong force becomes very weak at short distances, so that quarks and gluons are unbound inside the nucleus.

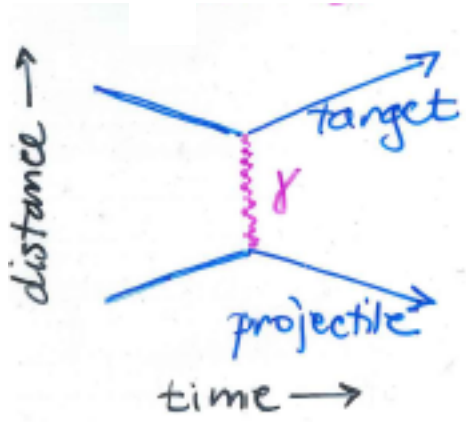
**The strong coupling "constant" is not constant** – *it varies with distance.*

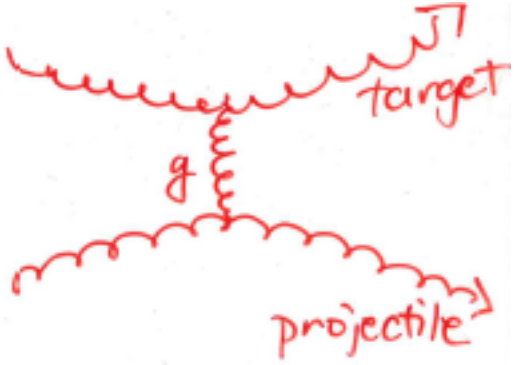
How this works...

Suppose that we want to study the strengths of the various forces.
"Strength" = magnitude of the coupling, which is proportional to the magnitude of the charge.

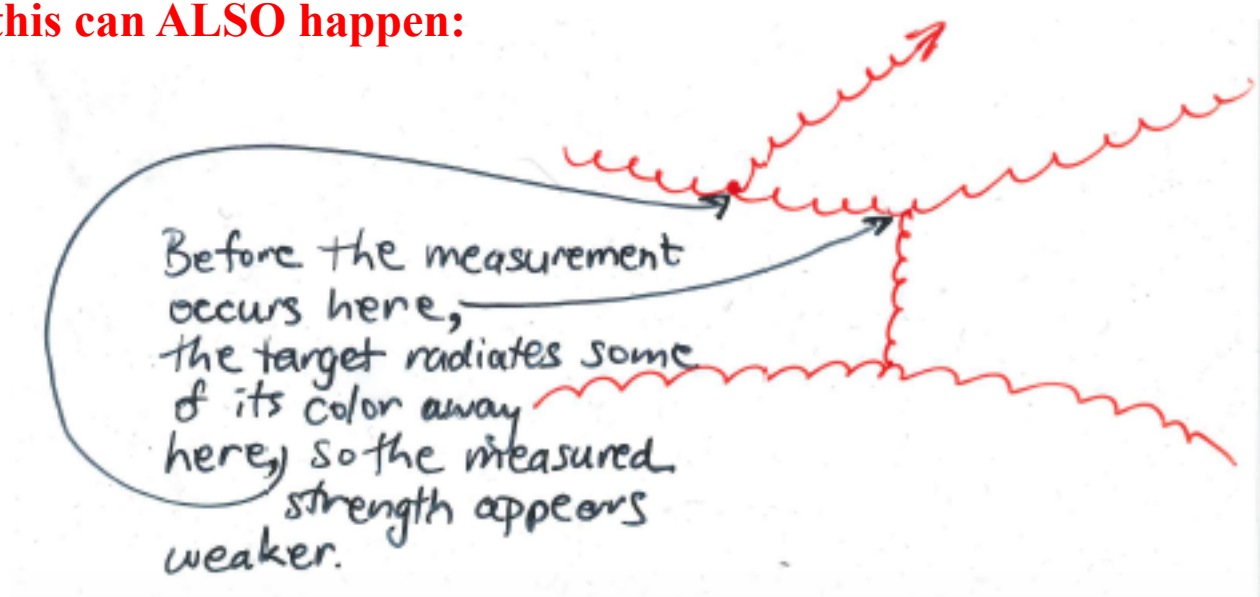
The strength indicates the probability that the particle under test will emit the kind of boson (photon, gluon...) that carries that force.

| | EM force | Strong force |
|---|---|---|
| coupling "constant": | $\alpha \equiv \frac{e^2}{\hbar c}$ | $\alpha_s$ |
| proportional to: | electric charge "$e$" | color charge "$g$" |
| Direct a projectile | electron | gluon |
| at a target | electron | gluon |
| So this can happen: | The target emits photons | The target emits gluons |



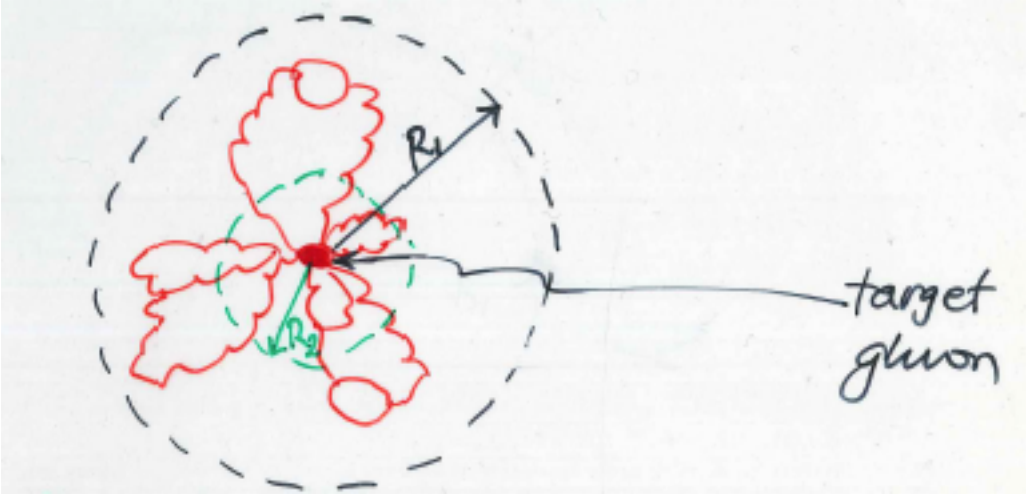
**But in the case of the strong force, this can ALSO happen:**




The radiated gluons always eventually connect back to the target (there are no permanently free quarks or gluons) but the loops they make can be large:


*So the closer the projectile comes to the target, the more likely it is to miss sensing some of the color.*

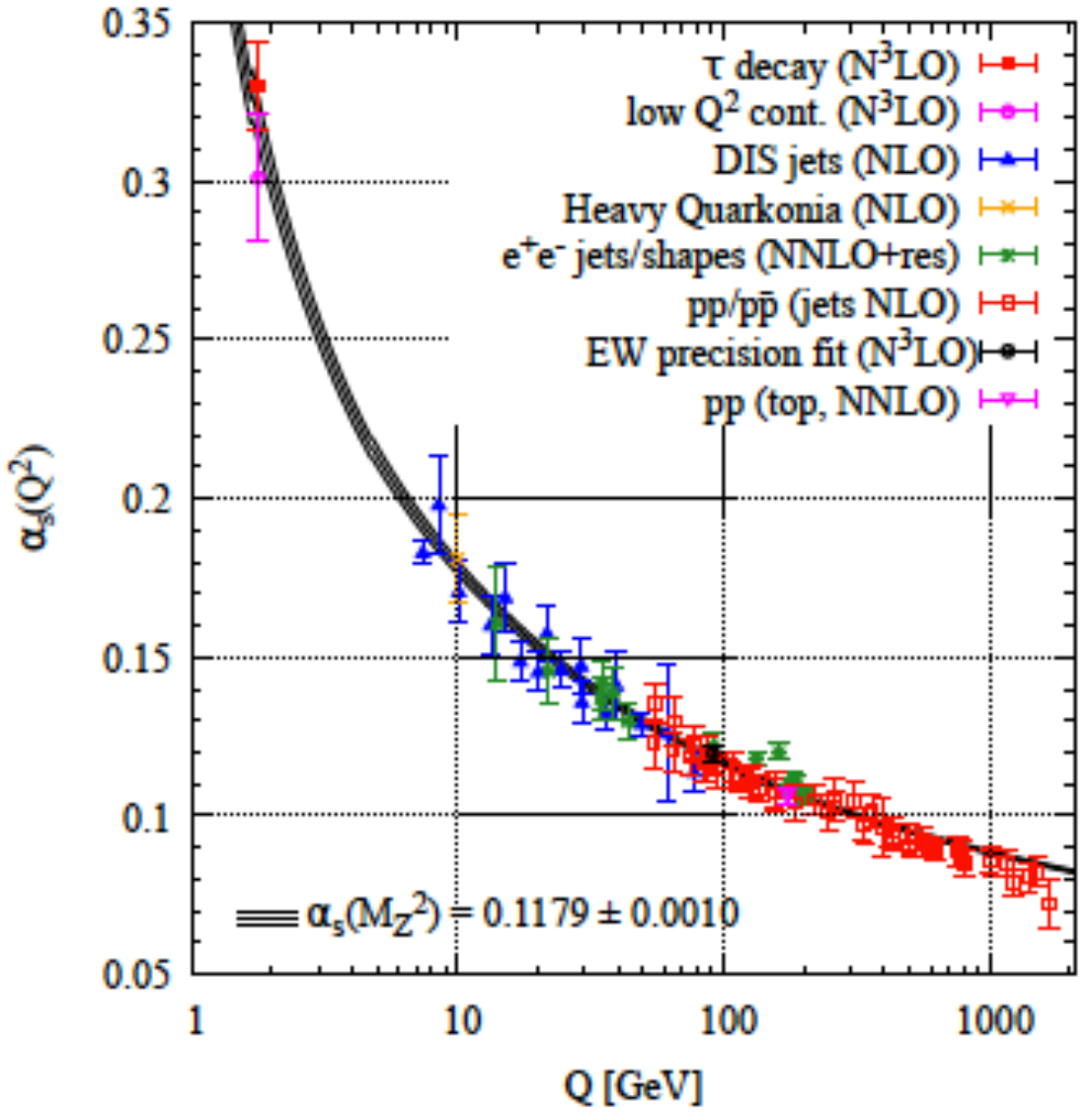
A projectile that recoils at radius $R_1$ must have less energy than one that recoils at radius $R_2$.




A projectile that recoils at radius $R_1$ will sense all the color, whereas a projectile that gets to radius $R_2$ will miss the gluons that are temporarily in the volume outside of $R_2$.

The higher the projectile's energy is, the closer it gets to the target before deflection. So we **expect $\alpha_s$, to decrease as momentum transfer Q increases.** That's what's observed:

We say: **"$\alpha_s$ runs with energy"**




←far from target near to target→

## Summary

1. Introduce the idea of symmetry, implying invariance, generalizing from geometrical configurations to mathematical expressions. Noether's Theorem relates symmetry in a system to conservation of a quantity that is measured in that system.

2. Gauge invariance of a set of equations: multiple simultaneous transformations leave the form preserved.

3. Local gauge invariance, with "scalar phase," (this is U(1)) produces the electromagnetic force.

4. Local gauge invariance, with "non-commuting matrix phase," (this is SU(3)) produces the strong force.

5. The correctness of the SU(3) choice is experimentally confirmed in the observation of asymptotic freedom and the running of the $\alpha_s$ coupling.

Maybe the deepest truths about nature are not the physical laws, which have certain symmetries among their properties.

**Maybe the deepest truths are the symmetries themselves, which, in order to be realized in our universe, generate the forms of the physical laws.**